MECHANICS

FOR

PRACTICAL MEN.

BY

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Author of a Dictionary of Mechanical Science, and a Treatise on the Elements of Algebra.

TREATISES

ON THE COMPOSITION AND RESOLUTION OF FORCES; THE CENTRE OF GRAVITY; AND THE MECHANICAL POWERS.

Illustrated by Examples and Diagrams.

"There are many calculations, in which the introduction of algebraical symbols, at a certain stage, is, practically speaking, absolutely indispensable; but we have always observed that the further the verbal reasoning, or the geometrical representation, could be carried, the more simple, elegant, and satisfactory was the solution; and on the other hand, that the unnecessary adoption of literal characters has almost uniformly tended to divert the mind from the true state of the inquiry, and to suspend the exercise of the judgment, while the eye and the memory only were occupied in the mechanical process of manufacturing a work of science."

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MECHANICS FOR PRACTICAL MEN.

INTRODUCTION.

MECHANICS is the science which enquires into the laws of equilibrium and the motion of solid bodies; into the forces by which bodies, whether animate or inanimate, may be made to act upon one another; and into the means by which these forces may be increased so as to overcome those which are more powerful. The term *Mechanics* was originally applied to the doctrine of equilibrium. It is now, however, extended to the motion and equilibrium of all bodies, whether solid, fluid, or aeriform. In this volume, the term is used in its primitive signification; the adjunct by which we have designated our work, is meant to convey the idea of a book that is self-instructing, or intended to teach those who have not had the benefit of a regular academic education, how they may pursue the study of MECHANICS; of which we here offer the first three subjects, viz.—

The Composition and Resolution of Forces,
The Centre of Gravity, and the
Mechanical Powers.

The complete arrangement of Mechanics is now made to embrace besides, the Pressure and Tension of Cords, the Equilibrated Polygon, the Catenarian Curve, Suspension Bridges, the Equilibrium of Arches and the Stability of their Piers, the Construction of Oblique Arches, the Equilibrium of Domes and Vaults with Revetments, the Strength of Materials whether they be of Wood or of Iron, Dynamics, or the Science of Moving Bodies, with Hydrostatics, Pneumatics and Hydraulics.

Some of these topics belong to the mechanics of antiquity; others of them date their origin from the age of Galileo, and from that of

Newton; and some others of them have but recently engrossed the contemplation of philosophers and mathematicians.

We are indebted to Archimedes for having preserved to us some of the knowledge which the ancients possessed respecting mechanics, theoretical and practical. But the age of Archimedes belongs not to antiquity, properly so called, for he lived in the consulship of Marcellus, and that was only about 200 years before the Christian æra. We must look back upon Asia, as the birth-place of science; and Egypt, as the cradle of the arts. And it is, therefore, no difficult matter to account for the silence of remote antiquity respecting the theory of mechanics; for of the existence of the lever, the wheel and axle, the pulley, the inclined plane, and the screw, it were folly to doubt. But in those countries the learned and the scientific were separated from the rest of the people by the language of mystery in which they communicated knowledge to even the initiated. And even if mechanical science had been taught without metaphor, the wreck of ages, which has spread desolation over Babylon and Nineveh, Memphis, and Ægyptian Thebes, would have swept away all records of those principles for which we stand indebted to the philosopher of Syracuse. How much soever Archimedes might improve mechanics, we must go back to ages before his birth, for the invention of those machines which abridge and facilitate manual labour. Babylon was a circuit of thirty miles; Nineveh a journey of three days around; its walls, an hundred feet high, were broad enough to allow three chariots to be driven abreast on their summit; and the towers thereon, fifteen hundred in number, were two hundred feet in height. Of Babylon and Nineveh no trace remains; the pyramids attest the grandeur of Egypt; and its hieroglyphical writing leaves no doubt that the elements of such science as then existed were wrapped up in impenetrable mystery from the understanding of the common people. All these facts respecting the civilization of a very remote antiquity, make us regret the loss of that knowledge which prevailed when the wise became philosophers for their amusement; but our regrets are diminished by the recollection that the curiosity of the human mind has outlived the existence of powerful kingdoms, whose men of science it would be unwise to depreciate by exalting the contemporary of Marcellus.

We are bound, however, to believe that Archimedes greatly improved the geometry of the Greeks, and that his discoveries in statics, or the equilibrium of solid bodies, rank him above all the ancients as a writer on mechanical philosophy. These discoveries he applied practically in the defence of Syracuse, when the Roman engineer, Appius, directed the assault of the besieging army with its well-appointed galleys; but we can say nothing of Archimedes that is not already known to every reader of biography; we shall however, in the sequel, give a short sketch of his knowledge and application of the principles of mechanics theoretical and practical.

We find little improvement in statics from the time of Archimedes till the sixteenth century, when SIMON STEVINUS, a Dutch mathematician, contributed greatly to the advancement of the doctrine of equilibrium. In a work upon Sluices, he introduced the novel but now celebrated problem of the composition of forces, and thus appears as the earliest writer on mechanics who had ventured to consider the composition of pressures as different from the mere composition of motions. In his *Statics*, published in 1586, he demonstrates, that

> "If a body be urged by two forces in the direction of "the sides of a parallelogram and proportional to these "sides, the combined action of these forces is equivalent "to a third force acting in the direction of the diagonal of "the parallelogram, and having its intensity proportional "to that diagonal."

The solution of Stevinus has been supplanted by the demonstrations of Daniel Bernouilli,* D'Alembert,† Riccati, Fonseneix,‡ Belidor,§ Frisius,|| Poisson,¶ La Place,** and Dr. Robison. But the merit of the discovery belongs exclusively to the Dutch engineer, who has, besides, the honour of illustrating other parts of statics, and discovering the laws of equilibrium in bodies placed on an inclined plane, and, in the French edition of his works published in 1634, an enlarged appendix is given, in which he treats of the rope machine, and of pulleys acting obliquely.

* Comm. of the Imp. Acad. of Sciences at St. Petersburg.
† Mem. of Acad. Paris, 1769.
‡ Mem. of Acad. of Turin.
§ Ingenieur Français.
|| Cosmographia.
¶ Mecanique.
** Mecanique Celest. tom. 1. § 1.

The opinion which is entertained of the different demonstrations of this problem may be very briefly summed up. Bernouilli has given the first complete solution, which we must not confound with his son's plagiarism on Newton's doctrine of deflecting forces. D'Alembert greatly simplified and improved the demonstration of D. Bernouilli; the solutions of Riccati and Fonseneix, analytical and concise as they are, require an acquaintance with the higher mathematics; the demonstration of Belidor is taken from the *Journal des Scavans* for 1764. Frisius' solution may be easily comprehended by an ordinary mathematical reader; the demonstrations of Poisson and La Place are ingenious, but elaborate; and with one or two exceptions, none of these can be accounted elementary: they are, besides, very remote from the train of reasoning by which the truth was originally discovered by Simon Stevinus, whose name should have obtained a greater celebrity than it has yet enjoyed.

Dr. Robison, whose demonstration, purely geometrical, is the most expeditious and simple that has yet appeared, has blended together the methods of Daniel Bernouilli and D'Alembert, limiting his inquiry entirely to pressures, without at all considering and employing the motions which they may be supposed to produce. Dr. Gregory, whose mechanics have obtained a distinguished reputation, employs, if we mistake not, the method of Francœur, who simplified D'Alembert's demonstration.

The problem of the composition and resolution of forces may be resolved by a graphical construction, and the result obtained in this way has been thought sufficiently accurate for the ordinary purposes of mechanics. But science requires greater nicety than can be obtained from any operation purely mechanical, or altogether manual. Hence have originated the elaborate analytical demonstrations of the French and Italian mathematicians, whom men of science in Britain have too servilely followed, without adopting their principles and remodelling the problem to render it of general practical utility to those who are not deeply read in pure and mixed mathematics. Besides, if we desired a text-book for public instruction upon this individual branch of mechanics, where shall we find a popular treatise combining the means with the end for such a

laudable undertaking? We speak with reverence, when we affirm, that there is no treatise, except the one which we produce, that embraces, to the same extent, and in such varied application, the twofold properties of precept and example in this important problem of the parallelogram of forces.

The method which we have adopted is altogether different from any that we have seen. We have taken hold of all the essential elements of the problem, each of which has been demonstrated with rigorous exactness, and familiarly illustrated by examples in such a manner as to make its details available in any course of public or private instruction, while it is generally and particularly applicable to the mechanics of practical men. It is thus, we conceive, that the problem of the parallelogram of forces—itself of boundless extent—may be rendered the great basis on which the whole doctrine of mechanical science can, with general application, be founded. For, when the resolution of this problem is once established, in its simplest form, it conducts by a continual chain of reasoning through all the complicated phenomena of physical science, building one truth upon another till the whole rises into a sublime and massive pile, involving the various laws and operations of the material world.

This treatise upon the parallelogram of forces, the reader will find has been drawn up as a separate and independent work; and that its theory has been deduced from the principles upon which the subject is founded, unconnected with any other, except in those general propositions which are applicable alike to all branches of the mixed mathematics. This plan precludes the necessity of reference to any treatise beyond the one immediately under consideration, while at the same time, from its own materials, it permits the reader to make himself perfectly master of the subject, and thereby prepare himself for ultimately reading the whole of mechanics, as arranged in scientific order under their respective heads. And upon this principle have the several treatises which we have written upon mechanics been constructed. The advantage of this plan to practical men is, that each treatise becomes systematically a text-book, and by its examples a practical book of reference in that individual subject upon which it treats.

To accomplish these objects, it became necessary to handle the several subjects very amply; but in the developement of their principles we are not aware that we have introduced more theory than would serve for the complete illustration of those principles, and the combination of such practical materials as experience or observation had placed at our disposal. For, to have extended our theory beyond this object, or to have involved with a course of reading designed to be generally useful, subjects of mere speculative inquiry, would have added nothing advantageous to our plan, nor interesting among its details. On the contrary, in the several articles, our aim has been to supply a desideratum in the philosophy of mechanical science, by furnishing expeditious and practical methods of operation in its general and every-day business. Whether we have succeeded effectually in every part of our task, the intelligent reader will judge and pronounce. Except in their principles, he will find our treatises original. This assertion is verified by the matter and manner in which these treatises differ from the compositions of any of our predecessors and contemporaries.

The following are the points of distinction to which we allude.

In the first place, having established the principles theoretically, we have illustrated every conclusion to which our investigations led by practical examples. Constructed after this fashion, each article becomes didactic; for it is thereby rendered a *manual of principles* to the directing engineer, and of practical utility to the operative machinist. Every principle in the theory is brought under the form of an equation, generally of the first or second degree; and this formula is translated, if we may so phrase our thought, into a practical rule, to which are attached examples worked out at full length, so that any person who understands the ordinary course of arithmetic, and the use of logarithmic tables, may trace for himself the algebraical or geometrical investigations which have preceded the examples, and which are expressed by the general rule. In almost all these questions, geometrical constructions, illustrated step by step, have preceded the numerical operations; and therefore he who will become master of the subject by both these methods, must by means of the manual operations, necessarily acquire the dexterity of a draftsman, and, by the intellectual process which accompanies it,

the address and facility of a practical calculator. We know of no other work on mechanics which enables its reader to do this.

In the second place, our investigations are not crowded by a multiplicity and variety of symbols. We have very sparingly introduced the fluxional calculus, or any of the higher departments of modern algebra; hence the simplicity of our notation, and the originality of our plan for bringing every separate proposition into a concise equation, accompanied by its rule, and illustrated by examples, embracing as they do a great variety of practical operations, found in various departments of mechanical science, but which have hitherto been omitted, as exemplifying the principles which we employ them to elucidate.

Of the Plates it will be sufficient to observe, that the diagrams which they contain have all been drawn according to the conditions of the questions upon which they are founded; and reference is made from the diagrams in the plates to the particular examples upon which they have been constructed.

It will be observed that four of the figures on Plate III. of the CENTRE OF GRAVITY are counterparts of the wood engravings in the body of the work; but they have been repeated in this instance because of their great practical importance, and in order to show how such figures ought to be constructed, when it is desired to obtain accurate working drawings. Some of the diagrams of the pulley, White's for example, have been repeated on the copper-plates; and without this process we could not have exhibited the tension on the cords, and equilibrium of the system.

All the copper-plate diagrams are independent figures, as will become manifest by inspection of their details; take, for example, those of the PARALLELOGRAM OF FORCES: every diagram is itself a solution or exposition of the question on which it is founded; for, by reason of its geometrical construction and literal description, it presents a faithful analysis and composition of the algebraic and arithmetical demonstration, incomparably more satisfactory, when thus detached, than if placed in the text along with its corresponding example, and delineated simply by the wood engraving; and what is true of the two plates of diagrams belonging to the PARALLELOGRAM OF FORCES, is true also of the three belonging to the

Centre of Gravity, and those remaining plates which so beautifully exhibit the various orders of Levers, the Wheel and Axle, and the all-prevailing Pulley. And pointing to the remark with which this volume ends, I cannot help recommending to students of mechanics to study *Euclid's Geometry*; which is "a better key to a great quantity of useful reading" than any other Elements of Geometry in existence; and which is one of the grand handmaids of sound knowledge in all the mixed sciences, especially mechanics. A good geometrician can rarely fail to be a good draftsman. But to proceed:

As our examples are original, so are the diagrams; which Mr. Lowry has executed with his well-known ability, and knowledge of the subjects to which they relate.

These Plates are,—

Two on the Composition and Resolution of Forces.

Three on the Centre of Gravity of Planes and Solids.

Four on the Mechanical Powers, viz.—

Two on the different orders of the Lever, Wheel and Axle, and

Two on the various combinations of the Pulley.

The other powers are left for the reader's exercise; the diagrams in wood being all that our investigations require; which is true also of the other subjects throughout this volume.

In conclusion ὡς ὑπ' εὐκλείᾳ θάνω, I have now to acknowledge the assistance which, in the composition of the work, I have received from an engineer and a mathematician,

Quam, scit uterque, libens (censebo) exerceat artem.

Our united labours in the preparation of the whole of the MSS., before a sheet of the work was printed, should be some guarantee with the public for the accuracy of the calculations which every where abound.

A. J.

COMPOSITION AND RESOLUTION OF FORCES.

CONTENTS.

MECHANICS FOR PRACTICAL MEN.

OF THE COMPOSITION AND RESOLUTION OF FORCES.

1. The Composition of Forces is that process by which we determine the *magnitude* and *direction* of a single force, the energy of which shall be equivalent to the energy of two or more given forces.

The force thus obtained or determined, is called the *resultant*, or the *equivalent* of the given forces, and the given forces themselves are called the *component* or *composant* forces.

The *magnitude or intensity of a force*, is the measure of that quantity by which it is expressed; and the *direction of a force*, is the straight line over which that force would urge a material point in a given time.

2. For the complete solution of this problem, we shall divide it into four distinct cases, in the following order.

The first case, is that in which the component and resultant forces are situated in the same plane, and concur in the same point.

The second case, is that in which the component and resultant forces are situated in different planes, but concur in the same point.

The third case, is that in which the component and resultant forces are situated in the same plane, but applied to different points of the same body.

And the fourth case, is that in which the component and resultant forces are situated in different planes, and applied to various parts of the same body.

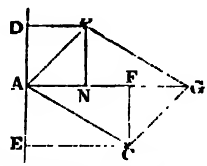
In our investigation of these cases, we shall endeavour to avoid all prolixity, and confine our inquiries to the simplest combinations that necessarily belong to each, giving the subject, as it unfolds itself in our investigations, all the generality of which it is susceptible.

CASE I.

Section First.

OF FORCES SITUATED IN THE SAME PLANE, AND CONCURRING IN THE SAME POINT.

3. The general principle for this case is expressed in the following Proposition. *Any two forces whatever, have their resultant or equivalent represented in magnitude and direction, by the diagonal of the parallelogram constructed on the lines which represent the given forces.*



For, let the two straight lines BA and CA, represent the magnitudes and directions of any two forces which concur in the same point A, and upon these lines describe the parallelogram ABGC, join AG, upon which demit the perpendiculars BN, CF; through the point A draw DAE at right angles to AG, and through the points B and C, draw BD, CE, parallel to AG.

4. We have to shew, first, that the *resultant* is represented in *magnitude by the diagonal* AG.

To this end, divesting ourselves of all considerations of the force AB, let us suppose the two forces AD, AN, to act upon the point A, at right angles to one another, and that the joint energy of these forces, is equal to the energy of some single force (R) which is yet unknown to us.

Then, it is evident, that, if the energy of AD be equal to nothing, the energy of R is equal to the energy of AN, and consequently R is equal to AN; for equal forces produce equal energies, and the contrary.

Also, if the energy of AN be equal to nothing, the energy of R is equal to that of AD, and consequently R is equal to AD; and, if both these forces vanish, or, if each becomes equal to nothing, then R also becomes equal to nothing, for its energy is equal to nothing.

Now, because R is unknown, we may suppose it equal to the square root of AN squared, added to AD squared, and multiplied by some unknown quantity x. This put into the form of an equation will read

$$\mathrm{R} = \sqrt{\{\mathrm{AD}^2 + \mathrm{AN}^2\}}\, x.$$

This being the general expression for R corresponding to all values of the forces AN, AD, acting at right angles to one another; if we now suppose either of these forces, as AD, to become equal to nothing, then

$$\mathrm{R} = \sqrt{(\mathrm{AN}^2)}\, x = (\mathrm{AN}.x);$$

and if only the force AN should become equal to nothing,

$$\mathrm{R} = \sqrt{(\mathrm{AD}^2)}\, x = (\mathrm{AD}.x);$$

but it has been shewn above that, under both these circumstances,

$$\mathrm{R} = \mathrm{AN}, \text{ and } \mathrm{R} = \mathrm{AD},$$

$$\text{therefore } \mathrm{AN}.x = \mathrm{AN}, \text{ and } \mathrm{AD}.x = \mathrm{AD};$$

whence it is evident, that x = unity, and consequently that

$$R = \sqrt{\{AN^2 + AD^2\}}.$$

But $AD = BN$; therefore $R = \sqrt{\{AN^2 + BN^2\}} = AB$;

which shews us that the joint energy of the forces AN and AD is equal to the energy of AB, the diagonal of the parallelogram constructed on the lines which represent these forces.

In the same way it may be demonstrated, that the energies of the forces AE and AF are equal to the energy of the force AC.

Therefore, since the energy of the force AB, is equal to the joint energies of the forces AN, AD, and the energy of the force AC, is equal to the joint energies of the forces AF, AE; by equal additions the energies of the forces AB, AC, are equal to the energies of the four forces AN, AF, AD and AE.

But since the triangle ADB is identical to the triangle CFG, therefore $AD = FC = AE$; and since $AC = BG$, therefore $AF = GN$; and consequently, since the forces AD, AE, are equal and in opposite directions, their energies destroy one another, and because the line AF is equal to the line GN, it follows, that the joint energy of the forces AB and AC, is equal to the energies of the forces AN and AF, or of the single force AG.

Therefore, the *resultant* of the forces AB, AC, *is equal in magnitude* to the *diagonal* of the parallelogram constructed upon the lines which represent these forces.

5. We are next to prove that the direction of the *resultant* cannot be otherwise than in the *direction of the diagonal* AG.

Here the forces AD, AE, being equal and opposite, destroy one another; consequently the intensity of the forces AN, AF, is equal to the intensity of the forces BA and CA, each to each, and since

$$AB : AN :: \sin. ANB : \sin. ABN :: \sin. NAD : \sin. BAD;$$

therefore, forces of equal intensities acting upon a point in a straight line, are to each other inversely as the sines of the angles of their inclination to that line.

Now, let any force (x) in the same plane with AB, AC, act upon the point A with an intensity equal to that of AG, and having its direction inclined to DE in the angle φ, then shall

$$x : AG :: \sin. GAD, \text{ or radius} : \sin. \varphi;$$

and since our inquiry is to determine if AG can act upon the point A with the same intensity in any other direction, except in that of the diagonal of the parallelogram ACGB, we have only to suppose the magnitude of the force (x) to be equal to that of AG; therefore, as the two first terms of the last proportion are equal, the two last terms must necessarily be equal, and consequently, sine φ is equal to radius, or the angle φ is equal to a right angle, equal to the angle GAD; therefore, the force AG, which is the resultant of the forces AB, AC, lying in the same plane with these, cannot act upon DE with an intensity equal to the intensity of these forces, in any other direction except in the direction of the diagonal of the parallelogram constructed upon the lines which represent these forces.

6. To complete our demonstration, we have finally to shew that the *resultant* must always be in *the plane* of the forces.

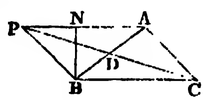
For if it be supposed to be on either side of the plane of the forces, another line may be conceived to lie on the other side of the plane similarly situated to this line; and since there is no reason why the resultant should be situated in one of these lines, rather than in the other, it is therefore in neither of them, unless we admit that it is situated in both, which is evidently absurd; for then the same forces acting in the same manner would produce two diſtinct effects; in other words, if it be supposed to be out of the plane, and on either side of it, let its inclination to the plane be denoted by (θ), then, the effect of the resultant in the plane of the forces, is equal to $AG.\cos.\theta$, for the same reason that the effect of the force AB upon the right line AD is equal to $BD = AB \times \cos.\angle ABD$: wherefore, if AG lying without the plane of the forces AB, AC, can possibly produce the same effect as when it lies in that plane, we have

$$AG.\cos.\theta = AG \quad \therefore \cos.\theta = \frac{AG}{AC} = 1 = \text{radius};$$

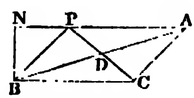
that is, the angle θ = nothing, and therefore the force AG cannot lie out of the plane of the forces AB, AC.

Hence the composition of two forces resolves itself into the following very simple problem, the solution of which we shall now proceed to give at full length:

7. PROB. 1. *Given the magnitude and direction of two forces represented by the contiguous sides of a parallelogram, to determine the resultant of these forces, or the diagonal of the parallelogram.*



From B, the extremity of one of the forces, let fall the perpendicular BN on the direction of the other, and it will fall on AP, or on AP produced, according as the angle APB is acute or obtuse, both of which are represented in the annexed figures.



First, if the angle APB be acute, we have $AB^2 = AP^2 + BP^2 - 2\,AP.PN$. by the principles of Geometry.

But if the angle APB be obtuse, then, as above, we shall have $AB^2 = AP^2 + BP^2 + 2AP.PN$.

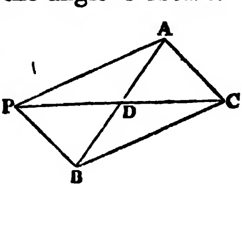
In both cases we have

$$\text{Radius, or } 1 : \cos.APB :: BP : PN;$$

$$\text{therefore, } PN = BP.\cos.APB,$$

which value being substituted for PN, in the preceding equations, gives

1. $AB^2 = AP^2 + BP^2 - 2AP.PB.\cos.APB$, when the angle is acute.
2. $AB^2 = AP^2 + BP^2 + 2AP.BP.\cos.APB$, when the angle is obtuse.



And, since the signs of the third members on the right-hand side of these equations, depend entirely on the magnitude of the angle APB, therefore, generally,

$$AB^2 = AP^2 + BP^2 \mp 2AP.BP.\cos.APB,$$

according as the angle APB is acute or obtuse.

Let us now represent each of the lines PA, PB, PC, and AB, by single letters, and the angle APB by φ in the following manner, for the purpose of simplifying the equation we have just now handled; that is,

Put $\left.\begin{array}{l} a=\text{PA} \\ b=\text{PB} \end{array}\right\}$ the lines which indicate the magnitudes and directions of the given forces.

$\varphi=$ APB, the angle of their inclination,

$r=$ PC, the resultant or equivalent,

and $d=$ AB, the diagonal of the parallelogram opposite the angle P.

Then, by substituting, in the foregoing equation, we have

$$d^2=a^2+b^2\mp 2\,a\,b\ \cos.\varphi.$$

And because the diagonals of a parallelogram bisect each other, AD$=$DB; consequently,

$$\tfrac{1}{2}\,d^2=\tfrac{1}{2}\{a^2+b^2\mp 2\,a\,b\ \cos.\varphi\}.$$

Now it is shewn by the writers on geometry, that in any plane triangle

Twice the square of the line drawn from the vertex to the middle of the base, together with twice the square of half the base, is equal to the sum of the squares of the two sides;

consequently we have

$$2\,(a^2+b^2)=a^2+b^2\mp 2\,a\,b\ \cos.\varphi+r^2$$

which by transposition becomes

$$r^2=a^2+b^2\pm 2\,a\,b\ \cos.\varphi$$

and extracting the square root of both members of the equation, we finally obtain

$$r=\sqrt{\{a^2+b^2\pm 2\,a\,b\ \cos.\varphi\}}\ \ldots\ldots (a)$$

The truth of this is manifest from the triangle PAC, in which the angles PAC and APB are supplemental to each other; and it is known from the principles of trigonometry, that an angle and its supplement have the same cosine, but they are affected with contrary signs.

Hence, the expression for one diagonal of a parallelogram, must be the same as that for the other, with the exception of the sign belonging to that term in which the cosine of the angle of inclination occurs.

If the angle be acute, the upper or positive sign must be employed, when that diagonal is sought which divides the angle of inclination into two parts. When, however, it is required to find the diagonal which subtends the acute angle, the lower or negative sign must be taken. And, in both cases, the contrary will take place when the angle is obtuse; that is to say, the signs will be inverted.

8. If (φ) the angle of inclination vanishes, then cosine $\varphi=1$, and consequently equation (a) becomes

$$r=\sqrt{\{a^2+b^2+2\,a\,b\}}=a+b\ \ldots\ldots (b)$$

which is as much as to say, (and which is axiomatical,) that the resultant, both in magnitude and direction, is equivalent to the sum of two forces, by which, in this instance, it is represented. And we infer from this equation, that

Two forces acting in the same straight line, and in the same direction, are equivalent to a single force the magnitude of which is expressed by their sum.

Hence, two given forces produce the greatest effect when they act in the same direction.

9. If (φ) the angle of inclination is equal to 180 degrees, then cosine $\varphi = -1$, and equation (a) becomes

$$r = \surd\{a^2 + b^2 - 2\,a\,b\} = a - b \quad . \; . \quad (c)$$

which is the same as saying, that the resultant, both in magnitude and direction, is equivalent to the difference of the two forces which, in this case it represents. And we infer from this equation (c), that

Two forces acting in the same straight line, but in different directions, are equivalent to a single force, the magnitude of which is expressed by their difference.

Hence, two given forces produce the least effect when they act in opposite directions.

10. If $a = b$, or in other words, if the lines which represent the magnitude and direction of the given forces be equal, and φ the angle of their inclination be 180 degrees, then cosine $\varphi = -1$, and equation (a) becomes

$$r = \surd\{2\,a^2 - 2\,a^2\} = o \quad . \; . \; . \; . \; . \quad (d)$$

From which we infer, that

Two equal forces acting in the same straight line, but in opposite directions, destroy one another.

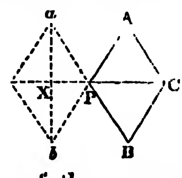
11. If $a = b$, and $(\varphi) = 120°$; then, cos. $\varphi = -\frac{1}{2}$, and equation (a) becomes

$$r = \surd\{2\,a^2 - a^2\} = a \quad . \; . \; . \; . \; . \; . \; . \quad (e)$$

From which we infer, that

If two equal forces are inclined to one another in an angle of 120°, the magnitude of the resultant is equal to that of either of the forces.

Consequently, when the forces PA and PB are equal to one another, the resultant must be expressed in magnitude and direction by the diagonal of a rhombus; therefore if a rhombus Pacb be constructed in the reverse order, its diagonal Pc will be equal and opposite to the resultant PC of the forces, and consequently will sustain them in equilibrio (Equation d). Now, the direction of the forces being inclined to each other in an angle of 120 degrees, the diagonal Pc of the reverse rhombus will be inclined to each of the forces PA, PB, in the same angle, and we have shewn before, that it is equal in magnitude to either of them, therefore,



If three equal forces are inclined to each other in angles of 120 degrees, any one of them will balance the united efforts of the other two.

This is obvious; for, as any one of the forces mutually destroys or balances the united efforts of the other two, none of them can prevail.

12. If $\varphi = 90$ degrees, and the forces (a) and (b) of any magnitude whatever, then cosine $\varphi = o$, and equation (a) becomes

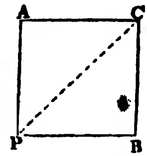
$$r = \sqrt{\{a^2 + b^2\}}, \ldots\ldots\ldots (f),$$

which implies, (and which we have already demonstrated), that

Two forces, PA *and* PB, *the directions of which coincide with the base and perpendicular of a right-angled triangle, have for their resultant a force represented in magnitude by the hypothenuse of that triangle; or, which is the same thing, by* PC, *the diagonal of the rectangle constructed on the lines which represent these forces.*

13. If $a = b$, and $\varphi = 90$ degrees, then cos. $\varphi = o$, and equation (a) becomes

$$r = a\sqrt{2} \ldots\ldots\ldots (g)$$



From which we infer, that

Two forces AP *and* BP *represented in magnitude and direction by the sides of a square, have for their resultant a force represented in magnitude and direction by the diagonal of that square.*

14. When the forces (a) and (b) are equal, and (φ) an angle of any magnitude whatever, then equation (a) becomes

$$r = 2a\sqrt{\{\tfrac{1}{2}(1 \pm \cos.\varphi)\}}$$

but $\sqrt{\{\tfrac{1}{2}(1 + \cos.\varphi)\}}$, or $\sqrt{\{\tfrac{1}{2}(1 - \cos.\varphi)\}}$ when (φ) is greater than 90 degrees, is equal to cos. $\tfrac{1}{2}\varphi$*; consequently by substitution,

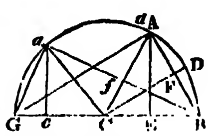
$$r = 2a \cos.\tfrac{1}{2}\varphi \ldots\ldots\ldots (h)$$

15. Hence it is manifest, that equation (a) resolves generally the problem of the composition of two forces situated in the same plane, and concurring in the same point, while equation (h), which is

* *Note* A. When the forces (a) and (b) are equal, and the angle (ϕ) of any magnitude whatever, then equation (a) becomes

$$r = 2a\sqrt{\{\tfrac{1}{2}(1 \pm \cos.\phi)\}}$$

But in this case, $\sqrt{\{\tfrac{1}{2}(1 \pm \cos.\phi)\}} = \cos.\tfrac{1}{2}\phi$, which we shall illustrate in the following manner:—



On GB describe a semicircle, and let AB and BD be any two arcs, of which AB is double of BD; and let the arc AB be denoted by ϕ; then will the arc BD $= \tfrac{1}{2}\phi$. Draw the chords AB, aG, and from the point A, in the circumference, let fall the perpendicular AE upon the diameter GB, then we have

$$\text{CE} = \cos.\phi, \text{ and } \text{CF} = \tfrac{1}{2}\,\text{AG} = \cos.\tfrac{1}{2}\phi;$$

And if the radius CB or CG be assumed equal to unity,

$$\text{BG} = 2, \text{ and } \text{GE} = 1 + \cos.\phi;$$

and because AG is double CF; $\text{AG}^2 = 4\,\text{CF}^2 = 4\cos^2.\tfrac{1}{2}\phi$.

But $\text{AG}^2 = \text{BG}\,.\,\text{GE} = 2(1 + \cos.\phi)$; therefore $4\cos^2.\tfrac{1}{2}\phi = 2(1 + \cos.\phi)$;

$$\text{and } \cos.\tfrac{1}{2}\phi = \sqrt{\{\tfrac{1}{2}(1 + \cos.\phi)\}}.$$

Next for the small letters, when B$a = \phi$, we shall have

$$\text{C}e = -\cos.\phi;\ \text{C}f = \tfrac{1}{2}\,a\text{G} = \cos.\tfrac{1}{2}\phi;\ \text{and G}e = 1 - \cos.\phi$$

In this case also, $a\text{G}^2 = 4\,\text{C}f^2 = 4\cos^2.\tfrac{1}{2}\phi$

$$\begin{aligned}
\text{But} \ldots\ldots\ a\text{G}^2 &= \text{BG}\,.\,\text{G}e = 2(1 - \cos.\phi), \\
\therefore\ 4\cos^2.\tfrac{1}{2}\phi &= 2(1 - \cos.\phi), \\
\therefore\ \cos.\tfrac{1}{2}\phi &= \sqrt{\{\tfrac{1}{2}(1 - \cos.\phi)\}}.
\end{aligned}$$

And, therefore, for the different values of ϕ we get

$$\cos.\tfrac{1}{2}\phi = \sqrt{\{\tfrac{1}{2}(1 \pm \cos.\phi)\}}. \qquad \text{Q. E. D.}$$

simpler, applies only to the limited case of two equal forces. And hence arises the following most general proposition, which is thus expressed by Dr. Robison:—

"*If a material point be urged at once by two pressures, whose intensities are proportional to the sides of any parallelogram, and which act in the directions of those sides, it is affected in the same manner as if it were acted on by a single force, whose intensity is measured by the diagonal of the parallelogram, and which acts in its direction.*

"*Or, two pressures, having the direction and proportion of the sides of a parallelogram, generate a pressure, having the direction and proportion of the diagonal.*"

In a subsequent part of our work, this proposition is condensed and exhibited in a more definite and simple form.

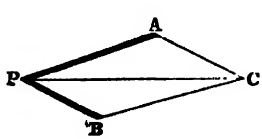
We shall now endeavour to illustrate this doctrine by a few select examples, prefaced by rules which may be considered as popular or arithmetical translations of the formulæ (a) and (h). By this means we shall enable such of our readers as are unacquainted with algebra, to calculate expeditiously and correctly the magnitude of the resultant of any two given forces, whether those forces are equal or unequal, and whatever may be the inclination of their directions to one another.

16. The practical rule afforded by equation (a), is as follows

Rule. *To or from the sum of the squares of the two contiguous sides of the parallelogram, or lines which represent the magnitudes of the forces, according as the angle of inclination contained by their directions is acute or obtuse, add or subtract twice their product multiplied by the natural cosine of the contained angle; then, the square root of the sum or difference, will be the diagonal of the parallelogram, or of the resultant of the two forces.*

Examples for practice under this rule.

17. Example 1. Suppose a force equivalent to the pressure of 83 cwt. acting at a given point in a body, and in a certain direction is assisted by another force of 56 cwt. acting at the same point but in a direction inclined to the former in an angle of 45 degrees; what must be the magnitude of a single force to produce the same effect?



Construction. Let AP $=83$, taken from any scale of equal parts; BP $=56$, taken from the same scale, and with a chord of $60°$ make the angle APB $=45°$. Complete the parallelogram ACBP, and PC the diagonal will be the equivalent, which, being taken from the same scale from which AP or BP was taken, will measure $128\cdot83$ equal parts.

Arithmetical solution agreeably to the rule. Here we have the natural cosine of $45° = \cdot707$.

$$a^2=83\times 83 \qquad = \left.\begin{matrix}6889\\3136\end{matrix}\right\}\text{add}$$
$$b^2=56\times 56$$

$$a^2+b^2 \qquad =10025 \qquad \text{[angle } \varphi \text{ is acute.}$$
$$2\,ab\cos.\varphi=2\times 83\times 56\times\cdot 707 \qquad = 6572\cdot 272, \text{ add because the}$$

$$\text{Hence } a^2+b^2+2\,ab\cos.\varphi \qquad =16597\cdot 272$$
$$\text{And } \sqrt{\{a^2+b^2+2\,ab\cos.\varphi\}} \qquad =\sqrt{16597\cdot 272}=128\cdot 83 \text{ cwt.}$$

consequently, a single force capable of producing the same effect as the two given forces of 83 and 56 cwt. respectively, is 128·83 cwt.

18. Example 2. Suppose that two forces act at a material point in a body, the one equal to 20, and the other equal to 30 horses' power, and that their directions incline to one another in an angle of 75° 30′; what is the magnitude of their resultant, or of a single force equivalent to their joint energy?

Here we have $a=20$, $b=30$, and $\varphi=75°\ 30'$, the natural cosine of which is ·25038, therefore by rule 1, equation (a), we have

$$a^2=20\times 20 \qquad = 400$$
$$b^2=30\times 30 \qquad = 900 \qquad \text{[angle } \varphi \text{ is acute.}$$
$$2\,ab\cos.\varphi=2\times 20\times 30\times\cdot 2538 \qquad = 300\cdot 456, \text{ add because the}$$

then $a^2+b^2+2\,ab\cos.\varphi = 1600\cdot 456$, and by extracting the square root, we have for the diagonal of the parallelogram, or the resultant of the given forces,

$$r=\sqrt{1600\cdot 456}=40 \text{ horses' power nearly.}$$

19. Example 3. Suppose a ship to sail N.N.E. 40 miles in a current that sets S.S.E. 25 miles in the same time, what is the absolute distance passed over by the ship?

Here we have given $a=40$, $b=25$; and from a table of rhombs we find that φ, the angle which the ship's course makes with the direction of the current, is 12 points of the compass, or 135 degrees; its natural cosine is —·70711; then by equation (a), or its rule, we have

$$a^2=40\times 40 \qquad = 1600$$
$$b^2=25\times 25 \qquad = 625 \qquad [\varphi \text{ is obtuse.}$$
$$-2ab\cos.\varphi=2\times 40\times 25\times -\cdot 70711 = 1414\cdot 214, \text{ sub. because angle}$$

Then $a^2+b^2-2\,ab\cos.\varphi = 810\cdot 786$, and by extracting the square root of both sides of the equation, we have for the magnitude of the resultant, or the ship's distance,

$$r=\sqrt{\{810\cdot 786\}}=28\cdot 456 \text{ miles.}$$

20. *Corol.* If the sides of the paralellogram, or the magnitudes of the given forces, be estimated in inches, feet, yards, or any other linear measure, as in the last example in miles, then the resultant must be estimated in the same measure. The forces in the first and second examples are represented by weights or pressures, and the resultant is accordingly represented by so many of the same weights. But this is, indeed, the simplest manner in which a force can be represented, and it accords perfectly with our definition of

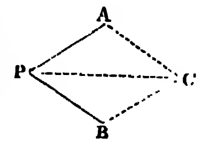
force; consequently, having established the principles of composition from the consideration of lines, it will be sufficient hereafter, in all practical operations, or the examples we may introduce, to contemplate the forces as being represented by weights, and this will very materially assist the conceptions we form of equilibrium, or balanced rest, and greatly facilitate the numerical processes; for, without all doubt, weight is the simplest representative of force, since it is the property of heaviness or lightness which all bodies must possess.

21. The practical rule afforded by equation (h), is as follows.

Rule. *Multiply the sum of the given forces by the natural cosine of half the angle contained by their directions, and the product will be the diagonal of the rhombus, or the resultant of the given forces.*

Examples to equation (h).

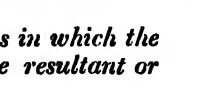
22. Example 1. Suppose two forces, each measured by a weight of 6 tons, act simultaneously on a material point, in directions inclined to each other in an angle of 58° 36′; what must be the magnitude of the resultant, or of a single force, to produce the same effect?



Half the angle of inclination is 29° 18′; its natural cosine is 872, therefore,

$6+6\times \cdot 872 = 10\cdot 464$ tons, the equivalent single force.

23. Example 2. Let the forces remain as above, but suppose the angle of inclination to be 156° 12′; what then is the magnitude of the resultant?



Half the angle of inclination is 78° 6′, its natural cosine is ·2062, therefore,

$6+6\times \cdot 2062 = 2\cdot 4744$ tons, the magnitude of the resultant, from which we infer that

The greater the inclination of the directions in which the forces act, the less is the magnitude of the resultant or equivalent force.

But this inference might have been previously drawn; for it has been shown (equation d), that when the angle of inclination is 180 degrees, the resultant vanishes.

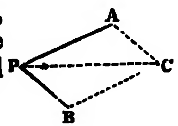
24. *Corol.* One thing will have become obvious to the reader in these examples. It is this. We take no account of the figure of any bodies producing equilibrium. We consider only the mutual action of the forces as directed to single particles or physical points, as if all the matter of which any body is composed, (whether weighing one grain or one thousand tons), were concentrated in a single point. If one force balance another, the energy of the one must be measured by the energy of the other. If the one force act by pressure, its weight must be estimated only by pressure; if it act by traction, its weight or force must be estimated by traction. But, in order

that these measures may be accurate, they must invariably be connected with the magnitudes which they are employed to measure, and so connected, that, if we transpose them, their mutual energy must remain the same.

Dr. Robison very justly remarks, "We do not perceive force as a separate existence from the magnitude. Our measures of force, therefore, are necessarily connected with the magnitude which they measure, and their proportions are the same, because the one is always an inference from the other, both in species and degree."* Nor can we have any knowledge of the forces which may balance two or more bodies, different from our knowledge of their effects. Every denomination, therefore, of force is descriptive merely of its effects: these denominations—as the terms, components, resultant, &c.—are mere names of reference to the substances in which the forces are supposed to reside. And whenever a force of a given measure, or magnitude, is opposed by another equal force, the existence, energy, and intensity of the antagonist force, is detected and measured by means of the force exerted. The quiescent state of the body—its position in balanced rest—results, as our examples have proved, and as the subsequent examples in this case will most amply demonstrate — from the known action of one power over another, whose measure, indication, or characteristic, is ascertained in this way. Thus, forces are recognized, not only by the effects they produce, but also by the effects or consequences which they prevent. We have not, however, yet arrived at that branch of our investigation of the Paralellogram of Forces, where the disquisitions, which enable us to investigate forces or energies, are susceptible of those relations from which we may draw such conclusions as shall prove, not merely that statics is a demonstrative science—one of the *disciplinæ accuratæ*—but that, in fact, it is the basis of natural philosophy. Our inquiry hitherto applies only to the determination of the resultant, or equivalent of two given forces, and questions of this nature generally propose a double object, which may scientifically be expressed in the following—

25. Proposition. *To find the magnitude and direction of a single force whose energy shall be equivalent to that of two given forces.*

When the solution of this problem is conducted graphically, both these objects are accomplished at once by the manual construction; but in a numerical process, each must be determined separately from the other. We proceed, therefore, to inquire by what means the direction of the resultant is to be discovered.



For this purpose, let $x = \text{APC}$, the angle which PC, the direction of the resultant, makes with PA, the direction of one of the forces, pressures, or dead weights; then, by plane trigonometry,

we have $\sin. x : \sin. \text{PAC} :: \text{AC} : \text{PC}$;

* Mec. Phil. vol. i. p. 20. Brewster's Edition.

but $AC = b$; sine $PAC = \sin. \varphi$ because the sine of an arc is the same as the sine of its supplement, and $PC = \sqrt{\{a^2 + b^2 \pm 2ab \cos. \varphi\}}$; consequently, by substitution, we obtain

$$\sin. x : \sin. \varphi :: b : \sqrt{\{a^2 + b^2 \pm 2ab \cos. \varphi\}},$$

and this, by converting the analogy into an equation, becomes

$$\sin. x \, (a^2 + b^2 \pm 2ab \cos. \varphi)^{\frac{1}{2}} = b \sin. \varphi;$$

Therefore, by division, the expression for the value of sin. x is as follows, viz.

$$\sin. x = \frac{b \sin. \varphi}{\sqrt{\{a^2 + b^2 \pm 2ab \cos. \varphi\}}} \quad \ldots\ldots (i)$$

In this equation, it is supposed that PC, the resulting direction of the two forces AP, BP, and which in this case, we call the magnitude of the resultant, or the diagonal of the parallelogram, has been previously ascertained, the denominator of the fraction or radical quantity indicating its measure, as has been already shewn in equation (a); but, by the nature of the subject, it is not necessary to make any previous calculation; for the angle APC or ACP can easily be found in terms of a, b, and φ, without previously knowing the value of the resultant r.

26. If the forces a and b are equal, and φ less than a right angle, then equation (i) becomes

$$\sin. x = \frac{\sin. \varphi}{2\sqrt{\{\frac{1}{2}(1 + \cos. \varphi)\}}};$$

and it has already been shewn (art. 14), that

$$\sqrt{\{\tfrac{1}{2}(1 + \cos. \varphi)\}} = \cos. \tfrac{1}{2}\varphi;$$

wherefore, by substitution, we obtain

$$\sin. x = \frac{\sin. \varphi}{2 \cos. \frac{1}{2}\varphi};$$

but by the arithmetic of sines,

$$\frac{\sin. \varphi}{2 \cos. \frac{1}{2}\varphi} = \sin. \tfrac{1}{2}\varphi^*$$

consequently we obtain,

$$\sin. x = \sin. \tfrac{1}{2}\varphi; \text{ and } x = \tfrac{1}{2}\varphi.$$

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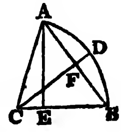
* *Note* B. Which may be thus demonstrated. Let AB and BD be two arcs, of which AB is double of BD, and let AB and BD be respectively represented by ϕ and $\frac{1}{2}\phi$. Draw AB, and from the point A let fall the perpendicular AE; then we have $AE = \sin. \phi$; $CF = \cos. \frac{1}{2}\phi$; and $AB = 2 \sin. \frac{1}{2}\phi$; therefore, if BC be assumed equal to unity, we have, from similar triangles BAE and BCF, the following analogy:



$$1 : 2 \sin. \tfrac{1}{2}\phi :: \cos. \tfrac{1}{2}\phi : \sin. \phi;$$

or by making the product of the mean terms equal to the product of the extremes, we obtain

$$2 \sin. \tfrac{1}{2}\phi \cos. \tfrac{1}{2}\phi = \sin. \phi,$$

consequently, by division, we have the equation

$$\sin. \tfrac{1}{2}\phi = \frac{\sin. \phi}{2 \cos. \frac{1}{2}\phi}$$

which it was our purpose, by this note, to demonstrate.

Hence we infer,

That the resultant of two equal forces concurring in a point, bisects the angle of their inclination;

and generally, that

The resultant of any two forces concurring in a point, divides the angle of their inclination into two parts, which are such, that the sines of the parts are to each other, directly as the magnitude of the forces.

Therefore, to determine the direction of the resultant of two given forces, it is only necessary to resolve the following very simple problem.

27. PROBLEM. *To divide a given angle into two such parts, that their sines may be to one another in a given ratio,*

Let as before, $x=$the angle APC, then will $(\varphi-x)=$ the angle ACP or CPB, and by trigonometry we have

$$\sin. x : \sin. (\varphi-x) :: b : a,$$

a proportion which agrees with the general inference given above; therefore, by converting the analogy into an equation, by multiplying severally the extremes for one member and the means for the other member, we obtain

$$b \sin. (\varphi-x)=a \sin. x;$$

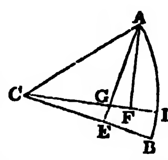
but $\sin. (\varphi-x)=\sin. \varphi \cos. x-\cos. \varphi \sin. x$,* therefore by substitution, transposition, and division we finally obtain

$$\cot. x=\frac{a}{b}\text{cosec.} \varphi+\cot. \varphi.$$

This equation gives that segment of the angle adjacent to the force a, but if that adjacent to the force b were required, the equation would become

$$\cot. x = \frac{b}{a}\text{cosec.} \varphi+\cot. \varphi$$

And if φ the given angle of inclination be obtuse or greater than a



* *Note* C. This will be obvious from the following demonstration.

Let AB and BD be any two arcs of which BA is greater than BD, and let BA and BD be respectively denoted by the letters ϕ and x; then is AD, the difference of BA, BD, equal to $(\phi-x)$. Draw the radii CB, CD and CA, and from the point A let fall the perpendiculars AE and AF; then AE$=\sin. \phi$; CE$=\cos. \phi$ and AF$=\sin. (\phi-x)$, consequently by similar triangles we have

$$\text{rad.} : \text{AG} :: \cos. x : \text{AF}$$

But AG$=$AE$-$GE; that is, AG$=\sin. \phi-\cos. \phi \tan. x$; and we have shown above that AF$=\sin. (\phi-x)$ hence our analogy becomes

$$\text{rad.} : (\sin. \phi-\cos. \phi \tan. x) :: \cos. x : \sin. (\phi-x).$$

Therefore, by equating the products of the extreme and mean terms, and putting radius equal to unity, we have by the arithmetic of sines,

$$\sin. (\phi-x)=\sin. \phi \cos. x-\cos. \phi \sin. x$$

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right angle, then cot. φ is negative, and the equation in a general form for either segment of the angle becomes

$$\cot. x = \left\{ \begin{array}{l} \frac{a}{b} \operatorname{cosec}. \varphi \pm \cot. \varphi \\ \frac{b}{a} \operatorname{cosec}. \varphi \pm \cot. \varphi \end{array} \right\} \dots . (k)$$

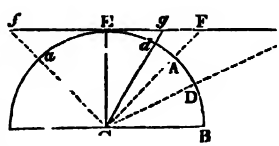
28. If the forces a and b are equal, and φ less than a right angle, then $\cot. x = \operatorname{cosec}. \varphi + \cot. \varphi.$
But $\operatorname{cosec}. \varphi + \cot. \varphi = \cot. \frac{1}{2}\varphi$;* therefore $\cot. x = \cot. \frac{1}{2}\varphi$, and $x = \frac{1}{2}\varphi$, the same as we have shewn it to be in the case of equal forces from equation (i), article 26.

If φ were taken greater than a right angle, the resultant would, notwithstanding, be the same; this is evident from the construction; for when the forces are equal, the figure is a rhombus, and the diagonals of a rhombus bisect the opposite angles.

29. The Algebraic formula for calculating the direction of the resultant, or the angle of its inclination, with the direction of either force, may be expressed in the following practical rule, which we shall illustrate by a series of examples.

Rule. *Divide the magnitude of one of the given forces by the magnitude of the other, and multiply the quotient by the natural cosecant of the given angle of inclination; then, to or from the product, according as the given angle is less or greater than a right angle, add or subtract its natural cotangent, and the sum or remainder will be the natural cotangent of an angle, which expresses the inclination of the resultant to the direction of that force whose magnitude is made the dividend in the first step of the operation.*

30. Example 1. The magnitude of one force is expressed by a weight of 46 tons, and that of another concurring in the same point, but inclined to the former in an angle of 78° 36′, is expressed by a



* *Note* D. It will be satisfactory to demonstrate that $\operatorname{cosec}.\phi + \cot.\phi = \cot.\frac{1}{2}\phi$. For this purpose, let BCE be a quadrant of a circle, BA any arc represented by ϕ, and BD another arc equal to the half of BA be represented by $\frac{1}{2}\phi$. Through the point E draw the straight line EG parallel to CB, and produce the radius CA to F and CD to G; then it is evident from the definitions of Trigonometry that

$$\text{CF} = \operatorname{cosec}. \phi;\ \text{EF} = \cot. \phi;\ \text{and EG} = \cot. \tfrac{1}{2}\phi.$$

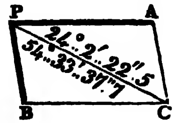
Since the straight line EG is by construction parallel to CB, the angle BCG = the angle FGC; but BCG = FCG, each of them being equal to half of the angle ACB; therefore the angle FCG is equal to the angle FGC, and consequently the side FG is equal to the side CF; that is, FG is equal to cosec. ϕ; hence,

$$\operatorname{cosec}. \phi + \cot. \phi = \cot. \tfrac{1}{2}\phi.$$

The same demonstration will hold for the small letters, only it must be observed that Ef = cot. ϕ is negative, because it is taken in the contrary way to EG.

weight of 92 tons; what angle does the direction of the resultant make with the direction of either force?

Here we have $92 \div 46 = 2$ for AP÷BP, or $a \div b$, Natural cosecant of 78° 36′=1·02012 Natural cotangent of 78° 36′=0·20163, consequently by the rule it is



$(1·02012 \times 2)+0·20163=2·24187$, which is the natural cotangent of 24° 2′ 22″ 5.

Now, in this example the measure of the greater force was made the dividend, hence, the direction of the resultant is inclined to that of the greater force in an angle of 24° 2′ 22″ 5, and of course its inclination to the direction of the lesser force is 54° 33′ 37″ 5.

Therefore by the general inference to equation (i) we have

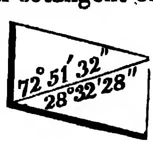
sin. 24° 2′ 22″ 5 : sin. 54° 33′ 37″ 5 :: 46 : 92.

From this also the magnitude of the resultant can easily be determined. For, in the triangles PAC or PBC, we have given all the angles and two of the sides, from which to determine the other side. Therefore, by plane trigonometry, it is

sin. 24° 2′ 22″½ : sin. 78° 36′ :: 46 : 110·69 tons, the measure of the resultant. The operation at length is as follows:

24° 2′ 22″½	log cosec. . . .	0·390013
78° 36′	log sine	9·991346
46	log	1·662758
Natural number 110·69, sum of the logs . .		2·044117

31. EXAMPLE 2. If the given angle of inclination had been 101° 24′ the supplement of 78° 36′, then the natural cotangent of 101° 24′ would be subtractive, consequently, the inclination of the resultant to the direction of the greater force would be 28° 32′ 28″, and its inclination to the direction of the lesser force would be 72° 51′ 32″, the magnitude of the resultant in that case being 94·37 tons.



32. EXAMPLE 3. Suppose that at a material point in a body, two forces act, the one equal to 20, the other equal to 30 horses' power, and inclining to each other in an angle of 75° 30′; what is the inclination of the resultant to the greater force?

Here we have given $a=20$, $b=30$, and $\varphi=75°\ 30'$, whose natural cosecant is 1·0329, and its natural cotangent is ·2586, then, by equation (k) or its rule, we have $30 \div 20 = 1·5$, consequently

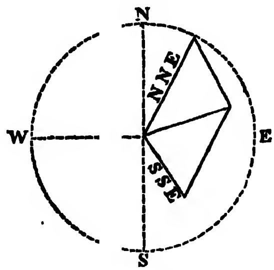
$$\frac{a}{b}\,\text{cosec.}\ \varphi + \text{cot.}\ \varphi = (1·5 \times 1·0329) + ·2586 = 1·80795 =$$

natural cotangent of 28° 56′ 51″, being the quantity, or angle, which the resultant makes with the greater force.

33. EXAMPLE 4. Suppose that a ship sails N.N.E. 40 miles in a current that sets S.S.E. 25 miles in the same time; what is the ship's true course, or her deviation from the apparent course?

Here we have given $a=40$, $b=25$, and $\varphi=135°$; the natural cosecant of which is 1·4142, and its natural cotangent is -1. Then, by equation (k) or its rule, we have $40 \div 25 = 1·6$.

Consequently $\frac{a}{b}$ cosec. φ — cot. $\varphi = 1·6 \times 1.4142 - 1 = 1·26172 =$



natural cotangent of 38° 32′ 36″, which falling eastward, increases the ship's course by its quantity; hence the angle between the ship's way and the meridian is

$$22° \; 30' + 38° \; 22' \; 36'' = 60° \; 52' \; 36''$$

And this in nautical phraseology is 5 points 4° 37′ 36″; or N.E. by E. 4° 37′ 36″ E.

34. EXAMPLE 5. If the influence of the moon on the waters of the ocean be to that of the sun as 9 to 2; when the sun and moon are in quadrature at the time of the equinox; how many degrees is the moon distant from the meridian of the place where it is full tide, supposing these celestial bodies to be situated in the plane of the equinoctial?

This is simply to find the angle contained between the line which marks the elevation of the tide, and that in which the attraction of the moon is exerted, for we have given $a=9$, $b=2$, and $\varphi=90°$;

Its natural cosecant is 1, and its natural cotangent is 0, consequently by equation (k) or its rule, we have

$$9 \div 2 = 4·5; \text{ therefore}$$

$$\frac{a}{b} \text{ cosec. } \varphi + \text{cot. } \varphi = 4·5 \times 1 + 0 = 4·5 = \text{nat. cot. } 12° \; 31' \; 44'';$$

hence the moon is 12° 31′ 44″ distant from the meridian of the place of full tide,

and the sun is 77° 28′ 16″ distant from it.

35. If, instead of supposing the sun and moon to be situated in the plane of the equinoctial, we were to admit them to have any other position, then it is necessary to know the sun's place in the ecliptic, and the moon's place in her path through the heavens, which data can be found from the Nautical Almanac for any instant of time.

Now, supposing the centres of the sun, moon, and earth to be the angular points of a rectilinear triangle, the plane of this triangle being produced, will become a great circle in the heavens, and the equivalent of the forces of the sun and moon will make an angle with the moon's direction such that

$$\text{cot. } x = 4·5 \text{ cosec. } \varphi \pm \text{cot. } \varphi.$$

This angle may be considered as the portion of a great circle, (intercepted between the moon's place and the meridian), whose plane passes through the centres of the sun, the moon, and the earth; and the difference between this angle and φ, will evidently be the distance between the meridian and the sun.

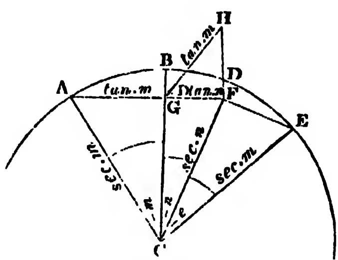
36. In the preceding equation however, φ is unknown, but if we have the places of the sun and moon at the instant to which our inquiry refers, its determination involves nothing beyond the solution of that case of spherical trigonometry, where two sides and the contained angle are given, and it is required to find the other side.

Now, the two sides of the triangle are, the sun and moon's distances from the pole of the ecliptic, and the included angle is the difference of their longitudes; and the third side found by the process, is that portion of a great circle passing through the places of the sun and moon, which measures the angle φ between the directions of the attracting forces.

Put D = the difference between the longitude of the sun and moon,
and l = the moon's latitude; then,
the formula of calculation for the value of φ is as follows, viz.

$$\cos. \varphi = \cos. D \cos. l.$$

Before we proceed further, it may be expedient to furnish a demonstration of this equation, though in doing so we shall be led into a slight digression from the more rigid form in which our general investigation proceeds.*



* *Note* E. Draw the straight line CA, of any convenient length whatever, and about the centre C, with the distance CA, describe the circle ABDE, on which lay off the portions AB and BD, respectively equal to the given sides of the spherical triangle, and join CB and CD; then are the angles ACB and BCD angles at the centre of the sphere, measured by the arcs AB and BD on its surface.

From the point A, let fall the perpendicular AG meeting CB in the point G; produce AG to meet CD in F, and at the point G make the angle FGH equal to the given angle contained between the planes ACB and BCD, whose line of common section is CB; make GH equal to GA, and join HF; about F, as a centre with the distance FH, describe an arc cutting the circle ABDE in the point E, and draw EC; then is the arc DE, which measures the angle DCE at the centre of the sphere, the required side of the spherical triangle.

For, if the planes ACB and DCE be made to turn about the lines CB and CD, till CA coincide with CE, the solid thus formed is a trilateral pyramid, whose vertex C is at the centre of the sphere, and whose base is a portion of the spheric surface, bounded by the circular arcs AB, BD, and DE, which are obviously the sides of a spherical triangle measuring the angles ACB, BCD, and DCE at the centre of the sphere.

Again, if the triangle GHF be made to turn about the line GF till GH coincide with GA, and FH with FE, then will the plane HGF be perpendicular to the plane BCD, and consequently perpendicular to the line CB; the angle FGH therefore measures the inclination of the planes ACB, BCD, and of course the inclination of the arcs AB and BD.

Let m = the angle ACB, or arc AB, one of the given sides,
n = the angle BCD, or arc BD, the other given side,
D = the angle FGH, contained between the sides AB and BD,
and φ = the angle DCE, or arc DE, the required side.

This expression evidently indicates a right-angled spherical triangle, in which the base and perpendicular D and l are given, to find φ the hypothenuse; and accordingly, by referring to the circles on the sphere, we find that the distance of the moon from the ecliptic, measured on a circle of celestial latitude, constitutes the perpendicular, while that portion of the ecliptic intercepted between the secondaries passing through the centres of the sun and moon, is the base, and the distance between the centres of the sun and moon is manifestly the hypothenuse. This latter view of the subject adds much to its simplicity, for the value of φ, the angle contained by the directions of the attracting forces, is then found by simply adding together two logarithmic cosines, without the necessity of any lengthened trigonometrical operation.

If, however, the places of the luminaries be referred to the equinoctial instead of the ecliptic, this abridged method of calculation will not hold, for then the triangle which involves the conditions of the problem becomes oblique, and in consequence, must be resolved by the formula appropriated to that case of spherical tri-

Now, if the straight line CG be considered as radius, and assumed equal to unity, then,

$$\text{CA} = \sec. m, \text{ and GA} = \tan. m,$$
$$\text{CF} = \sec. n, \text{ and GF} = \tan. n;$$

but by construction, CE is equal to CA, and GH equal to GA; consequently we have

$$\text{CE} = \sec. m, \text{ and GH} = \tan. m,$$

respectively, as marked in the figure; then, in the triangle FGH we have given the two sides GH $= \tan. m$; GF $= \tan. n$, and the contained angle FGH $=$ D; to find the side HF equal to FE.

Therefore, by what we have demonstrated in Article 7, we get

$$\text{HF or FE} = \sqrt{\tan.^2 m + \tan.^2 n - 2 \tan. m \tan. n \cos. \text{D}};$$

then in the triangle FCE, we have given the sides CE $= \sec. m$, CF $= \sec. n$, and FE as determined above, to find the angle FCE, or arc DE $= \phi$.

Wherefore (by the principles of Trigonometry) we get

$$\cos. \phi = \frac{\sec.^2 m + \sec.^2 n - \tan. m - \tan.^2 n + 2 \tan. m \tan. n \cos. \text{D}^2}{2 \sec. m \sec. n};$$

but $\sec^2. m - \tan^2. m + \sec^2. n - \tan^2. n = 2 \text{ rad}^2. = 2$; therefore, by substitution and division, it is

$$\cos. \phi = \frac{1 + \tan. m \tan. n \cos. \text{D}}{\sec. m \sec. n}; \text{ that is,}$$

$$\cos. \phi = \cos. m \cos. n + \sin. m \sin. n \cos. \text{D}. \qquad (1)$$

This is the general expression for the value of $\cos. \phi$, attention being paid to the algebraic signs, according to the magnitudes of the given arcs m, n, and D; but to accommodate the formula to our present purpose, we must

put $m = 90^\circ$, the sun's distance from the pole of the ecliptic,
$n = 90^\circ - l$, the moon's co-latitude,
and D = the difference of the longitudes of the sun and moon;

then our general expression becomes

$$\cos. \phi = \cos. 90^\circ \cos. (90^\circ - l) + \sin. 90^\circ \sin. (90^\circ - l) \cos. \text{D}$$

but $\cos. 90^\circ = 0$; $\sin. 90^\circ = 1$, and $\sin.(90^\circ - l) = \cos. l$; consequently, by substitution, we have $\cos. \phi = \cos. l \cos. \text{D}. \qquad (2)$

Q. E. D.

angles, in which two sides and the contained angle are given, and we are required to find the side opposite to the given angle. Here, the distances between the equinoctial and the centres of the sun and moon, or the codeclinations, are the sides of the triangle given, and the difference of the right ascensions of these luminaries is the measure of the contained angle.

Let d = the declination of the sun,

δ = the declination of the moon,

and D′ = the polar angle, or the difference between the right ascensions.

Then, if d and δ, with their proper trigonometrical terms, be substituted for m and n in equation (1) [article 35, note E] we shall obtain

$\cos.\varphi = \cos.(90^\circ \pm d)\cos.(90^\circ \pm \delta) + \sin.(90^\circ \pm d)\sin.(90^\circ \pm \delta)\cos.D'$;

but $\cos.(90^\circ \pm d) = \sin.d$; $\cos.(90^\circ \pm \delta) = \sin.\delta$; $\sin.(90^\circ \pm d) = \cos.d$, and $\sin.(90^\circ \pm \delta) = \cos.\delta$;

consequently, by substitution, the above equation becomes,

$$\cos.\varphi = \sin.d\sin.\delta + \cos.d\cos.\delta\cos.D'.$$

The equation thus transformed, is better adapted to the purpose of calculation than the one from which it is derived, as it precludes the necessity of employing the polar distances, the declinations themselves being sufficient; but very particular attention must be paid to the signs of the quantities, if one or both of the declinations happens to be south, or towards the depressed pole.

In equation (2) [article 35, note E], we have put l to denote the moon's latitude, and D for the difference between the longitudes of the sun and moon; the sun's latitude being so small, it does not sensibly affect the rectangular character of the triangle. If, then, D denote the difference of longitudes in one case, and D′ the difference of right-ascensions in the other, the two equations which refer the bodies to the planes of the ecliptic and equinoctial, are as follow, viz.:

I. When the bodies are referred to the ecliptic,

$$\cos.\varphi = \cos.l\cos.D.$$

II. When the bodies are referred to the equinoctial,

$$\cos.\varphi = \sin.d\sin.\delta + \cos.d\cos.\delta\cos.D'.$$

An example for each supposition will be necessary in this place, as the solution of the primary problem is incomplete till we have determined the value of φ.

37. Example 1. If, when the sun's longitude is $163^\circ\ 22'\ 15''$, that of the moon is $203^\circ\ 55'\ 44''$, at what point does the meridian of the place where it is high-water intersect the great circle passing through the centres of the sun and moon, supposing the moon's influence on the waters of the ocean to be to that of the sun in the ratio of 9 to 2, and that her latitude at the instant of observation is $4^\circ\ 18'\ 28''$?

Here we have given, $l = 4^\circ\ 18'\ 28''$ and $D = (203^\circ\ 55'\ 44'' - 163^\circ\ 22'\ 15'') = 40^\circ\ 33'\ 29''$, and the bodies are referred to the plane of the ecliptic; hence we get

$$\cos.\ \varphi = \cos.\ 4^\circ\ 18'\ 28''\ \cos.\ 40^\circ\ 33'\ 29'';$$

or, by logarithms, it is

$l = 4^\circ\ 18'\ 28''$	- -	log. cos. 9·998771
$D = 40\ \ 33\ \ 29$	- -	log. cos. 9·880670
$\varphi = 40\ \ 44\ \ 48$	- -	log. cos. 9·879441

Then, to find the point where φ is intersected by the meridian of the place where it is high-water, we have from equation (k)

$$\cot.\ x = 4\cdot 5\ \operatorname{cosec.} \varphi + \cot.\ \varphi;$$

but the natural cosecant of 40° 44′ 48″ is 1·53206, and natural cotangent is 1·16069; therefore we get

$$\cot.\ x = 4\cdot 5 \times 1.53206 + 1\cdot 16069 = 8\cdot 05496 = \text{nat. cot. } 7^\circ\ 4'\ 37'';$$

consequently, the place where φ is intersected by the meridian is 7° 4′ 37″ distant from the centre of the moon, and 33° 40′ 11″ distant from the centre of the sun; or rather, the line of high-water is inclined to the lines of attraction in those angles.

38. EXAMPLE 2. If, when the sun's declination is 6° 32′ 32″ north, his right-ascension is 164° 40′ 45″; required the same as in the last example, the moon's right-ascension at that instant being 203° 44′ 30″, and her declination 5° 17′ 8″ south, the ratio of the attractive influences remaining as before.

Here we have given, $d = 6^\circ\ 32'\ 32''$; $\delta = 5^\circ\ 17'\ 8''$, and $D' = (203^\circ\ 44'\ 30'' - 164^\circ\ 40'\ 45'') = 39^\circ\ 3'\ 45''$, and the bodies are referred to the plane of the equinoctial; hence we get

$$\cos.\ \varphi = -\sin.\ 6^\circ\ 32'\ 32''\ \sin.\ 5^\circ\ 17'\ 8'' + \cos.\ 6^\circ\ 32'\ 32''\ \cos.\ 5^\circ\ 17'\ 8''\ \cos.\ 39^\circ\ 3'\ 45'';$$

or, by logarithms, it is

6° 32′ 32″ - -	log. sin. 9·056658		6° 32′ 32″ - -	log. cos. 9·997162	
5 17 8 - -	log. sin. 8·964352		5 17 8 - -	log. cos. 9·998149	
—·01050	log. 8·021010		39 3 45 - -	log. cos. 9·890118	
			0·76812	log. 9·885429	

consequently, we have

$$0\cdot 76812 - 0\cdot 0105 = 0\cdot 75762 = \text{nat. cos. } 40^\circ\ 44'\ 48'',$$

and the point of intersection is found as exhibited above.

39. Having now shewn the method of calculating the inclination of the resultant to the direction of each of the component forces, independently of its magnitude, and also how to calculate the magnitude without adverting to the direction; it therefore only remains to generalize the subject by including both conditions in one simple and definite problem, as follows:—

PROBLEM. *Given the length of two contiguous sides of a parallelogram, with the angle of their inclination, to find the length of the diagonal drawn from that angle, and also the segments of the angle made by the diagonal.*

This is the problem to which we referred in Article 15, and which is in verity what the writers on mechanics, our predecessors and contemporaries, have denominated the *parallelogram of forces*;

a problem the most important in the theory of statics; and we have on this account given an independent determination of each condition in the equations (a) and (k), and another of a mixed and more complex character is indicated in equation (2), Note E; but, on the whole, we are persuaded that our readers will find the independent operations the most instructive; and, with this conviction, we therefore conscientiously recommend their adoption. The examples introduced under the several rules justify this remark. We shall now therefore proceed to shew that this problem extends beyond the composition of two forces. For, since the combined action of two forces puts the material point or body into the same state as if their equivalent alone had acted upon it, we may suppose this to have been the case, and then the action of a third force will produce a change on this equivalent pressure. Hence the resulting force will be the same as if only this third force, and the equivalent of the other two, had acted on this body.

Section Second.

THE COMPOSITION OF THREE FORCES, SITUATED IN THE SAME PLANE, AND CONCURRING IN ONE POINT, THE MAGNITUDES AND DIRECTIONS OF THE FORCES BEING RESPECTIVELY GIVEN.

40. The general principle for this branch of the first case may be expressed in the following

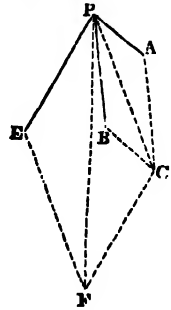
Proposition. *The resultant or equivalent of any three given forces situated in the same plane, and concurring in one point, is represented in magnitude and direction by the diagonal of a parallelogram, constructed on the lines which represent the magnitude and direction of one of the forces, and the resultant of the other two.*

This is only a simple extension of the principle formerly laid down for the composition of two forces, acting in the same plane and applied to the same point. For since the resultant of any two forces, when applied to the same point with them and properly directed, produces the same effect as the forces themselves, this resultant may be considered as a single force of determinate magnitude and direction acting in concert with another force, whose magnitude and direction are given; consequently, the resultant of these two forces, being determined in magnitude and direction, must be the resultant of the three given forces.

The same principle, it is obvious, will extend to the composition of any number of forces whatever; but because the investigation of formulæ for a greater number than three, although proceeding on similar principles, would, if pursued, become excessively prolix, we think it preferable to confine our labours to the composition of three forces only.

PROBLEM I. *To find the magnitude of the resultant of three given forces.*

41. Let the lines PA, PB and PE represent the magnitudes and directions of three forces, acting simultaneously on a point at P given in position. Upon the lines PA and PB, construct the parallelogram PACB, and draw the diagonal PC; then will PC represent the magnitude and direction of a single force, compounded of the two given forces PA and PB, and whose energy is equal to that of the two forces of which it is compounded.



Again, on the lines PC and PE, construct the parallelogram PCFE, and draw the diagonal PF; then will PF represent the magnitude and direction of a single force compounded of the two given forces PC and PE; but the single force PC has already been shown to be compounded of the two forces PA and PB; consequently, the single force PF is compou 'ed of the three forces PA, PB, and PE, and its energy is equivalent to the united energies of all the three.

Let a = PA, one of the forces,
b = PB, another of the forces,
c = PE, the third force;
φ = APB, the angle of inclination of the forces a and b,
φ' = BPE, the angle of inclination of the forces b and c;
r = PC, the resultant of the two forces a and b,
and R = PF, the resultant of the three forces a, b, and c.

Find first by equation (k), the angle CPB $= x$, such that $\cot. x = \frac{b}{a} \operatorname{cosec.} \varphi \pm \cot. \varphi$; then will $(\varphi + x) =$ the angle CPE; and by equation (a)

$$\text{PC or } r = \sqrt{a^2 + b^2 \pm 2\,ab \cos. \varphi};$$

consequently, by the same equation (a), we must have

$$\text{PF or R} = \sqrt{c^2 + r^2 \pm 2\,cr \cos. (\varphi' + x)}.$$

If the value of r and r^2 as indicated in equation (a), be substituted instead of them in this last equation, we shall obtain

$$R = \sqrt{a^2 + b^2 + c^2 \pm 2ab \cos. \varphi \pm 2c \cos. (\varphi' + x) \sqrt{a^2 + b^2 \pm 2ab \cos. \varphi}}. \quad (l)$$

This equation is exceedingly complex, but will, it is presumed, be rendered sufficiently intelligible, by tracing attentively the operation for the following numerical example.

42. Example 1. The magnitudes of three forces, a, b and c, acting in one plane, and applied to the same point of a body, are respectively represented by lines of 8, 12, and 16 inches; what must be the magnitude of their common resultant, supposing the middle force to be inclined to each of the others in an angle of 45 degrees?

Here we have $a=8$; $b=12$; $c=16$, and $\varphi=\varphi'=45°$, of which the natural cosine is $\cdot 70711$: hence it is [See remark under equat. (a.)
$2ab\cos.\varphi=2\times8\times12\times\cdot70711=135\cdot76512$, add because φ is acute.

$a^2+b^2=\overline{8\times8}+\overline{12\times12}=208,$

therefore, $a^2+b^2+2ab\cos.\varphi=\overline{343\cdot76512}$

and $\sqrt{a^2+b^2+2ab\cos.\varphi}=\sqrt{343\cdot76512}=18\cdot54$ inches, for the measure of a single force, whose energy is equivalent to the united energies of the two forces a and b.

Again, $\cot.x=\frac{b}{a}\operatorname{cosec.}\varphi+\cot.\varphi$ (see equation k); but, $a=8$; $b=12$, and $\varphi=45°$, of which the cosecant and cotangent are respectively equal to $\sqrt{2}$ and 1; consequently, $\cot.x=1+1\cdot5\sqrt{2}=3\cdot12133$, the natural cotangent of 17° 45′ 51″; hence $(\varphi'+x)=62°\ 45'\ 51''$, and its natural cosine is $\cdot45766$; therefore, we get

$2c\cos.(\varphi'+x)\sqrt{a^2+b^2+2ab\cos.\varphi}=2\times16\times.45766\times18.54=271.52052$
but it has been shown above that $2ab\cos.\varphi=135.76512$
and moreover, $a^2+b^2+c^2=64+144+256=464$;

consequently,

$a^2+b^2+c^2+2ab\cos.\varphi+2c\cos.(\varphi'+x)\sqrt{a^2+b^2+2ab\cos.\varphi}=\overline{871.28564}$ and

$\sqrt{a^2+b^2+c^2+2ab\cos.\varphi+2c\cos.(\varphi'+x)\sqrt{a^2+b^2+2ab\cos.\varphi}}=\sqrt{871.28564}=$ 29.517 inches, for the measure of the common resultant.

43. Example 2. A ship was observed to sail S.S.W. at the rate of 10 miles an hour, in a current which set S.W. 8 miles an hour, and in a gale of wind which set S.E. 12 miles an hour; what was the rate of sailing per hour?

Here we have $a=8$; $b=10$; $c=12$;

$\varphi=22°\,30'$, and $\varphi'=67°\,30'$:—nat cos. $\varphi=\cdot92388$;

hence it is $2ab\cos.\varphi=2\times8\times10\times\cdot92388=147\cdot8208$, add because

$a^2+b^2=\overline{8\times8}+\overline{10\times10}=164\cdot$ [φ is acute.

therefore $a^2+b^2+2ab\cos.\varphi=\overline{311\cdot8208}\cdot$

and $\sqrt{a^2+b^2+2ab\cos.\varphi}=\sqrt{311\cdot8208}=17\cdot6584$ miles per hour, for the resultant of the current and the ship's natural motion.

Again, $\cot.x=\frac{b}{a}\operatorname{cosec.}\varphi+\cot.\varphi$, (see equation k); but $a=8$; $b=10$, and $\varphi=22°\ 30'$, of which, the natural cosecant and cotangent are respectively, $\sqrt{4+2\sqrt{2}}$, and $\sqrt{2}+1$; consequently, $\cot.x=\frac{10}{8}\sqrt{4+2\sqrt{2}}+\sqrt{2}+1=5\cdot6806$, the natural cotangent of 9° 59′ 2″; hence $(\varphi'+x)=77°\ 29'\ 2''$, and its natural cosine is $\cdot21671$; therefore, we get

$2c\cos.(\varphi'+x)\sqrt{a^2+b^2+2ab\cos.\varphi}=2\times12\times.21671\times17.6584=90.8421$
but it has been shown above, that $2ab\cos.\varphi=147.8208$
and moreover, $a^2+b^2+c^2=64+100+144=308$.

consequently,

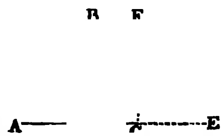
$a^2+b^2+c^2+2ab\cos.\varphi+2c\cos.(\varphi'+x)\sqrt{a^2+b^2+2ab\cos.\varphi}=\overline{546.6629}$

wherefore, by evolution, we get

$$R = \sqrt{546 \cdot 6629} = 23 \cdot 38 \text{ miles nearly.}$$

Hence it appears, that the conjoint effect of the current and the gale, considerably more than doubles the ship's natural motion.

44. Example 3. Suppose three men to pull at an inflexible hoop of iron; two of them at opposite points, and the other exactly in the middle between them. Now, if their strengths are to one another as the numbers 3, 4, and 5; what must be the magnitude of a single force to produce the same effect?



Here, since two of the forces PA $= a$, and PE $= c$ are directly opposed to each other, their effect is equivalent to a single force whose magnitude is expressed by their difference PC, (see equation c); and because the middle force PB acts at right angles to PC, the resultant of the other two, we have from equation (f),

$$R = \sqrt{r^2 + b^2};$$

but we have just stated that $r = c - a$; consequently, we have

$$R = \sqrt{(c-a)^2 + b^2} = \sqrt{(5-3)^2 + 4^2} = 4 \cdot 472$$

45. If φ and φ', the angles of inclination vanish, then x vanishes also, and equation (b) becomes

$$R = \sqrt{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc} = a + b + c, \qquad (m)$$

from which we infer, that

Three given forces acting in the same straight line and in the same direction, have for their resultant a single force, whose magnitude is equal to the sum of the given forces.

46. If φ and φ', the angles of inclination, are each 180 degrees, then is x, 180 degrees also, consequently equation (l) becomes

$$R = \sqrt{a^2 + b^2 + c^2 - 2ab + 2ac - 2bc} = a + c - b, \qquad (n)$$

From which we infer, that

Three given forces acting in the same straight line but in opposite directions, have for their resultant a single force, whose magnitude is equal to the sum of two of the forces diminished by the third.

47. If φ and φ', the angles of inclination, are each equal to 90°, then cos. $\varphi = o$, and equation (l), becomes

$$R = \sqrt{a^2 + b^2 + c^2 + 2c \cos.(\varphi' + x)\sqrt{a^2 + b^2}};$$

but cos. $(\varphi' + x) = -\sin. x$;

consequently, by substitution, the preceding expression for the value of the resultant becomes

$$R = \sqrt{a^2 + b^2 + c^2 - 2c \sin. x \sqrt{a^2 + b^2}} \qquad (o)$$

Let φ and φ' remain, and suppose the forces a, b, and c to be equal among themselves; then is $x = \frac{1}{2}\varphi = 45°$, and equation ($l$), becomes

$$R = \sqrt{3a^2 - 2a^2} = a. \qquad (p)$$

That is, the forces a and c, whose opposite directions are in the

same straight line, destroy each other's effects, and the resultant is equivalent to the remaining force.

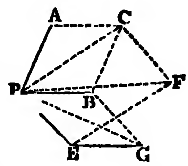
48. If the forces a, b, and c are equal to one another, and φ, φ' of any magnitude less than 90°, then $x = \frac{1}{2}\varphi$, and equat. (l), becomes

$$R = \sqrt{3a^2 + 2a^2 \cos.\varphi + 2a \cos.(\varphi' + \tfrac{1}{2}\varphi)\sqrt{2a^2 + 2a^2\cos.\varphi}};$$

but $\sqrt{2a^2 + 2a^2 \cos.\varphi} = 2a \cos.\frac{1}{2}\varphi$; (see equation, note A, p. 13); therefore by substitution, we have

$$R = a\sqrt{3 + 2\cos.\varphi + 4\cos.\tfrac{1}{2}\varphi \cos.(\varphi' + \tfrac{1}{2}\varphi)}. \qquad (q)$$

From this equation it is manifest, that when the directions of the extreme forces are inclined to the direction of the middle one, in angles of some magnitude less than 90 degrees, the resultant is greater than either of the forces.



Let PA, PB, and PE represent the magnitudes of the three equal forces, whose directions make respectively the angles APB and BPE, each less than a right angle; then is the resultant greater than either force.

Upon the lines PA and PB, the representatives of the forces a and b, construct the parallelogram PACB and draw PC; then is PC the resultant of the two forces a and b. In like manner, upon the line PB and PE, the representatives of the forces b and c, construct the parallelogram PBGE and draw PG; then is PG the resultant of the two forces b and c. But since the angles APB and BPE are each less than a right angle, the angles PAC and PEG, their supplements, must be greater than a right angle; consequently, the lines PC and PG are each of them greater than PA or PE. Again, upon the lines PG and PE, construct the parallelogram PCFE and draw PF; then is PF the resultant of the two forces PC and PE; but PC has been shown to be the resultant of a and b; consequently PF is the resultant of the three forces a, b, and c, and is greater than any of them: but they are equal in the present instance; therefore, the energy of PF is equivalent to the energy of $3a$. We shall illustrate this by a numerical example.

EXAMPLE 1. Suppose, for instance, that three equal forces, whose magnitudes are each represented by a line of 6 inches, act simultaneously in one plane, and on the same point of a body considered without weight; what must be the magnitude of a single force acting in the same plane and applied at the same point, such that its energy shall be equivalent to the united energies of the three given forces, the inclinations of their directions being respectively 45 and $67\frac{1}{2}$ degrees?

Here we have $(\varphi' + \frac{1}{2}\varphi) = 78° 45'$.	log. cos.	9.290236
$\frac{1}{2}\varphi = 33\ \ 45$.	log. cos.	9.919846
4	log.	0.602060
$4\cos.\frac{1}{2}\varphi\cos.(\varphi' + \frac{1}{2}\varphi) = 0.64884$	log.	9.812142
$2\cos.\varphi = 0.76536$		
$4\cos.\frac{1}{2}\varphi\cos.(\varphi' + \frac{1}{2}\varphi) + 2\cos.\varphi = 1.4142$		

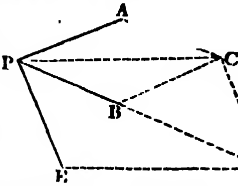
consequently, $R = 6\sqrt{3 + 1.4142} = 12.6$ inches, for the magnitude of the resultant. The magnitude of the lines PC and PG are each to be determined from the general equation (a) for the composition of two forces, but since the components are supposed to be equal, equation (b) is more appropriate; therefore, for the determination of PC we have

$$PC = 2\,a \cos.\tfrac{1}{2}\varphi = 2 \times 6 \times \cdot 83147 = 9.97764 \text{ inches.}$$

For the determination of PG, it is

$$PG = 2\,a \cos.\tfrac{1}{2}\varphi' = 2 \times 6 \times \cdot 92388 = 11\cdot 98656 \text{ inches.}$$

49. EXAMPLE 2. Suppose that three forces, each equal to 120lbs. act simultaneously at a material point of a body; what is the magnitude of a single force of equal intensity to that of the three equal forces, their inclinations being equal respectively to 45°?



Here we have $a = 120$; $\varphi = \varphi' = 45°$, and $(\varphi' + \frac{1}{2}\varphi) = (45° + 22°\ 30') = 67°\ 30'$; then,

$$4 \cos.\tfrac{1}{2}\varphi \cos.(\varphi' + \tfrac{1}{2}\varphi) = 4 \times \cdot 92388 \times .38268 = 1\cdot 41422$$
$$2 \cos.\varphi \quad = 2 \times \cdot 70711 \quad = 1\cdot 41422$$
$$3 + 2 \cos.\varphi + 4 \cos.\tfrac{1}{2}\varphi \cos.(\varphi' + \tfrac{1}{2}\varphi) = 5\cdot 82844;$$

consequently, we have by evolution and multiplication,

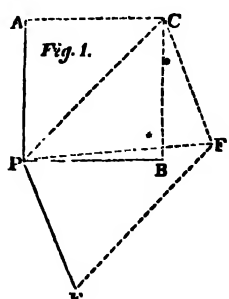
$$R = 120\sqrt{5\cdot 82434} = 289\cdot 7\text{lbs.}$$

50. When the forces a, b, and c are equal, and φ equal to 90 degrees, but φ' less than 90°; then $x = \frac{1}{2}\varphi = 45°$; and equation ($l$) becomes

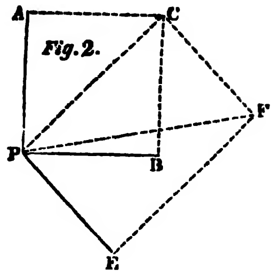
$$R = a\sqrt{3 \pm 2 \cos.(\varphi' + 45°)\sqrt{2}.} \qquad (r)$$

In the application of this formula, it must be observed, that if φ exceeds 45°, the negative sign of the second member under the vinculum must be employed; if $\varphi' = 45°$, the second member vanishes, and we get $R = a\sqrt{3}$; but if φ is less than 45°, the upper or positive sign comes into use. The above reasoning will become manifest from the following construction:—

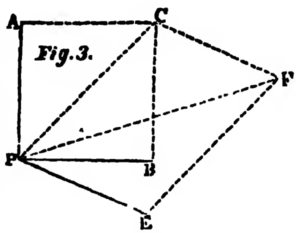
Let the straight lines, PA, PB, and PE, in all the figures, represent the magnitudes of the three equal given forces, of which PA and PB are perpendicular to each other, while the angle BPE in figure 1 is greater than 45°, in figure 2 equal to 45°, and in figure 3 less than 45°.



Upon PA and PB construct the square PACB, and join PC; on PC and PE construct the parallelogram PCFE, and join PF; then is PF the resultant or equivalent of the three given forces. Now, because in figure 1 the angle BPE, is by hypothesis



greater than 45°, the two angles BPE and BPC must together be greater than 90°, and consequently the cosine of the sum is negative; but the angle BPE is represented in the above equation by the letter φ', while the angle BPC is the constant angle 45°; therefore $(\varphi' + 45°)$ is greater than 90°, and cos. $(\varphi' + 45°)$ is a subtractive quantity; hence the second member of equation (r) when φ' is greater than 45° must be taken negatively.



In figure 2 the angle BPE, is by hypothesis equal to 45°; therefore, the two angles BPE and BPC are together equal to 90°, and its cosine is nothing; hence the second member of equation (r) under the radical sign vanishes, and the expression becomes $R = a\sqrt{3}$; but the same result is derived directly from the figure, without considering the evanescent value of the trigonometrical quantity which enters the equation; for, since the angle CPE is a right angle, the parallelogram PCFE is rectangular, and the resultant $PF^2 = PC^2 + CF^2$ by the property of the right-angled triangle; but $PC^2 = 2a^2$, and $CF^2 = a^2$; consequently $PF^2 = 3a^2$; that is, $R = a\sqrt{3}$.

Again, in figure 3, the angle BPE, is by hypothesis less than 45°; therefore the two angles BPE and BPC must together be less than 90°, and consequently the cosine of the sum, or that of the equivalent $(\varphi' + 45°)$, must be positive; therefore the second member under the vinculum must, when φ' is less than 45°, become an additive quantity.

A numerical example for the three cases will illustrate the application of the formula.

51. Example 1. Suppose that three equal forces, whose magnitudes are each represented by a line of six inches, act simultaneously in one plane, and on the same point of a body considered, without gravity; what is the common resultant of the three forces, the inclinations of their directions being 90° and $67\frac{1}{2}°$; 90° and 45°; and 90° and 22° 30'?

Here we have

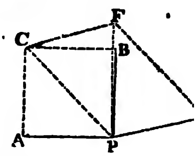
$(\varphi' + 45°) = (67° 30' + 45°) = 112° 30'$ ·	log. cos.	9·582840
2 ·	log.	0·301030
$\sqrt{2}$ ·	log.	0·150515
$-2\cos.(\varphi' + 45°)\sqrt{2} = -1·08239$ ·	log.	0·034385;

consequently, $R = a\sqrt{3 - 2\cos.(\varphi' + 45°)\sqrt{2}} = 6\sqrt{3 - 1·08239} =$ 8·3 inches, for the resultant when $\varphi' = 67° 30'$; again, when $\varphi' = 45°$; then cos. $(45° + 45° = 0$, and $R = 6\sqrt{3} = 10·392$ inches; but when $\varphi' = 22° 30'$, we have

$(\varphi'+45^\circ)=(22^\circ\ 30'+45^\circ)=67^\circ\ 30' \ldots$ log. cos. $9 \cdot 582840$
$2 \ldots$ log. $0 \cdot 301030$
$\sqrt{2} \ldots$ log. $0 \cdot 150515$
$+2\cos.(\varphi'+45^\circ)\sqrt{2}=+1 \cdot 08239 \ldots$ log. $0 \cdot 034385$;

consequently, $R = a\sqrt{3+2\cos.(\varphi'+45)\sqrt{2}}=6\sqrt{3+1 \cdot 08239}=$ $12 \cdot 12$ inches, for the resultant of the three given forces.

52. Example 2. Suppose that three forces, each equal to 100 lbs., and whose directions are inclined to each other in angles of 90 and 75 degrees, act simultaneously at a material point of a body; what is the magnitude of a single force, which being applied to the same point, and properly directed, shall produce the same effect?



Here we have $a=100$; $\varphi=90^\circ$, and $\varphi'=75^\circ$ then, according to equation (r), we get

$$R=100\sqrt{3 \pm 2\cos.(75^\circ+45^\circ)\sqrt{2}}$$

but $\cos.(75^\circ+45^\circ)=-5$; consequently,

$$R=100\sqrt{3-\sqrt{2}}=125 \cdot 92 \text{ lbs.}$$

53. When the forces a, b, and c are equal, and the angles φ and φ' supplements of each other, then is the angle $x=\frac{1}{2}\varphi$, and equation (l) becomes

$$R=\sqrt{3a^2+2a^2\cos.\varphi-2a\cos.(\varphi'+\tfrac{1}{2}\varphi)\sqrt{2a^2+2a^2\cos.\varphi}},$$

but by the arithmetic of sines

$$\sqrt{2a^2+2a^2\cos.\varphi}=2a\cos.\tfrac{1}{2}\varphi;$$

and because, by the supposition, the angles of inclination φ and φ' are supplements of each other,

$$\cos.(\varphi'+\tfrac{1}{2}\varphi)=\cos.\tfrac{1}{2}\varphi;$$

consequently, by substitution, the value of the resultant becomes

$$R=\sqrt{3a^2+2a^2\cos.\varphi-4a^2\cos.^2\tfrac{1}{2}\varphi},$$

and, moreover, since $2\cos.^2\frac{1}{2}\varphi=1+\cos.\varphi$, the last term of the radical quantity becomes $-2a^2(1+\cos.\varphi)$; if therefore we substitute $-2a^2(1+\cos.\varphi)$ for $-4a^2\cos.^2\frac{1}{2}\varphi$ in the expression for the resultant immediately preceding, we finally obtain

$$R=\sqrt{3a^2-2a^2}=a. \qquad (s)$$

54. If the forces a, b, and c are equal, and the angles of inclination φ and φ' be each equal to 90 degrees, the above result will notwithstanding be the same; hence generally,

When two of three equal forces act in the same line, but in contrary directions, they destroy each other's effects, and the resultant of the system is equal to the remaining force.

Again, when the forces a, b, and c are equal, and the angles of inclination φ and φ', each greater than a right angle, then is angle $x=\frac{1}{2}\varphi$, and equation (l) becomes

$$R=\sqrt{3a^2-2a^2\cos.\varphi-2a\cos.(\varphi'+\tfrac{1}{2}\varphi)\sqrt{2a^2+2a^2\cos.\varphi}}$$

but by the arithmetic of sines

$$\sqrt{2a^2+2a^2\cos.\varphi}=2a\cos.\tfrac{1}{2}\varphi,$$

therefore, by substitution, the expression for the resultant, when the angles of inclination are both greater than 90°, becomes

$$R=a\sqrt{3-2\cos.\varphi-4\cos.\tfrac{1}{2}\varphi\cos.(\varphi'+\tfrac{1}{2}\varphi)} \qquad (t)$$

Corol. From this equation it is evident, that when the directions of the extreme forces are inclined to that of the middle one, in angles greater than 90 degrees, the resultant is less than either of the forces, for the radical quantity $\sqrt{3-2\cos.\varphi-4\cos.\frac{1}{2}\varphi\cos.(\varphi'+\frac{1}{2}\varphi)}$ must in every case be less than unity; consequently, the value of the resultant R must always be less than a.

56. If φ and φ', the angles of inclination, are equal to each other, and each greater than 120 degrees, then the value of the resultant of the three given component forces is expressed in the following manner, viz.:—

$$R=a\,(2\cos.\varphi-1). \qquad (u)$$

When φ and φ', the angles of inclination, are each equal to 120 degrees, the value of the resultant vanishes; the three forces in that case mutually counteracting each other.

If φ and φ', the angles of inclination, are equal, but each less than 120 and greater than 90 degrees, the resultant appears on the other side of the point P in the direction of the middle force PB, of which it forms a part, and its value is then computed by the equation (l).

57. These deductions manifestly flow from the general equation (l), by introducing the consideration of angles greater than a right angle, with the appropriate signs of their trigonometrical values; but perhpas a proper conception of the truth will be facilitated, by attending to the following construction:—

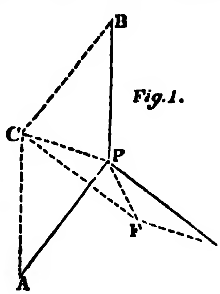


Fig. 1.

Let the straight lines PA, PB, and PE, in all the figures, represent the magnitudes of the three given and equal component forces, while the angles BPA and BPE, each greater than a right angle, are the angles of inclination which the directions of the extreme forces PA and PE, respectively make with PB the direction of the middle force.

Upon the straight lines PA and PB, the representatives of the first extreme and middle forces, construct the parallelogram PACB, and draw PC; then is PC the resultant of the two forces whose magnitudes are represented by the lines PA and PB.

Upon PC, the resultant just determined, and PE, the representative of the other extreme force, construct the parallelogram PEFC, and draw PF; then is PF the resultant of the two forces, whose magnitudes are denoted by the lines PC and PE; but PC is the resultant of the two forces represented by the lines PA and PB; consequently PF is the resultant of the three equal component forces, BA, PB, and PE, and is obviously less than either of them.

Now, in figure 1, since the angle BPA is greater than a right angle, and by hypothesis greater than 120 degrees, its supplement PAC or PBC must be less than a right angle, and of course less than

60 degrees; consequently, PC, the diagonal of the rhombus PACB, must be less than any of the sides.

Again, in the triangle CFP. figure 1, the angle CPF is greater than the angle PCF; consequently the side CF is greater than the side PF, but CF is equal to PE, one of the given and equal forces; therefore, when the angles of inclination are greater than 90 degrees, the magnitude of the resultant is less than the magnitude of any of the forces. The mugnitude of the resultant PC for the two equal forces PA and PB is, as we have already shewn in equation (h) expressed thus, $r=2\,a\cos.\frac{1}{2}\varphi$, and the angle CPE is equal to $360^\circ-(\varphi'+\frac{1}{2}\varphi)$; as is evident from the diagram; therefore by equation (a) we have

$$R=\sqrt{a^2+4\,a^2.\cos.^2\tfrac{1}{2}\varphi-4\,a^2\cos.\tfrac{1}{2}\varphi\cos.\{360^\circ-(\varphi'+\tfrac{1}{2}\varphi)\}};$$

now, the writers on Trigonometry have shewn, that

$$\cos.\{360^\circ-(\varphi'+\tfrac{1}{2}\varphi)\}=\cos.(\varphi'+\tfrac{1}{2}\varphi),$$

and by the arithmetic of sines, we have $2\cos^2.\frac{1}{2}\varphi=1+\cos.\varphi$; therefore, and because φ is greater than 90 degrees, $4\,a^2\cos^2.\frac{1}{2}\varphi=2\,a^2+2\,a^2\cos.\varphi$; hence, by substitution, the value of the resultant for three equal forces, when the angles of inclination are greater than a right angle, becomes

$$R=a\sqrt{3-2\cos.\varphi-4\cos.\tfrac{1}{2}\varphi\cos.(\varphi'+\tfrac{1}{2}\varphi)},$$

the same as we have shewn it to be in equation (t).

58. A similar mode of reasoning applied to figure 2 will prove that PF the magnitude of the resultant, is less than the magnitude of either of the component forces; and that its value is calculated according to equation (u), may be shewn in the following manner.

Because the angles BPA and BPE, fig. 2, are by hypothesis equal to each other, the line BP being produced, bisects the angle APE; consequently, the lines CA and PF are parallel. Through the point A, draw AG parallel to PE meeting PB produced in G, then is PG the diagonal of the rhombus; and because the line CF is equal to the line CA, it follows that GF is equal to GA, but GA is parallel and equal to PE, one of the given forces; therefore GF is equal to PE.

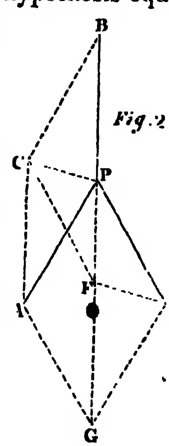


Fig. 2

Again, because the angle APF is the supplement of the angle $BPA=\varphi$; $\cos.APF=\cos.\varphi$, and since the diagonals of a rhombus besect each other at right angles, the diagonal PG is equal to $2\,a\cos.\varphi$; but the resultant PF is equal to PG diminished by GF; that is, $PF=PG-GF$; consequently, by restoring the proper symbols, we get

$$R=a\,(2\cos.\varphi-1).$$

the same as it was shewn to be in equation (u).

In figure 3, because the angles BPA and BPE are by hypothesis each equal to 120 degrees, the angle APE is also equal to 120 de-

grees; consequently, PC, the resultant of the two forces PA and PB, is equal and opposite to the third force PE, and therefore destroys its effects; hence the resultant of three equal forces inclined to each other in an angle of 120 degrees is nothing, and whatever may be the magnitude of the forces themselves, their joint energies cannot, while they continue to act in that position, produce the smallest mechanical advantage.

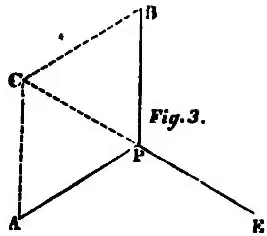


Fig. 3.

By equation (h) we have PC $= r = 2\,a \cos. \frac{1}{2}\,\varphi$; but $\varphi = 120^\circ$, therefore, $\cos. \frac{1}{2}\,\varphi = \cos. 60^\circ = \frac{1}{2}$; consequently $r = a$. Now the angle APE is equal to 180 degrees, because the lines PC and PE meeting at the point P form one continued straight line; therefore, the parallelogram constructed on the lines PC and PE vanishes, and its diagonal, or the resultant R, must vanish also, for it is expressed thus, $R = 2\,a \cos. \frac{1}{2}\,(180^\circ) = o$.

In figure 4 the angles BPA and BPE, are by hypothesis equal between themselves, but each less than 120 degrees; consequently, the angle APE is greater than 120 degrees, and the angle APE and APC together greater than two right angles; therefore, the parallelogram constructed on the lines PC and PE, which was evanescent in the last case, must now lie in the contrary direction, and its diagonal PF, which is the resultant of the three forces, will always coincide with PB, the middle force.

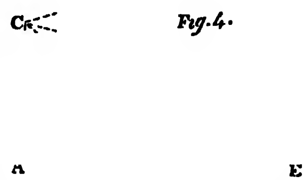


Fig. 4.

By equation (h), PC the resultant of the two forces PA and PB is, $r = 2\,a \cos. \frac{1}{2}\,\varphi$; and the angle CPE is equal to the two angles BPE and BPC, that is, equal to $(\varphi' + \frac{1}{2}\,\varphi)$; consequently PF, the resultant of the two forces PC and PE is, by equation (a),

$$R = \sqrt{a^2 + 4\,a^2 \cos^2. \tfrac{1}{2}\,\varphi - 4\,a^2 \cos. \tfrac{1}{2}\,\varphi \cos. (\varphi' + \tfrac{1}{2}\,\varphi)}\,;$$

but by the arithmetic of sines we have $2 \cos^2. \frac{1}{2}\,\varphi = 1 + \cos. \varphi$; therefore, by substitution, we get $4\,a^2 \cos^2. \frac{1}{2}\,\varphi = 2\,a^2 + 2\,a^2 \cos. \varphi$, and consequently, because φ is greater than 90°, the resultant becomes

$$R = a\sqrt{3 - 2 \cos. \varphi - 4 \cos. \tfrac{1}{2}\,\varphi \cos. (\varphi' + \tfrac{1}{2}\,\varphi)},$$

the same as we have shewn in equation (q), deduced from the general value of the resultant in equation (l); hence the truth of our deduction is manifest.

59. Having thus succeeded in proving that our theory is rigorous, we shall next endeavour to illustrate the application of the equations (t) and (u) by an example peculiar to each, in the same manner as we have illustrated the equations (o) and (p); and, that the whole may be rendered familiar, it will be proper to resolve an example in which neither of the forces, nor the angles that determine

their directions are equal. This will, of course, require the application of the equation in its general form, without admitting the principles of any particular case. No exercise, we are persuaded, can prove more valuable than this; because it will shew, that though, in certain instances, abbreviated forms can be employed with advantage, yet the general form ought more particularly to merit our attention, as being alone applicable to the conditions which examples of every variety may in this case embrace.

Examples to Equation (t).

60. EXAMPLE 1. Suppose that the magnitudes of three equal component forces, acting in the same plane, and directed to the same point of a body, are respectively measured by a load of 26 tons; what must be the measure of the resultant, or single force which shall produce the same effect, supposing the directions of the extreme forces to make with that of the middle one, respectively, angles of 134 and 122 degrees?

Here we have, angle of inclination $\varphi = 134°$

angle of inclination $\varphi' = 122$

$\frac{1}{2}\varphi = 67$ · log. cos. 9·591878

$(\varphi' + \frac{1}{2}\varphi) = 189$ · log. cos. 9·994620

4 · log. ·· 0·602060

$-4\cos.\frac{1}{2}\varphi\cos.(\varphi'+\frac{1}{2}\varphi) = -1\cdot54368$ log. ·· 0·188558

$-2\cos.\varphi = -1\cdot28558$

$-2\cos.\varphi - 4\cos.\frac{1}{2}\varphi\cos.(\varphi'+\frac{1}{2}\varphi) = -2\cdot82926$

consequently, $R = 26\sqrt{3-2\cdot82926} = 10\cdot738$ tons, for the magnitude of the resultant or single force, equivalent to three forces of 26 tons each.

61. EXAMPLE 2. Three ships equally distant from a battery send balls of equal weights to the same point of the wall in equal times. Now, supposing these balls to act simultaneously with a velocity of 300 feet per second, and the extreme lines of firing to incline to the middle one in angles of 100° and 120°; what is the velocity of a single ball to produce the same effect?

Here we have angle $\varphi = 100°$

$\varphi' = 120$

$\frac{1}{2}\varphi = 50$ · log. cos. 9·808067

$(\varphi' + \frac{1}{2}\varphi) = 170$ · log. cos. 9·993351

4 · log. 0·602060

$-4\cos.\frac{1}{2}\varphi\cos.(\varphi'+\frac{1}{2}\varphi) = -2\cdot5321$ log. 0·403478

$-2\cos.\varphi = -\ \cdot3471$

$-2\cos.\varphi - 4\cos.\varphi\cos.(\varphi'+\frac{1}{2}\varphi) = -2\cdot8792$;

consequently, we get

$$R = 300\sqrt{3-2\cdot8792} = 104\cdot1 \text{ feet nearly,}$$

the velocity sought.

Examples to Equation (u).

62. EXAMPLE 1. Suppose that the magnitude of each of three equal forces, disposed in one plane, and directed to the same point, is represented and measured by a line of 24 inches in length; what must be the magnitude of a single force whose energy shall be equivalent to that of all the three; supposing the direction of the extreme forces to be inclined to the middle one in angles each of 137 degrees?

Here we have the angles of inclination, φ and $\varphi' = 137°$, its cosine $= \cdot 73135$; consequently, $a\,(2\cos.\varphi - 1) = 24\,(1\cdot 4627 - 1) = 24 \times \cdot 4627 = 11\cdot 1048$ tons, for the magnitude of the resultant.

If the same example be performed by equation (t), it will stand thus:

$$\begin{array}{rll} \text{angle of inclination } \varphi = & 137° & \\ \text{angle of inclination } \varphi' = & 137 & \\ \tfrac{1}{2}\varphi = & 68:30 \text{ log.cos.} & 9\cdot 564075 \\ (\varphi' + \tfrac{1}{2}\varphi) = & 205:30 \text{ log.cos.} & 9\cdot 955488 \\ & 4 \text{ log.} & 0\cdot 602060 \\ \hline -4\cos.\tfrac{1}{2}\varphi\cos.(\varphi'+\tfrac{1}{2}\varphi) = & -1\cdot 32318 \text{ log.} & 0\cdot 121623 \\ -2\cos.\varphi = & -1\cdot 4627 & \\ \hline -2\cos.\varphi - 4\cos.\tfrac{1}{2}\varphi\cos.(\varphi'+\tfrac{1}{2}\varphi) = & -2\cdot 78588 & \end{array}$$

consequently, $R = 24\sqrt{3 - 2\cdot 78588} = 24 \times \cdot 4627 = 11\cdot 1048$ tons, for the magnitude of the resultant, the same as before.

63. EXAMPLE 2. Three forces, each 300 lbs. weight, and inclining to each other in angles of 135°, act simultaneously at the same point of a body; what is the magnitude of a fourth force that would sustain them at rest?

Here we have given $a = 300$, and $\varphi = \varphi' = 135°$; consequently, we get

$$R = 300\sqrt{2\cos.135° - 1} = 124\cdot 26$$

Examples to the General Equation (l).

64. EXAMPLE 1. Suppose that the magnitudes of three forces, disposed in the same plane, and directed to the same point of a body, are measured respectively by weights of 6, 23, and 14 tons; what must be the magnitude of the resultant or single force, whose energy shall be equivalent to the united energies of the three given forces, the inclinations being respectively 124° 56′ and 78° 34′?

By the question, it is to be understood that the greater force occupies the middle place, and that the direction of the first extreme, or least force, is inclined to the direction of the middle one in an angle of 124° 56′, while the inclination of the other extreme force is 78° 34′. This premised, the operation will stand as below.

Here we have the angle of inclination $\varphi = 124° 56'$, its natural cosine $= - \cdot 57262$,

$$a^2 = 6 \times 6 = 36$$
$$b^2 = 23 \times 23 = 529$$
$$a^2 + b^2 = 565$$
$$-2\,ab \cos.\varphi = -158 \cdot 04312, \text{ subtract, because } \varphi \text{ is obtuse;}$$

then $\sqrt{a^2+b^2-2\,ab\cos.\varphi} = \sqrt{406 \cdot 95688} = 20 \cdot 173$ tons, the resultant of the first extreme and the middle forces.

Again, to find the direction of this resultant to the middle force, we have by equation (k)

$$\begin{array}{llr}
b = 23 & \ldots \text{ log.} & 1 \cdot 361728 \\
a = 6 & \text{ar. co. log.} & 9 \cdot 221849 \\
\varphi = 124° 56' & \text{log. cosec.} & 0 \cdot 086282 \\
\hline
\text{natural number} = 4 \cdot 67582 & \text{log.} & 0 \cdot 669859
\end{array}$$

$$\varphi = 124° 56' \text{ natural cotangent } = -0 \cdot 69847$$
$$x = 14\ 7 \text{ natural cotangent } = 3 \cdot 97735$$

$$\begin{array}{rlll}
\text{consequently, the angle CPE} = (\varphi' + x) = & 92° 41' & \text{log. cos.} & 8 \cdot 670393 \\
\sqrt{a^2+b^2-2\,ab\cos.\varphi} = & 20 \cdot 173 & \text{log.} & 1 \cdot 304770 \\
2c = & 28 & \text{log.} & 1 \cdot 447158 \\
\hline
-2c\cos.(\varphi'+x)\sqrt{a^2+b^2-2\,ab\cos.\varphi} = & -\ 26 \cdot 4436 & \text{log.} & 1 \cdot 422321 \\
2\,ab\cos.\varphi = & -158 \cdot 04312 & &
\end{array}$$

$$\left.\begin{array}{r}-2\,ab\cos.\varphi - 2c\cos.(\varphi'+x) \\ \sqrt{a^2+b^2-2\,ab\cos.\varphi}\end{array}\right\} = -184 \cdot 48672, \text{ subtract.}$$
$$a^2+b^2+c^2 = 761$$
$$\left.\begin{array}{r}\{a^2+b^2+c^2-2\,ab\cos.\varphi-2c\cos. \\ (\varphi'+x)\sqrt{a^2+b^2-2\,ab\cos.\varphi}\}\end{array}\right\} = 576 \cdot 51328, \text{ difference.}$$

consequently we have $R = \sqrt{576 \cdot 51328} = 24$ tons nearly, the measure of the resultant of the three given forces.

65. EXAMPLE 2. Three cannons which discharge shots of 25, 30, and 35 lbs. respectively, were employed to demolish the walls of a garrison. Now, supposing the shots to proceed with the same velocity, and to act simultaneously on the same point of the wall, and the extreme lines of firing to incline to the middle one in angles of 30° and 45°; what is the weight of a single ball to produce the same effect? Here we have given $a = 25$; $b = 30$; $c = 35$; $\varphi = 30°$; and $\varphi' = 45°$; consequently, the angles φ and φ' being less than 90°, the general equation (l) becomes

$$R = \sqrt{a^2+b^2+c^2+2\,ab\cos.\varphi+2\,c\cos.(\varphi'+x)\sqrt{a^2+b^2+2\,ab\cos.\varphi}};$$

$$\text{but } a^2 = 25 \times 25 = 625$$
$$b^2 = 30 \times 30 = 900$$
$$a^2 + b^2 = 1525$$

$2ab\cos.\varphi = 2 \times 25 \times 30 \times \cdot 86603 = 1299 \cdot 045$; add, because φ is acute.

$$\sqrt{a^2+b^2+2ab\cos.\varphi}=\sqrt{1525+1299\cdot045}=53\cdot1417$$
$$a^2+b^2+c^2=625+900+1225=2750$$

hence we get

$$R=\sqrt{2750+1299.045+70\cos.(\varphi'+x)\times53.1417},$$

but cot. $x=\frac{b}{a}$ cosec. $\varphi+$cot. φ; see equation (k)

that is cot. $x=\frac{6}{5}\times2+\sqrt{3}=4.132=$ cot. $13°36$;

hence we get $(\varphi+x)=58°36'$; its nat. cos. $=.52101$,

therefore $R=\sqrt{8597.16}=77.5$ lbs.

Having thus illustrated the method of computing the resultant of three given composants, both in the general case when they are of different magnitudes, as indicated in equation (l), and in the particular cases when they are equal, as indicated in equations (o), (p), (t), and (n); we proceed to show in what manner the direction of the resultant is to be determined.

66. PROBLEM 2. *To determine the direction of the resultant of three given forces.*

For this purpose we must recur to the original diagram (page 30) for three forces, and calculate the resultant PC $=r$ for the two forces a and b, as indicated in equation (a), and the angle CPB $=x$ as indicated in equation (k); then will the angle CPE or CFE $=(\varphi'+x)$. Again, compute the angle FPE, such that

$$\text{cot. FPE}=\frac{c\ \text{cosec}\ (\varphi'+x)}{\sqrt{a^2+b^2\pm2\,ab\cos.\varphi}}\pm\text{cot.}\ (\varphi'+x). \qquad (v)$$

The angle FPE thus determined, expresses the direction of the resultant PF $=$ R with respect to the extreme force PE $=c$, from which its direction with respect to the other forces can easily be made known.

67. The application of this formula may be shown in all its generality by assigning the direction of the resultant with respect to the directions of the three given forces in the preceding example, art. 64.

Although the value of the resultant r and the angle x are each of them determined by the foregoing process, yet in order to show the whole of the operation that is required when the problem is to be resolved generally, it becomes necessary to recompute those elements.

$$\begin{aligned}a^2&=6\times6 & 36\\ b^2&=23\times23= & 529\\ a^2+b^2&= & 565\\ -\,2\,ab\cos.\varphi&= & -158.04312\\ a^2+b^2-2\,ab\cos.\varphi&= & 406.95688;\end{aligned}$$

[φ is obtuse. subtract because

consequently, $r=\sqrt{406.95688}=20.173.$

Again, for the value of the angle x, it is

$$\begin{array}{lll} b= & 23 \;.\;. & \log.\; 1{\cdot}361728 \\ a= & 6 \;.\;. & \log.\; 9{\cdot}221849 \\ \varphi= & 124^\circ 56' & \log.\ \text{cosec}\; \underline{0{\cdot}086282} \\ \text{natural number} & =4{\cdot}67582\; - & \log.\; 0{\cdot}669859 \end{array}$$

$\varphi=124^\circ 56'$ natural cotang. $= -0{\cdot}69847$
$x=\;14\;\;7$ natural cotang. $=\;\;3{\cdot}97735$;

hence we have $(\varphi'+x)=92^\circ 41'$.

The equation now assumes the following form, viz.

$$\text{cot. FPE}=\frac{14\ \text{cosec.}\ 92^\circ 41'}{20{\cdot}173}-\text{cot.}\ 92^\circ 41'.$$

and its reduction is simply as below,

$$\begin{array}{lll} & 14 \;.\;.\;.\;.\;. \log.\;.\;. & 1{\cdot}146128 \\ & 92^\circ{\cdot}41 \;.\;. \log.\text{cosec.} & 0{\cdot}000476 \\ & 20{\cdot}173\ \text{ar.co.log.}\;.\;. & \underline{8{\cdot}695230} \\ \text{natural number}= & 0{\cdot}69486 \quad \log.\;.\;. & 9{\cdot}841834 \end{array}$$

$92^\circ\,51$ natural cotang. $= \underline{-0{\cdot}04686}$

$57\;\;3$ natural cotang. $=\;\;0{\cdot}648$; consequently, the inclination of the resultant to the extreme force PE is $57^\circ 3'$; its inclination to the middle force PB is $78^\circ 34'-57^\circ 3'=21^\circ 31,'$ and its inclination to the first extreme PA is $21^\circ 31'+124^\circ 56'=146^\circ 27'$.

The above is the method of resolving the equation numerically, and that which all persons conversant with the language of algebra will most probably adopt and prefer, but as some practical mechanics are ignorant of the meaning of algebraic combinations, it will perhaps be advisable for such to construct the figure geometrically, this being the only check against the admission of error in estimating the position of the resultant with respect to the directions of the several forces.

68. If the angle FPE be represented by x'; then, the algebraic expressions for the relative positions of the resultant with respect to the directions of the forces, are x'; $(\varphi' \backsim x')$, and $\varphi \pm (\varphi' \backsim x')$; that is, when the resultant falls between the middle and either of the extreme forces; but when it falls in the reverse direction between the forces PA and PE, then the expressions become x'; $180^\circ \backsim (\varphi'+x')$, and $360^\circ-(\varphi+\varphi'+x')$.

If the forces are equal among themselves,

$$\text{cot.}\ x'=\frac{\text{cosec.}\ (\varphi'+\frac{1}{2}\varphi)}{\sqrt{2\pm 2\cos.\ \varphi}}\pm\text{cot.}\ (\varphi'+\tfrac{1}{2}\varphi).$$

In this equation the forces do not occur, by reason of the supposed equality, the factors in the denominator of the general equation (v) become equal to those in the numerator in so far as the forces are concerned, and therefore they mutually cancel one another.

We give the following example for elucidation.

69. EXAMPLE 1. Suppose that the directions of three equal component forces are inclined to each other, as follows: viz. the first to the second in an angle of 67 degrees 30 minutes, and the second to the third in an angle of 45 degrees; what is the inclination of the common resultant to each of the forces?

Here we have $\varphi = 67^\circ 30'$
$\varphi' = 45$
and $\frac{1}{2}\varphi = 33\ 45$;

consequently, $(\varphi' + \frac{1}{2}\varphi) = (45^\circ + 33^\circ 45') = 78\ 45$. log.cosec. $0{\cdot}008426$
$(2 + 2\cos.\varphi) = 2{\cdot}76536$ $\frac{1}{2}$ log. $0{\cdot}220875$
natural number $= 0{\cdot}61313$ log. $9{\cdot}787551$
$(\varphi' + \frac{1}{2}\varphi) = 78\ 45$ nat. cot. $= 0{\cdot}19891$;

therefore, nat. cot. $x' = 0{\cdot}61313 + 0{\cdot}19891 = 0{\cdot}81204$, and $x' = 50^\circ 55'$.

Here, because x' is greater than φ', and both the angles of inclination less than 90 degrees, it is evident that the resultant must fall between the middle and first extreme force; that is, between the forces a and b, whose magnitudes are represented by the lines PA and PB; its positions, therefore, relative to the directions of the three forces, are as below, viz.

With respect to PE $= c$, it is $50^\circ 55'$, found by the process,
With respect to PB $= b$, it is $5\ 55 = 50^\circ 55 - 45^\circ\ 0'$,
With respect to PA $= a$, it is $61\ 35 = 67\ 30 - 5\ 55$.

We shall propose another example of a general nature, and such that the resultant may fall in the reverse direction, between the extreme forces PA and PE, thus.

70. EXAMPLE 2. Suppose that three forces, whose magnitudes are respectively represented by straight lines of 12, 9, and 14 inches in length, act simultaneously in the same plane, and concur in the same point: what must be the magnitude and direction of a single force applied at the same point, to produce an equivalent effect to all the three given forces acting together, supposing their inclinations to be respectively 135 and 120 degrees?

This question demands both the magnitude of the resultant, and its direction with respect to the directions of the given forces; now, since in equation (f), we have shown how to obtain the magnitude without regarding the direction, and in equation (l), how to find the direction independently of the magnitude, it may perhaps be an useful exercise to find each of them by its appropiate formula, and then to deduce them respectively from each other.

First, for the magnitude of the resultant.

$a^2 = 144$
$b^2 = \ \ 81$
$a^2 + b^2 = 225$ —— $\varphi = 135^\circ$, its cosine $= -70711$
$-2ab\cos.\varphi = -\ 152{\cdot}73576$, subtract because φ is obtuse,

then $\sqrt{a^2 + b^2 - 2ab\cos.\varphi} = \sqrt{72{\cdot}26424} = 8{\cdot}5$ inches, the resultant of the first extreme and middle forces.

Then by plane trigonometry, or equation (d) we have

$$8\cdot5 : \sin.\ \varphi :: 12 : \sin.\ x,$$

or by equating the products of the extreme and mean terms, it is

$$8\cdot5 \sin.\ x = 12 \sin.\ \varphi;$$

therefore, by introducing the numerical value of sin. φ and dividing by 8.5, we get sin. $x = \cdot99826$; wherefore $x = 86°37'$. Now, the angle CPE $= 360 - (\varphi' + x) = 153°23'$, consequently it is

$$\begin{array}{rrr}
153°23 \ldots & \text{log. cos.} & 9\cdot931349 \\
\sqrt{a^2+b^2-2\,ab\cos\varphi} = 8.5 & \text{log.} & 0\cdot929419 \\
2\,c = 28 & \text{log.} & 1\cdot447158 \\
\end{array}$$

$$-2c\cos.\Big(360° - (\varphi' + x)\Big)\sqrt{a^2+b^2-2ab\cos.\varphi} = -212\cdot778 \text{log.} 2.327926$$

$$-2ab\cos.\varphi = -152\cdot73576$$

$$\left.\begin{array}{r} -2ab\cos.\varphi - 2c\cos. \\ \Big(360°(\varphi'+x)\Big)\sqrt{a^2+b^2-2ab\cos.\varphi} \end{array}\right\} = -365\cdot51376 \text{ subtract.}$$

$$a^2+b^2+c^2 = \quad 421$$

$$\left.\begin{array}{l} \{a^2+b^2+c^2-2ab\cos.\ \varphi - 2c\cos. \\ \Big(360° - (\varphi'+x)\Big)\sqrt{a^2+b^2-2ab\cos.\varphi} \end{array}\right\} = \quad 55\cdot48624 \text{ diff.}$$

$$\text{consequently, } R = \quad \sqrt{55\cdot48624} = 7\cdot449$$

inches, the magnitude of the resultant.

Then, for the direction with respect to the other forces, we have by Trigonometry,

$$7\cdot449 : \sin.\ 153°23' :: 8\cdot5 : \sin.\ x';$$

$$\text{that is, } \sin.\ x' = \frac{8\cdot5 \times \cdot44802}{7\cdot449} = \cdot51123\cdot$$

hence we have $x' = 30°45'$; therefore, the directions of the resultant with respect to the directions of the several component forces are as under.

With respect to PE $= c$, it is $30°45°$, found by the process.
With respect to PB $= b$, it is $29\ 15 = 180° - (120 + 30°45')$.
With respect to PA $= a$, it is $74\ 15 = 360 - (135 + 120 + 30°\ 45')$.

It may here be necessary to remark, that the directions given above are what may be called the acute directions, or those in which the resultant, or its production, makes with the given forces angles respectively less than a right angle. By viewing the subject in this light, we learn, that either the resultant, as determined by the operation or its production, will produce the same effect as the forces themselves, or, if acting co-temporaneously, will sustain the system at rest.

71. In the foregoing process we have first determined the magnitude of the resultant from equation (f), and then deduced the direction from the magnitude; but in what follows, we shall first find the direction from equation (l), and then deduce the magnitude from the direction.

First then for the angle of direction.

$$a^2 = 144$$
$$b^2 = \ \ 81$$
$$a^2 + b^2 = \overline{225} \qquad \varphi = 135, \text{ its cos.} = -\cdot 70711$$
$$-2ab\cos.\varphi = -152\cdot 73576, \text{ subtract because } \varphi \text{ is obtuse.}$$

$\sqrt{a^2+b^2-2ab\cos.\varphi} = \sqrt{72\cdot 26424} = 8.5$ inches, the measure of the resultant PC $= r$; then by Plane Trigonometry, we have

$8\cdot 5 : \sin. 135^\circ :: 12 : \sin. x = \cdot 99826$; therefore $x = 86^\circ\ 37$

Now, the angle CPE $= 360^\circ - (\varphi' + x) = 153^\circ 23'$ . . log. cos. $0\cdot 348703$

$c = \ 14$ . . . . log. . . $1\cdot 146128$

$\sqrt{a^2 + b^2 - 2ab\cos.\varphi} = \ 8\cdot 5$ ar. co. log . . $9\cdot 070581$

natural number $= 3\cdot 6763$ log. . . $0\cdot 565412$

$153^\circ\ 23'$ natural cotang. $= -1\cdot 9955$

$30\ \ 45$ natural cotang. $= \ \ 1\cdot 6808$; consequently the inclination of the resultant to the extreme force PE is $30^\circ 45'$; this is the same inclination as was found by the foregoing process; hence, to find the magnitude of the resultant from the angle of inclination just determined, we have

$$\sin. 30^\circ\ 45' : 8.5 :: \sin. 153^\circ\ 23' : 7\cdot 449 \text{ inches.}$$

The inclination of the resultant to the direction of the other forces may be found as before.

72. Thus have we fully and clearly developed the method of finding the magnitude and direction of the resultant corresponding to two and to three composants, by means of what is generally called the *parallelogram of forces;* but there is another method, somewhat different from the former, by which the same things can be accomplished, called the method of *rectangular co-ordinates;* to which we wish to call the reader's attention, but having dwelt so long on the method by the parallelogram of forces, we shall be very brief in our remarks on this latter method.

Previously, however, to our developing the principles of the composition of forces by the method of rectangular co-ordinates, it will be necessary to show in what manner a single force may be decomposed into two or more forces, whose united energies shall be equivalent to the energy of the given force. This is called the *Resolution of Forces,* and the mode of procedure is by a problem exactly the reverse of that which we have been investigating in the foregoing sections.

SECTION THIRD.

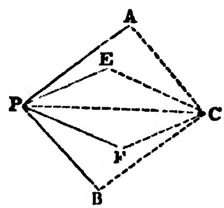
OF THE RESOLUTION OF FORCES.

73. It has been shown that the resultant or single equivalent of any two given forces, situated in the same plane and directed to the same point, is represented in magnitude and direction by the diagonal of a parallelogram constructed on the lines that represent the forces; and, moreover, that the resultant or single equivalent

of any three given forces, situated in the same plane, and directed to the same point, is represented in magnitude and direction by the diagonal of a parallelogram constructed on the line which represents one of the forces and the resultant of the other two; and the same principle of composition would manifestly extend to any number of forces whatever: hence conversely,

A single force may, ad libitum, be decomposed or resolved into any number of other forces, whose united energies shall just be equivalent to the energy of the given force.

This is evident; for the single force PC, is equivalent to, and composed of the two forces PA and PB, which are the contiguous sides of the parallelogram PACB, whose diagonal is the given force PC; but the single force PC will likewise be equivalent to and composed of any other two forces PE and PF, which are also the contiguous sides of the parallelogram PECF, having for its diagonal the given force PC; it consequently follows, that the single force PC may be decomposed into as many pairs of component forces, as there can be parallelograms constructed upon it as a diagonal; that is, an unlimited number. And since each of the component forces may in like manner be decomposed into two, and each of these again into two, it is obvious that the primitive force PC may be conceived to be resolved into others without end. This mode of decomposition, or analysis, is very frequently employed in mechanical inquiries from its great usefulness in what is called



The Reduction of Forces,

or that process, by which we estimate the effects of forces in a given direction, when their action is solely confined to a different direction.

But although, as we have stated, any single force can be resolved into an unlimited number of other forces, whose united energies shall only be equivalent to the energy of the force proposed; yet in practical cases, a limit to the number of composants must always be assigned; and in order to bring the subject within the powers of calculation, a sufficient number of data must likewise be supplied. Thus for instance, to resolve a single force into two others of determinate magnitude, it is necessary to have given, besides the magnitude of the force proposed, the angles of its inclination to the directions of the required forces; or if the magnitude of one component and the inclination of their directions, with the magnitude of the resultant or single equivalent force be given, the magnitude of the other component can easily be found. Similar remarks will apply to the cases of three components, but our inquiries shall be confined to two.

74. It is shown in equation (a), that the expression for the magnitude of PC, the resultant of the two forces PA and PB, is

$$r=\sqrt{a^2+b^2\pm 2ab\cos.\varphi};$$

consequently, if the values of the given quantities be substituted for their representatives in this equation, the value of the unknown term will thence be ascertained; but it will perhaps be better to resolve the question in its general form, and then to substitute the given quantities in the result according to their powers and combinations.

75. Let it, for example, be proposed to find the magnitude of the force (a), the magnitudes of b and r being given, together with φ, the angle of inclination of the directions of a and b.

If φ, the given angle of inclination is acute, then by squaring both sides of the equation, we obtain

$$r^2=a^2+b^2+2ab\cos.\varphi;$$

but if φ, the given angle of inclination is obtuse, it is

$$r^2=a^2+b^2-2ab\cos.\varphi.$$

In either case, however, the method of solution is the same, and in both cases the component a, has a positive and a negative value, unless when b is greater than r, and φ the angle of inclination obtuse; in that case, both the values of a will either be positive or impossible.

First then, for the acute value of φ, transpose and we get

$$a^2+2ab\cos.\varphi=r^2-b^2,$$

complete the square, and we get

$$4a^2+8ab\cos.\varphi+4b^2\cos.^2\varphi=4\{r^2-b^2+b^2\cos^2.\varphi\},$$

evolve both sides and it becomes

$$2a+2b\cos.\varphi=\pm\sqrt{4\{r^2-b^2+b^2\cos.^2\varphi\}},$$

then by transposition and division, we get

$$a=\pm\sqrt{r^2+b^2(\cos.^2\varphi-1)}-b\cos.\varphi, \qquad (w)$$

Next for the obtuse value of φ, transpose and we get

$$a^2-2ab\cos.\varphi=r^2-b^2,$$

complete the square, and we have

$$4a^2-8ab\cos.\varphi+4b^2\cos.\varphi^2=4\{r^2-b^2+b^2\cos.^2,\varphi\}$$

evolve both sides, and it becomes

$$2a-2b\cos.\varphi=\pm\sqrt{4\{r^2-b^2+b^2\cos.^2\varphi,\}}$$

then by transposition and division we obtain

$$a=\pm\sqrt{r^2+b^2(\cos.^2\varphi-1)}+b\cos.\varphi, \qquad (x)$$

Examples to equation (w)

76. EXAMPLE 1. Suppose the magnitude of one component to be measured by a load of 22 tons, and the magnitude of the resultant by a load of 39 tons; what must be the magnitude of the other component, supposing the inclinations of the directions to be 82° 30′?

By substituting the given numbers in equation (w), we get

$$a=\sqrt{39^2+22^2(\cos.^2 82°30'-1)}-22\cos. 82°30';$$

but the natural cosine of 82° 30′ is 0·13053, and its square is 0.01704, from which if unity be subtracted there remains −0·98296; consequently, we have

$a=\sqrt{1521-475\cdot75264}-2\cdot87=29\cdot46$ tons very nearly, for the magnitude of the required component.

77. Example 2. A heavy body is drawn in one direction by a force equal to 30 horses' power, what force should be attached to this body, and inclining to the former in an angle of 60 degrees; so that their joint efforts may be equal to 60 horses' power?

Here we have given $a=30$, $r=60$, and $\varphi=60°$; consequently, by equation (w) we get

$$b=\sqrt{r^2+a^2(\cos.^2\varphi-1)}-a\cos.\varphi.$$

that is $b=\sqrt{3600+900\,(\cdot25-1)}-15=39\cdot083$ horses' power.

Examples to Equation (x).

78. Example 1. Let the magnitudes of the given component and resultant forces remain as above, and suppose φ, the angle of inclination, to be 112° 30′; what then, will be the magnitude of the other component?

By substituting the given numbers in the equation, we get

$$a=\sqrt{39^2+22^2(\cos.^2 112°30'-1)}+22\cos. 112°30'$$

but the natural cosine of 112° 30′ is −0·38268, and its square is 0·14644, from which, if unity be subtracted, there remains −0·85356; consequently, we have

$a=\sqrt{1521-413\cdot12304}+8.419=41\cdot704$ tons very nearly, for the magnitude of the required component.

If the component a were given instead of b, and b required, the resulting equations for the acute and obtuse values of φ would be perfectly symmetrical, and the operation would be the same as above; it is therefore unnecessary to furnish examples in the case of a being the given quantity, since performing the operation would cast but very little if any new light on the subject.

79. Example 2. We shall, however, resolve the foregoing example, by substituting the given numbers in the original equation, instead of transferring them to the reduced expressions as we have done in the preceding case: therefore,

By subsituting the given numbers in equation (a) it becomes

$$\sqrt{a^2+44\,a\cos. 82\;30'+22^2}=39,$$

by squaring both sides and transposing we obtain

$$a^2+44\,a\cos. 82°\;30'=1037,$$

or by introducing the natural cosine of 82° 30′, it becomes

$$a^2+5\cdot74332\,a=1037,$$

complete the square, and we have

$$a^2+5\cdot7433a+8\cdot2464=1045\cdot2464,$$

evolve both sides, and we get

$$a+2\cdot87166=32\cdot33$$

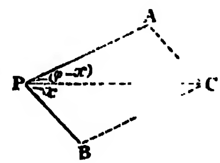
transpose, and we obtain

$$a = 32.33 - 2.87 = 29.46 \text{ tons},$$

for the magnitude of the required component, the very same as before:—and exactly in the same manner, would the equation be resolved for the obtuse value of φ the angle of inclination, with this exception, that the sign of the term into which φ enters is to be taken negatively.

Such is the method of calculating the components corresponding to a given resultant, when the magnitude of one of them and the inclinations of their directions are known; we shall now endeavour, in the remaining branch of this section, to show how the components are to be determined when there are given, first, the magnitude of the resultant, and secondly, the angle which it makes with the direction of each component.

80. PROBLEM. *Given the magnitude of the resultant and the angle which it makes, with the direction of each composant, to determine or resolve these composants.*



Since the whole angle BPA is represented by the Greek letter φ, and the part of it BPC by x, the other part APC is represented by $(\varphi - x)$; therefore, the algebraic representatives of the two given angles are $(\varphi - x)$ and x.

Then, by Plane Trigonometry, we have

$$\sin.\varphi : \sin.x :: r : a = \frac{r \sin.x}{\sin.\varphi},$$

$$\sin.\varphi : \sin.(\varphi - x) :: r : b = \frac{r \sin.(\varphi - x)}{\sin.\varphi}.$$

But these results would be better adapted for calculation, if the denominators of the fractions were removed; and this is effected by considering that $\frac{1}{\sin.\varphi} = \text{cosec.}\,\varphi$; therefore, by substituting cosec. φ in the numerators instead of sin. φ in the denominators, we shall get

$$\left.\begin{array}{ll} 1. & a = r \sin.x.\,\text{cosec.}\,\varphi \\ 2. & b = r \sin.(\varphi - x)\,\text{cosec.}\,\varphi \end{array}\right\} \qquad (y)$$

The practical rule afforded by these equations, is simply as under.

81. Rule. *Multiply the given resultant by the natural cosecant of the sum of the given angles, and that product being again multiplied by the natural sine of either angle, will give the magnitude of the force opposite to that angle.*

82. EXAMPLE. Suppose a single force whose magnitude is represented by a line of 15 inches in length, is to be resolved into its components, in such a manner that their directions shall be inclined to it, respectively, in angles of 24 and 46 degrees?

Here we have, $r=15$ inches; $\varphi=70°$, and $(\varphi-x)=24°$, x being $46°$; therefore by the rule we have

$a=15\times$ cosec. $70°\times$ sin. $46°=15\times1{\cdot}06415\times{\cdot}71934=11{\cdot}48$ inches,

$b=15\times$ cosec. $70\times$ sin. $24=15\times1{\cdot}06415\times{\cdot}40674=6{\cdot}49$ inches.

83. These are the results, or magnitudes of the components for the acute value of φ, but when φ is obtuse, one at least or both the components may be greater than the proposed resultant; for which reason this case merits a separate

Example.

Let the resultant remain as above, and suppose the angles to be respectively 68 and 82 degrees; what then is the magnitude of the components?

Here we have $r=15$ inches; $\varphi=150°$; $x=82°$, and $(\varphi-x)=68°$; therefore, by the rule we have

$a=15\times$ cosec. $150°\times$ sin. $82°=15\times2\times{\cdot}99027=29{\cdot}7+$ inches.

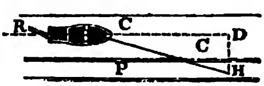
$b=15\times$ cosec. $150\times$ sin. $68=15\times2\times{\cdot}92718=27{\cdot}8+$ inches.

84. If the composants act at right angles to each other, then $\varphi=90$ degrees and cosec. $\varphi=1$; therefore, the preceding equations become

$$\left.\begin{array}{l}1.\ a=r\ \text{sin.}\ x,\\ 2.\ b=r\ \text{cos.}\ x.\end{array}\right\}\qquad (z)$$

These are the equations applicable to the reduction of forces, and their use will be rendered manifest by the following familiar and very important Example.

85. Let C, C, be a portion of a canal or river, B a boat or other vessel afloat thereon, R its rudder, P the towing path, and H the position of the horse dragging the vessel along in the direction of the dotted line BD. Now, it is obvious, that if the horse could be brought to act at D in the direction BD, the boat or vessel would glide along in the same direction without the least tendency to deviate from the straight course towards the one side or the other; but since the horse cannot act on the water, and in a line perpendicular to the beam of the head at midships, it is necessary to place him on the towing path, which brings the line of traction into the oblique position BH, and in consequence of this obliquity, the vessel has a tendency to approach that side of the canal or river on which the power acts; but in order to prevent this approach, and keep the boat on the straightforward course, the rudder R is turned towards the other side, which, operating on the water as a fulcrum, keeps the vessel's head in the same direction; but the power still acting in the oblique tract BH, counteracts the operation of the rudder, and forces the vessel forward in the dotted line BD.



If the rudder be maintained in a position always parallel to the line of traction, the vessel would have no more tendency to deviate from the direction of the line BD, than it would have, if the power operated in that direction, but the resistance to the power is much greater.

Now, the question is, how much more force the horse must exert in the direction BH, than would be required of him in the direction BD, supposing that he operates at H with a force equal to 150 pounds, the obliquity of traction being $22\frac{1}{2}$ degrees, and the resistance of the water wholly set aside?

From the point H let fall the perpendicular HD, then will BD and HD represent the magnitudes of the composants corresponding to the resultant BH, which is here supposed to represent the power of the horse; but the force HD acts perpendicularly to BD, and therefore neither promotes nor counteracts the effect of the force in that direction; consequently, the power of the horse reduced to the direction BD is represented in magnitude by the line BD, and its numerical value obtained from the first of equation (q) is 138·582 pounds.

Section Fourth.

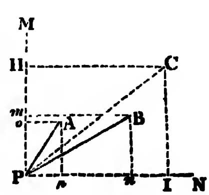
OF THE COMPOSITION OF FORCES BY THE METHOD OF RECTANGULAR CO-ORDINATES.

86. We must now resume the subject of the composition of forces, and endeavour to expound the principles of that curious and important mechanical problem by the method of "*rectangular co-ordinates*," a method which, although not distinguished by the elegant simplicity of construction appertaining to the parallelogram of forces, is extremely fertile in unfolding numerous beautiful and interesting properties and formulæ, which may on many occasions be employed with immense advantage. If for this reason it claims our attention, how much more so by the knowledge which it will impart?

87. Problem I. *To determine the magnitude of the resultant, whether the forces be equal or unequal, provided the magnitude of the forces and the inclinations of their directions to the co-ordinates be known.*

Let the straight lines PA and PB represent the magnitudes and directions of the two given forces a and b, acting in the same plane and having the same point of application P.

Through P, the common point of application, draw the indefinite straight line PN in any determinate direction with respect to PA and PB, and erect PM perpendicular to PN; then are PM and PN the axes of rectangular co-ordinates by which PC, the magnitude of the resultant, is to be determined.



Upon the straight lines PA and PB, as diagonals, construct the rectangular parallelograms PoAp and PmBn; then are the forces PA and PB considered as resultants, respectively resolved into the components Po, Pp, and Pm, Pn.

Make mH equal to Po and nI equal to Pp, and on PH, PI construct the rectangular parallelogram PHCI, and draw PC; then is PC the resultant of the two given forces a and b, whose magnitudes are represented by the straight lines PA and PB.

This is evident, for Pp and Pn represent the magnitudes of the forces PA and PB when reduced to the direction PN, and because Pp and Pn act in the same straight line and in the same direction, they are equivalent to a single force whose magnitude is expressed by their sum.

Again, Po and Pm represent the magnitudes of PA and PB when reduced to the direction PM, and because they act in the same straight line and in the same direction, they are equivalent to a single force whose magnitude is expressed by their sum; but PI is the sum of Pp and Pn, and PH is the sum of Po and Pm; consequently, PC is the resultant or equivalent of the two forces, PA and PB.

Put $\pi = \text{BP}n$, the angle which the force b or its representative PB makes with the axis or ordinate PN,

$(\varphi+\pi) = \text{AP}p$, the angle which the force a, or its representative PA makes with the same axis or ordinate, φ being as before,

and $x = \text{CPI}$, the angle which the resultant r, or its representative PC makes with the axis or ordinate PN, on which the effects of the forces a and b are estimated.

Then by Plane Trigonometry, we have

$$Pn = b \cos. \pi, \text{ and } Pm = b \sin. \pi$$

$$Pp = a \cos. (\varphi+\pi), \text{ and } Po = a \sin. (\varphi+\pi);$$

therefore, $\text{PI} = b \cos. \pi + a \cos. (\varphi+\pi)$, and $\text{PH} = b \sin. \pi + a \sin. (\varphi+\pi)$; but $\text{PI} = r \cos. x$, and $\text{PH} = r \sin. x$; consequently, by substitution we obtain

$$\left.\begin{array}{l} 1.\ r \cos. x = b \cos. \pi + a \cos. (\varphi+\pi) \\ 2.\ r \sin. x = b \sin. \pi + a \sin. (\varphi+\pi) \end{array}\right\} \quad (\text{A})$$

If the forces a and b are equal, then we have

$$r \cos. x = a \{\cos. \pi + \cos. (\varphi+\pi)\},$$

$$r \sin. x = a \{\sin. \pi + \sin. (\varphi+\pi)\};$$

but by the arithmetic of sines, $\cos. \pi + \cos. (\varphi+\pi) = 2 \cos. (\frac{1}{2}\varphi+\pi) \cos. \frac{1}{2}\varphi$, and $\sin. \pi + \sin. (\varphi+\pi) = 2 \sin. (\frac{1}{2}\varphi+\pi) \cos. \frac{1}{2}\varphi$; therefore, by substitution, we have

$$\left.\begin{array}{l} 1.\ r \cos. x = 2a \cos. (\frac{1}{2}\varphi+\pi) \cos. \frac{1}{2}\varphi \\ 2.\ r \sin. x = 2a \sin. (\frac{1}{2}\varphi+\pi) \cos. \frac{1}{2}\varphi \end{array}\right\} \quad (\text{B})$$

88. From these equations, then, it is easy to determine the magnitude of the resultant of two forces, either in the general case, where they are unequal, as indicated in equation (B), or in the particular case, where they are equal, as indicated in equation (A), provided that the magnitude of the forces, and the inclination of their directions to the co-ordinates, are known. For, notwithstanding that x, the inclination of the resultant to the co-ordinate PN, is not supposed to be given, it can easily be found in terms of the forces and the angles of their inclinations to that co-ordinate.

Thus, for example,

89. Suppose it is required to determine the inclination of the resultant PC to the co-ordinate PN, having given the magnitude of the components a and b, and the inclinations of their directions to that co-ordinate.

Let the second of equation (A), be divided by the first, and we obtain

$$\frac{r \sin. x}{r \cos. x} = \frac{b \sin. \pi + a \sin. (\varphi+\pi)}{b \cos. \pi + a \cos. (\varphi+\pi)}$$

but $\frac{r \sin. x}{r \cos. x} = \tan. x$; consequently it is

$$\tan. x = \frac{b \tan. \pi + a \sec. \pi \sin. (\varphi+\pi)}{b + a \sec. \pi \cos. (\varphi+\pi)} \qquad \text{(c)}$$

The application will be best elucidated by the following numerical examples.

90. Example 1. Suppose that the magnitudes of two component forces a and b, are respectively represented by straight lines of 11 and 21 inches in length, and that the angles, which their directions make with one of the rectangular co-ordinates, are 50° 25′ and 32° 12′; at what angle is the resultant of those forces inclined to the same co-ordinate?

Here we have $a=11$ inches; $b=21$ inches; $\pi=32°\ 12'$ and $(\varphi+\pi)=50°\ 25'$; let these numbers be substituted in equation (c), and it becomes

$$\tan. x = \frac{21 \tan. 32°\ 12' + 11 \sec. 32°12' \sin. 50°25'}{21 + 11 \sec. 32°12' \cos. 50°25'},$$

and the operation for its reduction is as follows:

$(\varphi+\pi)=50°\ 25'$. log. sin. 9·886885
π = 32 12 . . log. sec 0·072530 . . . . log. tan. 9·799157
a = 11 inches. log. . . . 1·041393 . . b = 21 in. log. . . 1·322219
nat. num. = 10·018 log. . . 1·000808 . . . 13·224 log. . . 1·121376;

hence, we have for the value of the numerator of the fraction in equation (c),

$b \tan. \pi + a \sec. \pi \sin. (\varphi+\pi) = 23·242$
again, $\pi = 32°\ 12'$ . . log. sec. . . 0·072530
$(\varphi+\pi) = 50\ 25$ . . log. cos. . . 9·804276
$a = 11$ inches . log. . . . . 1·041393
natural number = 8·283 . log. . . . . 0·918199

consequently, the value of the denominator in equation (c) is

$$21 + a \sec. \pi \cos. (\varphi+\pi) = 29·283,$$

and the numerical value of the whole fraction is therefore

$$\tan. x = \frac{23·242}{29·283} = ·82176 = \text{nat. tan. } 35°\ 24'\ 43''.$$

91. Example 2. Suppose that at a material point of a body two forces act, equal to 60 and 80 tons respectively, and inclining to

each other in an angle of 15°; what will be the inclination of the resultant to a line which is in the plane of these forces, and inclining to the direction of the nearest force in an angle of 45°?

Here we have given, $a=60$; $b=80$, and let b be the force whose inclination to the line is given; then we have by equation (c)

$$\tan. x=\frac{b \tan \pi+a \sec. \pi \sin. (\varphi+\pi)}{b+a \sec. \pi \cos. (\varphi+\pi)}.$$

$$\text{that is } \tan. x=\frac{80 \tan. 45 + 60 \sec. 45^\circ \sin. 60^\circ.}{80+60 \sec. 45^\circ \cos. 60^\circ}$$

but $\tan. 45=1$; $\sec. 45=\sqrt{2}$; $\sin. 60^\circ=\frac{1}{2}\sqrt{3}$ and $\cos. 60^\circ=\frac{1}{2}$, hence we get

$$\tan. x=\frac{80+60\sqrt{2}\times\frac{1}{2}\sqrt{3}}{80+60\sqrt{2}\times\frac{1}{2}} \text{ that is}$$

$$\tan. x=\frac{80+30\sqrt{6}}{80+30\sqrt{2}}=1{\cdot}25325=\text{nat. } \tan. 51^\circ 25'$$

Having in the preceding process determined the inclination of the resultant to the rectangular co-ordinates, and consequently its inclination to each of the forces, or more properly to the lines by which the magnitudes of the forces are represented, we shall now proceed in the following

92. PROBLEM II. *To show in what manner the angle of inclination between the resultant and the co-ordinates is to be found, in order that its value may be incorporated with the given quantities, for the purpose of ascertaining, not the direction but the magnitude of the resultant.*

The equations for determining the magnitude of the resultant, supposing x to be known and incorporated with the other quantities, are

$$\left.\begin{array}{ll} 1. & r=\{b \cos. \pi+a \cos. (\varphi+\pi)\} \sec. x \\ 2. & r=\{b \sin. \pi+a \sin. (\varphi+\pi)\} \operatorname{cosec.} x \end{array}\right\}. \qquad (\text{D})$$

These values of the resultant r, it is evident, are derived from equation (A), by simply dividing both sides of each expression, by the respective Trigonometrical value of x occurring in that expression, and the application of those derivative formulæ, will, it is presumed, be sufficiently exemplified by the solution of the following numerical examples.

93. EXAMPLE 1. Suppose the magnitudes of two component forces to be respectively represented by straight lines of 72 and 96 inches in length, and let the angles which their directions make with one of the rectangular co-ordinates, be 88° 44′, and 56 °58′; what must be the magnitude of the single force or resultant, whose effort shall be equivalent to the united efforts of the two given forces?

Here we have given, $a=72$ inches; $b=96$ inches; $(\varphi+\pi)=$ 88° 44′, and $\pi=56^\circ 58'$;

then, the value of the angle x is found by the following operation.

$(\varphi+\pi)=88\ 44'$ log. sin. 9·999894

$\pi=56\ 58$ log. sec. 0·263502 . . . . . log. tan. 0·186930

$a=72$ inches log. 1·857332 $b=96$ inch. log. 1·982271

nat. num. = 132·074 log. 2·120728 nat. num. = 147·639 log. 2·169201

then, 132·047 + 147·639 = 279·686, the numerator;

again $(\varphi+\pi)=88^\circ\ 44'$ log. cos. 8·344504

$\pi=56\ \ 58$ log. sec. 0·263502

$a=72$ inches log. 1·857332

nat. number $=2·919$ log. 0·465338

then $96+2.919=98·919$, the denominator;

therefore, $\tan. x=\frac{279·686}{98.919}=2·82741=$ nat. tan. $70^\circ 31'20''$,

Now, if the value of x thus obtained, be employed in either form of equation (D), say No. 1, we shall have

$r=\{96\cos. 56^\circ 58'+72\cos. 88^\circ 44'\}\sec. 78^\circ 31'20''$,

which expression indicates the following process,

$\pi=56^\circ\ 58'$ log. cos. 9·736498 $(\phi+\pi)=88^\circ 44'$ log. cos. 8·344504

nat.] $b=96$ inch. log. 1·982271 $a=72$ inch. log. 1·857332

num. $=52·3321$ log. 1·718769 nat. number $=1·5916$ log. 0·201836

then $52·3321+1·5916=53·9237$ . . . log. 1.731780

$70^\circ 31'20''$ . . log. sec. 0·476981

natural number $=161·719$ . . . log. 2·208761;

consequently, the required resultant is represented in magnitude by a line of 161·719 inches in length; and its directions, with respect to the directions of the two components, are as beneath, viz.

With respect to PA, it is $(\varphi+\pi)-x=88^\circ 44'-70^\circ 31'20''=18^\circ 12'40''$

With respect to PB, it is $x-\pi\ \ =70\ 31\ 20-56\ 58=13\ 33\ 20$.

94. EXAMPLE 2. A ship sails N.E. 120 miles, in a current which sets N. 60°, E., 80 miles in the same time; what is the absolute distance sailed by the ship on account of the compound action?

Here, we have given $a=120$; $b=80$; $\pi=30^\circ$, and $(\varphi+\pi)=45^\circ$ therefore we get

$$\tan. x=\frac{80\tan 30^\circ+120\sec. 30^\circ\sin. 45^\circ}{80\ +\ 120\sec. 30^\circ\cos. 45^\circ};$$

or by substituting the numerical values of the angular quantities, we get

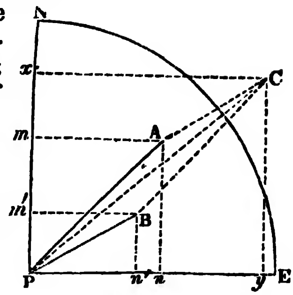
$$\tan. x=\frac{\{80+169·704\}\sqrt{3}}{240+120\sqrt{6}}=·8157=\text{cot. course}=\text{cot. } 50^\circ\ 51';$$

then to find the magnitude of the resultant, or the distance sailed by the ship, we have

$$r=\frac{80\cos. 30^\circ+120\cos. 45^\circ}{\cos. 51^\circ 51'};$$

$$\text{that is } r=\frac{154·1356}{·7864}=196 \text{ miles}.$$

The following is the method of construction.



Let NP represent the meridian of the place from which the ship sails, PE a parallel of latitude passing through that place; then are NP and PE the axes of rectangular co-ordinates originating at P.

With the chord of 60° describe the quadrant of a circle NE, and make the angle NPA equal to 45° and NPB equal to 60°; then is BPE equal to 30° and NPC is the required bearing.

Make PA equal to 120 and PB equal to 80 miles, each taken from a scale of equal parts of any convenient magnitude whatever; complete the parallelogram APBC and join PC; then is PC, the ship's way, equal to 196 miles, and NPC, the ship's course, equal to 51° 51′.

Or it may be otherwise performed, thus: resolve each of the forces PA and PB into their composants Pm, Pn, and Pm', Pn', respectively parallel to the axis of co-ordinates; make Px equal to Pm+Pm', and Py equal to Pn+Pn', complete the rectangular parallelogram xPyC and join CP; then does CP represent the magnitude and position of the resultant of the two forces PA and PB.

The above, then, is the whole of the process required to determine the magnitude and direction of the resultant of two given composant forces, in the general case, when they are unequal between themselves; but in the particular case, where the composants are equal, the operation is greatly simplified, as in the following

95. PROBLEM III. *To determine the magnitude and direction of the resultants when the composants are equal.*

If the second of equation (B), be divided by the first, we get

$$\tan. x = \tan. (\tfrac{1}{2}\varphi + \pi)$$

consequently, $x = (\frac{1}{2}\varphi + \pi)$; this is obvious from the diagram; for if PA is equal to PB, the resultant PC bisects the angle BPA $=\varphi$, and therefore, the angle CPB $=\frac{1}{2}\varphi$; but the angle BPn $=\pi$ by the notation, wherefore, CPE $= x = (\frac{1}{2}\varphi + \pi)$. If this value of x be substituted in equation (B), the two forms become assimilated, and the expression for the resultant is

$$r = 2a \cos. \tfrac{1}{2}\varphi;$$

this agrees with what we have already laid down in equation (h).

If we square both sides of the first and second forms of equation (A), we shall have

1. $r^2 \cos.^2 x = b^2 \cos.^2 \pi + 2ab \cos. \pi \cos. (\varphi + \pi) + a^2 \cos.^2 (\varphi + \pi)$,
2. $r^2 \sin.^2 x = b^2 \sin.^2 \pi + 2ab \sin. \pi \sin. (\varphi + \pi) + a^2 \sin.^2 (\varphi + \pi)$.

The sum of these is

$$r^2 (\cos.^2 x + \sin.^2 x) = b^2 (\cos.^2 \pi + \sin.^2 \pi) + 2ab \{\cos. \pi \cos. (\varphi + \pi) + \sin. \pi \sin. (\varphi + \pi)\} + a^2 \{\cos.^2 (\varphi + \pi) + \sin.^2 (\varphi + \pi)\},$$

and because $\cos.^2 + \sin.^2 = \text{rad}^2$; if we assume the radius equal to unity, we obtain

$$r^2 = a^2 + b^2 + 2ab\,\{\cos.\pi \cos.(\varphi+\pi) + \sin.\pi \sin.(\varphi+\pi)\},$$

but by the arithmetic of sines we get

$$\{\cos.\pi \cos.(\varphi+\pi) + \sin.\pi \sin.(\varphi+\pi)\} = \cos.\{(\varphi+\pi)-\pi\} = \cos.\varphi;$$

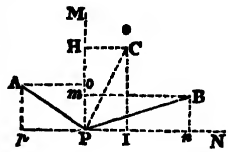
therefore, by substitution, we shall have

$$r^2 = a^2 + b^2 + 2ab\cos.\varphi,$$

and by evolving both sides of the equation, it becomes

$$r = \sqrt{a^2 + b^2 + 2ab\cos.\varphi}.$$

96. This agrees with what we have already laid down in equation (a), for the acute value of φ, the angle contained by the directions of the given forces; but when φ is obtuse, the mode of obtaining the fundamental equation will be a little different, as follows.



Let the straight lines PA and PB, represent the magnitudes and directions of the two forces, the measure of whose resultant is proposed to be ascertained.

Through P, the point of application, draw the indefinite straight line PN, making any angle whatever with PB, the direction of the nearest force b; at the point P, erect the perpendicular PM, and produce NP to p in the contrary direction; then are PN and PM the rectangular co-ordinates by which the magnitude of the resultant is to be assigned.

Resolve the forces PA and PB, considered as resultants, into the composants Po, Pp, and Pm, Pn estimated on the rectangular co-ordinates Pm and Pn. Make mH equal to Po and nI equal to Pp taken negatively, because Pp falls in the contrary direction. On the right lines PI and PH, construct the rectangular parallelogram PHCI, and draw PC; then is PC the resultant of the two forces a and b, whose magnitudes are represented by the straight lines PA and PB.

Since the forces PA and PB, when reduced to the co-ordinate PN, are represented by Pn and Pp, acting in contrary directions with respect to P the point of application, it is obvious, that the effect of these forces in the direction of the greater, is equal to PI the difference of the lines Pn and Pp which represent the reduced magnitudes. Again, the straight lines Po and Pm represent the magnitudes of the forces PA and PB, when reduced to the co-ordinate PM, and because they act in the same straight line, and in the same direction, their efforts must be equivalent to that of a single force, whose magnitude is expressed by their sum; but PH is the sum of Po, and Pm and PI is the difference of Pn and Pp; consequently PC is the resultant of the forces PA and PB.

97. Retaining the former notation, we have by Plane Trigonometry,

Pn $= b\cos.\pi$, and P$m = b\sin.\pi$

Pp $= -a\cos.\{180^\circ - (\varphi+\pi)\}$, and P$o = a\sin.\{180^\circ - (\varphi+\pi)\}$;

therefore, PI $= b\cos.\pi - a\cos.\{180^\circ-(\varphi+\pi)\}$, and PH $= b\sin.\pi + a\sin.\{180^\circ-(\varphi+\pi)\}$; but because the tabular or numerical

values of the sin. and cosine of an arc and its supplement are the same, it follows, that cos. $\{180^\circ-(\varphi+\pi)\}$, and sin. $\{180^\circ-(\varphi+\pi)\}$, are respectively equal to cos. $(\varphi+\pi)$, and sin. $(\varphi+\pi)$; and since $(\varphi+\pi)$ is by hypothesis greater than a right angle, cos. $(\varphi+\pi)$ must be taken negatively, but a negative quantity subtracted is equivalent to a positive quantity added; therefore we have

$$1.\ r \cos. x=b \cos. \pi+a \cos. (\varphi+\pi)$$
$$2.\ r \sin. x=b \sin. \pi+a \sin. (\varphi+\pi)$$

The very same results as in equation (A); from which we conclude, that whether the directions of the components are inclined to each other in an angle greater or less than a right angle, the method of computing the magnitude of the resultant will nevertheless be the same, although the construction exhibits a slight variation.

98. If we multiply the first of the preceding equations by b sin. π, and the second by b cos. π, we obtain

$$1.\ br \sin. \pi \cos. x=b^2 \sin. \pi \cos. \pi+ab \sin. \pi \cos. (\varphi+\pi),$$
$$2.\ br \cos. \pi \sin. x=b^2 \sin. \pi \cos. \pi+ab \cos. \pi \sin. (\varphi+\pi);$$

let the first of these expressions be subtracted from the second, and we get

$$br(\sin. x \cos. \pi-\cos. x \sin. \pi)=ab(\sin. (\varphi+\pi) \cos. \pi-\cos. (\varphi+\pi) \sin. \pi);$$

but by the arithmetic of sines, we know that

$$\sin. x \cos. \pi-\cos. x \sin. \varphi=\sin. (x-\pi), \text{ and}$$
$$\sin. (\varphi+\pi) \cos. \pi-\cos. (\varphi+\pi) \sin. \pi=\sin. \{(\varphi+\pi)-\pi\}=\sin. \varphi;$$

consequently we have

$$br \sin. (x-\pi)=ab \sin. \varphi \quad \ldots\ldots\ldots \quad (\text{E})$$

This is a very curious deduction, and at the same time exceedingly well adapted for the determination of the resultant r; it indicates that the product of the lines PB and PC, drawn into the natural sine of their contained angle, is equal to the product of the lines PA and PB, drawn into the natural sine of their contained angle. Divide both sides of equation (E) by b sin. $(\varphi-\pi)$, and we get

$$r=a \sin. \varphi \operatorname{cosec}. (x-\pi) \quad \ldots\ldots\ldots \quad (\text{F})$$

The practical rule, afforded by this expression, is as follows:

99. RULE. *Find the direction of the resultant* PC *with respect to that of the force* PB, *by equation* (C), *or equation (k), in the parallelogram of forces; then multiply together the magnitude of the force* PA, *the natural sine of the angle contained between the directions of the forces, and the natural cosecant of the angle which the resultant makes with the directions of the force* PB, *and the product will be the magnitude of the resultant* PC.

If the direction of the resultant were determined relatively to that of the force PA; the formula would be symmetrical, and the rule would be the same, having the force PA instead of PB. The following examples will guide the reader to the practice of solution.

100. EXAMPLE 1. Two forces, whose magnitudes are respectively represented by straight lines of 18 and 28 inches, and whose direc-

tions are inclined to each other in an angle of 48° 12′ are required to be composed into a single force of equal effort; what must be the magnitude of that force, supposing the inclination of its components to one of the rectangular co-ordinates to be 22° 30′ and 70° 42′?

First, calculate $\cot.(x-\pi)=\frac{b}{a}\operatorname{cosec}.\varphi+\cot.\varphi$

	$b=28$ inches	log.	1·447158
	$a=18$ inches ar. co.	log.	8·744727
	$\varphi=48°\ 12'$.	log. cosec.	0·127566
natural number	2·08661	log.	0·319441
$\varphi=48°\ 12'$ nat. cotang.	0·89410		
sum	2·98071 = nat. cotangent of 18° 33′		

then by the rule

$a=18$ inches	log.	1·255273
$\varphi=48°\ 12'$	log. sin.	9·872434
$(x-\pi)=18\ 33$	log. cosec.	0·497393
natural number $=42\cdot18$ inches	log.	1·625100

consequently, the resultant of two forces according to the conditions proposed is measured by a line of 42·18 inches in length.

101. If we divide both sides of equation (E) by br, we get

$$\sin.(x-\pi)=\frac{a\sin.\varphi}{r};$$

but by the arithmetic of sines, $\sin.(x-\pi)=\sin.x\cos.\pi-\cos.x\sin.\pi$; therefore, by substitution,

$$\sin.x\cos.\pi-\cos.x\sin.\pi=\frac{a\sin.\varphi}{r},$$

and by transposition and division,

$$\cos.\pi\tan.x-\frac{a\sin.\varphi\sec.x}{r}=\sin.\pi:$$

but $\sec.x=\sqrt{1+\tan.^2x}$

hence we have

$$\cos.\pi\tan.x-\frac{a\sin.\varphi}{r}\sqrt{1+\tan.^2x}=\sin.\pi:$$ transpose

$\frac{a\sin.\varphi}{r}\sqrt{1+\tan.^2x}$ and $\sin.\pi$, and involve both sides, and we have

$$\cos.^2\pi\tan.^2x-\sin.2\pi\tan.x+\sin.^2\pi=\frac{a^2\sin.^2\varphi}{r^2}+\left(\frac{a^2\sin.^2\varphi}{r^2}\right)\tan.^2x;$$

transpose, and it is

$$\left(\cos.^2\pi-\frac{a^2\cos.^2\varphi}{r^2}\right)\tan.^2x-\sin.2\pi\tan.x=\frac{a^2\sin.^2\varphi}{r^2}-\sin.^2\pi,$$

an adfected quadratic equation, which, being resolved by the rules of algebra, will determine the direction of the resultant with respect to the direction of each of the forces.

102. Example 2. Two forces, whose magnitudes are respectively represented by weights of 30 and 40 tons, are inclined to each other in an angle of 60°, and they exert themselves at the same instant on

a material point of a body: what must be the magnitude of their resultant, supposing the composants to be inclined to one of the co-ordinates in angles of 15 and 75 degrees?

Here we have given

$$a=30\,;\quad b=40\,;\quad \varphi=60^\circ\,;\quad \pi=15,^\circ \text{ and } (\varphi+\pi)=75.^\circ$$

Find $(x-\pi)$ such that $\cot.\,(x-\pi)=\frac{b}{a}\operatorname{cosec.}\varphi+\cot.\,\varphi.$ see equat. (k)

$b=40$	log.	$\cdot 602060$
$a=30$	ar.co. log.	$8\cdot 522879$
$\varphi=60^\circ$	log. cosec.	$0\cdot 062469$
natural number $1\cdot 53959$	log.	$0\cdot 187408$
$\varphi=60^\circ$ nat. cotan. $0\cdot 57735$		

sum $=2\cdot 11694=$ nat. cot. $25^\circ\ 17.'$ Hence we get

$$r=30\times\sin.\,60^\circ\times\operatorname{cosec.}\,25^\circ\ 17'=60\cdot 83 \text{ tons.}$$

We have already been so diffuse in the illustration of our formulæ, that it is presumed the reader is, ere now, prepared to reduce the foregoing equation, and also to extend the principle of composition to three or more forces, according to the assigned conditions; we therefore quit this part of the subject, and proceed to consider the relations that subsist between the component and resultant forces under our second head; in which, it will appear, that in the composition of more than two forces, we are not limited to one plane.

CASE II.

103. The next branch of the subject that presents itself to our consideration, according to the order of arrangement, is that in which

the component and resultant forces are disposed in different planes, but directed to the same point of a body.

Section First.

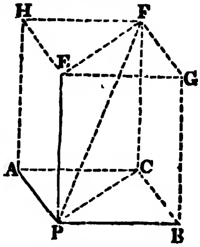
WHEN THE DIFFERENT PLANES IN WHICH THE COMPONENT AND RESULTANT FORCES ARE SITUATED, MAY BE AT RIGHT ANGLES ONE TO ANOTHER.

104. The general principle unfolding the solution of this case, is simply as follows, viz.:—

Proposition. *If three forces are represented in magnitude and direction by the three edges adjacent to the same angle of a parallelopiped, their resultant, or single equivalent, will be represented in magnitude and direction, by the diagonal which is drawn from that angle of the solid where the forces are applied.*

Indeed, any number of forces acting together on one particle of matter must be balanced by a force which is equal and opposite to their resulting force; for this force would balance their resulting force, which is equivalent to them in action.

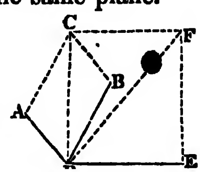
When this is duly considered, we perceive that each force is then in equilibrio with the equivalent of all the others; for a force can balance only what is equal and opposite to it.



105. Upon the straight lines PA, PB, and PE, which represent the magnitudes and directions of the three given composant forces a, b, and c, construct the solid parallelopipedon ABGH; draw PC, the diagonal of the parallelogram APBC; then shall PC be the resultant of the two forces PA and PB. Join EF and PF; then because by construction, CF is equal and parallel to AH, and AH to PE, CF is also equal and parallel to PE; but straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are themselves equal and parallel; wherefore EF is equal and parallel to PC, and the figure PEFC is a parallelogram, of which the diagonal PF represents in magnitude and direction the resultant of the two forces PC and PE; but PC is the resultant of the two forces PA and PB; therefore, PF is the resultant of the three forces PA, PB, and PE; and since P and F are the opposite corners of the parallelopipedon ABGH, the straight line PF which joins those opposite corners must be its diagonal; therefore PF is its diagonal, and it has been shewn that PF is the resultant of the three forces PA, PB, and PE; wherefore,

The resultant of three forces, situated in different planes, but concurring in the same point, is represented in magnitude and direction by the diagonal of the parallelopipedon constructed on the lines that represent the forces.

106. From which principle, we deduce the following method of construction for the resultant of three proposed composants, acting simultaneously at one point, but not disposed in the same plane.



Let the straight lines PA and PB, represent the magnitudes and directions of two component forces, situated in the same plane, and applied to the same point P:—and let the straight line PE represent the magnitude and direction of a third force, applied to the same point P, but situated in a plane anyhow inclined to that in which the other two forces exist.

Upon the straight lines PA and PB, construct the parallelogram PACB and join PC; then is PC the resultant of the two forces whose magnitudes and directions are represented by the straight lines PA and PB. Through the point C, draw CF parallel and equal to PE

and join PF; then is PF the resultant of the three proposed forces PA, PB, and PE, for it is obviously the diagonal of a parallelpipedon constructed on these lines:—

Put a=PA, } the two forces, disposed in the same plane PACB,
b=PB, }
c=PE, the third force not in the same plane with PA and PB,
φ=BPA, the angle which PA and PB, the directions of the forces a and b, make with each other,
r=PC, the resultant of the two forces PA and PB,
and R=PF, the resultant of the three forces, PA, PB, and PE.

Then, if the force PE be supposed to act at right angles to each of the two forces PA and PB, it is manifest that it acts at right angles to the resultant PC; but CF is by construction equal and parallel to PE; consequently, PCF is a right angle, and $\text{PF}^2=\text{PC}^2+\text{CF}^2$; now PC^2 is, by squaring equation (a), equal to $a^2+b^2\pm 2ab\cos.\varphi$; that is,

$$\text{PC}^2=r^2=a^2+b^2\pm 2ab\cos.\varphi,$$

to which, if $\text{CF}^2=\text{PE}^2=c^2$ be added, it becomes

$$\text{PF}^2=\text{R}^2=a^2+b^2+c^2\pm 2ab\cos.\varphi,$$

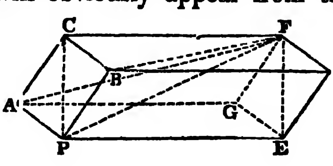
and by extracting the square root of both sides of this equation, we obtain for the value of the resultant

$$\text{R}=\sqrt{a^2+b^2+c^2\pm 2ab\cos.\varphi}. \qquad (\text{G})$$

107. This equation determines the magnitude of the resultant of three forces, when two of them, as a and b, are situated in one plane, and have their directions inclined to each other in an angle equal to φ, either less or greater than a right angle, while the third force c is perpendicular to the plane in which a and b exist, and consequently, has its direction inclined to each of them in an angle of 90 degrees. But if the directions of the three forces a, b, and c, are mutually inclined to each other in an angle of 90 degrees, then the solid in whose planes the forces exist, becomes rectangular, and the value of the resultant is

$$\text{R}=\sqrt{a^2+b^2+c^2} \qquad (\text{H})$$

This is manifest from equation (G), for when $\varphi=90^\circ$, $\cos.\varphi=0$, and the term $2ab\cos.\varphi$ vanishes, leaving for the magnitude of the resultant, the square root of the sum of the squares of the three given forces; but the same thing will obviously appear from the annexed diagram, where the three contiguous edges of the solid PA, PB, and PE represent the magnitudes or intensities* of the proposed componsants, being mutually inclined to each other in an angle of 90 degrees;



* *Intensity of a force* is the effort which it exerts in endeavouring to produce or to destroy motion.

that is, the angles APE, APB, *and* BPE *are all of them right angles*, and the solid PACBHFGE is rectangular.

108. Now, because PC is the diagonal of the rectangular parallelogram PACB, it is obviously the resultant of the two forces PA and PB, and for the same reason, PF is the resultant of the two forces PC, PE; consequently, the effort or intensity of PF is equivalent to the united efforts or intensities of the three composants PA, PB, and PE. But because the angle PCF is a right angle, $PF^2 = PC^2 + CF^2$; or since $PC^2 = PA^2 + PB^2$, and $CF^2 = PE^2$, we have, by substit. $PF^2 = PA^2 + PB^2 + PE^2$; that is, $R^2 = a^2 + b^2 + c^2$, and by evolution,

$$R = \sqrt{a^2 + b^2 + c^2}, \text{ the same as above.}$$

109. If the three forces a, b, and c are equal, the solid in whose planes they are supposed to act becomes a cube, and its diagonal, or the resultant of the given forces, is

$$R = a\sqrt{3} \qquad (I)$$

The following practical rules afforded by the equations (G), (H), and (I), will be found useful for their reduction, and two numerical examples to each will suffice for illustration.

The rule derived from equation (G) is as follows.

110. Rule. *To or from the sum of the squares of the three component forces, according as φ, the given angle of inclination, is acute or obtuse, add or subtract twice the natural cosine of the given angle drawn into the product of its containing forces; then the square root of the sum, or remainder, will give the magnitude of the resultant sought.*

When the φ is acute.

111. Example 1. The intensities of three forces, a, b and c, are measured respectively by weights of 13, 18 and 23 tons; what is the intensity of their single equivalent, supposing the directions of a and b to be inclined to each other in an angle of 62° 45′, and that of c inclined to each of the others in an angle of 90 degrees?

Here by the rule we have

$$\begin{aligned} a^2 &= 13 \times 13 = 169 \\ b^2 &= 18 \times 18 = 324 \\ c^2 &= 23 \times 23 = 529 \\ a^2 + b^2 + c^2 &= 1022 \end{aligned}$$

$2ab \cos.\varphi = 2 \times 13 \times 18 \times \cdot 45787 = 214 \cdot 28316$, add because φ is acute.

hence we have $R = \sqrt{a^2 + b^2 + c^2 + 2ab \cos.\varphi} = \sqrt{1236 \cdot 28316} = 35 \cdot 16$ tons, for the magnitude or intensity of the resultant required.

The magnitudes of the forces remaining, if φ the angle of inclination between the directions of a and b be 117° 15′; then, $R = \sqrt{1022 - 214 \cdot 28316} = 28 \cdot 42$ tons.

112. Example 2. The magnitudes or intensities of three forces, a, b, and c, which act simultaneously at the same point of a body, are respectively represented by weights of 24, 32, and 30 tons; what is the magnitude or intensity of their resultant, or single equivalent, supposing the forces a and b to be inclined to each

other in an angle of 165°, and the force c inclining to each of the others in an angle of 90°?

Here we have given, $a=24$; $b=32$; $c=30$, and $\varphi=165°$; its cosine is $-\cdot96593$; consequently, by equation (G) we obtain

$$\begin{aligned} a^2 &= 24\times 24 = 576, \\ b^2 &= 32\times 32 = 1024, \\ c^2 &= 30\times 30 = 900, \\ a^2+b^2+c^2 &= \overline{2500} \text{ and} \end{aligned}$$

$-2\,ab\cos.\varphi = 2\times24\times32\times-\cdot96593 = -1483\cdot66848$, subtract, because φ is obtuse:

therefore, $R=\sqrt{2500-1483\cdot66848}=\sqrt{1016\cdot33152}=31\cdot88$ tons.

The rule derived from equation (H) is as follows:

RULE. *Extract the square root of the sum of the squares of the given forces for the resultant required.*

113. EXAMPLE 1. Let the intensities of the given forces be the same as in the preceding example, and suppose their directions to be mutually inclined in an angle of 90 degrees; what then will be the magnitude of the resultant?

Here by the rule, we have $a^2 = 13\times13 = 169$

$$\begin{aligned} b^2 &= 18\times18 = 324 \\ c^2 &= 23\times23 = 529 \end{aligned}$$

hence we have, $R=\sqrt{a^2+b^2+c^2}=\sqrt{1022}=31\cdot96$ tons, the magnitude or intensity of the required resultant.

114. EXAMPLE 2. Suppose that three forces, the intensities of which are respectively equal to weights of 48, 64, and 60 tons, act simultaneously at one point of a body, and that each force is perpendicular to the plane of the other two; what is the magnitude or intensity of their resultant?

Here we have given $a=48$; $b=64$; $c=60$, and $\varphi=90°$: its cosine $=0$;

therefore by equation (H), we get

$$\begin{aligned} a^2 &= 48\times48 = 2304, \\ b^2 &= 64\times64 = 4096, \\ c^2 &= 60\times60 = 3600, \\ a^2+b^2+c^2 &= \overline{10000}; \end{aligned}$$

therefore, $R=\sqrt{10000}=100$ tons.

The rule derived from equation (I) is as follows.

115. RULE. *Multiply the square root of 3 or 1·732 by either of the equal forces, and the product will give the required resultant.*

116. EXAMPLE 1. Suppose the intensities of three equal forces to be each represented by a line of 24 inches in length; what must be the length of the line that represents the intensity of the resultant, the directions of the components being mutually inclined in an angle of 90 degrees?

Here by the rule we have $1 \cdot 732 \times 24 = 41 \cdot 568$ inches, the required measure of the intensity or magnitude.

117. EXAMPLE 2. The magnitudes or intensities of three equal forces, which act simultaneously at the same point of a body, are each equal to 60 horses' power; what is the magnitude of their resultant or single equivalent, supposing that the direction of each force is at right angles to the plane of the other two?

Here we have given $a = 60$, and $\varphi = 90°$; its cosine is $= 0$; therefore, by equation (I), we get

$$R = 60\sqrt{3} = 103 \cdot 923 \text{ horses' power.}$$

118. If the forces in equation (I) are equal, but φ, the given inclination of the directions of a and b, either less or greater than a right angle, then we have

$$R = a\sqrt{3 \pm 2 \cos. \varphi}. \qquad (K)$$

The practical rule which this expression affords is as follows.

119. RULE. *Increase or diminish the constant number 3, by twice the natural cosine of the given inclination, according as it is less or greater than 90°; then multiply the square root of the sum or remainder by either of the equal forces, and the product will be the resultant sought.*

120. EXAMPLE 1. Let the equal forces be the same as in the preceding example, and suppose the inclinations of the directions of a and b to be 76° 40′, and 124° 12′; what is the magnitude or intensity of the resultant in either case, the inclination of the direction of the third force to each of the others being 90 degrees?

Here $2 \cos. 76° 40' = \cdot 46124$; and $2 \cos. 124° 12' = -1 \cdot 12416$; then by the rule we have, when φ is acute,

$R = 24\sqrt{3 \cdot 46124} = 44 \cdot 6496$ inches, the measure of the resultant; and when φ is obtuse, we have

$R = 24\sqrt{3 - 1 \cdot 12416} = 32 \cdot 856$ inches, the measure of the resultant.

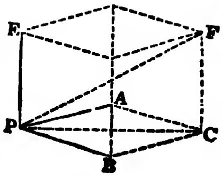
121. EXAMPLE 2. Suppose that three forces, each equal to 100 horses' power, act simultaneously on a material point of a body, two of the forces being inclined to each other in an angle of 30°, and the third inclined to the plane of the other two in an angle of 90°; what is the magnitude or intensity of the resultant?

Here we have given $a = 100$, and $\varphi = 30°$, and $\varphi' = 90°$; cosine $\varphi = \frac{1}{2}\sqrt{3}$, and $\cos. \varphi' = o$,

therefore, by equation (K), we get

$$100\sqrt{3 + \sqrt{3}} = 217 \cdot 53 \text{ horses' power.}$$

122. Geometrical construction to exhibit the conditions of action, while, at the same time, the magnitude of the resultant shall be truly represented by the diagonal of the figure.



Let the straight lines PA, PB, and PE, each equal to 100, represent the magnitudes of the forces, and let the angle APB be equal to 30 degrees, as specified in the question.

Complete the rhombus APBC, and draw the diagonal PC; then will PC be the resultant of the two forces PA and PB, whose actions are confined to the plane PAB. Draw PE at right angles to PC, and complete the solid PFB; join PF, then is the diagonal PF the resultant of the three given equal forces PA, PB, and PE.

This diagram affords the following equation for the magnitude of the resultant PF; that is,

$$\text{PF}=\text{R}=a\sqrt{4\cos^2.\tfrac{1}{2}\varphi+1}.$$

The truth of this equation is manifest, for $\text{PF}^2=\text{PC}^2+\text{CF}^2$, (Eucl. 47. 1.) and PC is, by equation (h), equal to $2a\cos.\frac{1}{2}\varphi$; consequently $\text{PF}^2=4a^2\cos^2.\frac{1}{2}\varphi+a^2$, which by evolution produces the preceding equation.

The preceding example wrought by this method is as follows, viz.

$$\text{R}=100\sqrt{4\cos^2.15^\circ+1};$$

but $\cos.15^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$; its square is $(\frac{1}{2}+\frac{1}{4}\sqrt{3})$; hence we get

$$\text{R}=100\sqrt{3+\sqrt{3}}=217\cdot53,\text{ the same as before.}$$

123. Recurring to the diagram under equation (H), art. 107, where it is shewn that $\text{PF}^2=\text{PA}^2+\text{PB}^2+\text{PE}^2$; if we denote the angle FPA by A, FPB by B, and FPE by C; that is, the angles which the resultant PF makes with the given primary forces, PA, PB, and PE; then, because the angles PAF, PBF, and PEF, are right angles, we shall have by Plane Trigonometry

$$\text{PA}=\text{PF}\cos.\text{A};\ \text{PB}=\text{PF}\cos.\text{B};\ \text{and}\ \text{PE}=\text{PF}\cos.\text{C};$$

wherefore, by squaring and substitution, we obtain

$$\text{PF}^2\{\cos^2.\text{A}+\cos^2.\text{B}+\cos^2.\text{C}\}=\text{PF}^2;$$

or, dividing both sides of this equation by the common factor PF^2, it becomes

$$\cos^2.\text{A}+\cos^2.\text{B}+\cos^2.\text{C}=1. \qquad (\text{L})$$

This is the equation employed by mathematicians for determining the direction of a force in space, and it enables us to draw accurate conclusions from very intricate observations. In the present case it is evident, that if we know the angles A, B, and C, which the direction of the resultant PF makes with each of the lines PA, PB, and PE, the position in space of the resultant can thence be ascertained; for the relation between the angles is such, that whatever may be the value of any two of them, the value of the third depends on the foregoing equation. Thus, for

124. EXAMPLE. Suppose that the angles A and B, which two contiguous sides of a rectangular parallelopipedon, make with the

diagonal drawn from the concurrent angle, are respectively 68° 18′ and 59° 12′; what angle does the third side make with that diagonal?

Here we have given, the angles A and B, and it is proposed to determine the angle C; now, from equation (L), by transposition, we obtain

$$\cos^2. C = 1 - \cos^2. A - \cos^2. B;$$

consequently, by evolution, we have

$$\cos. C = \sqrt{1 - \cos^2. A - \cos^2. B};$$

but $\cos^2. 68° 18' = \cdot 1367$, and $\cos^2. 59° 12' = \cdot 26209$; therefore, $\cos. C = \sqrt{1 - \cdot 39879} = \cdot 77538$; hence, the angle $C = 39° 10'$.

125. By knowing the angles A, B, C, and the resultant PF, it is easy to determine the values of PA, PB, and PE, the primary forces which compose the resultant PF; for, as we have shewn above, we have only to multiply the natural cosines of the given angles respectively by the resultant, and the several products will give the required forces.

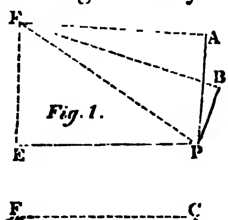
126. EXAMPLE 1. Suppose the diagonal of a rectangular parallelopipedon to be 36 inches in length, and let the angles which it makes with the three exterior edges of the solid contiguous to the angle where it originates, be respectively 60°, 50°, and 54° 31′; what is the length of each edge of the solid, or, in other words, what are the magnitudes or intensities of the composant forces of which the given diagonal is the resultant?

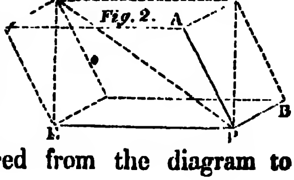
Here we have

$$PA = 36 \cos. 60°00' = 36 \times \cdot 5 \quad = 18 \cdot \ldots \text{ inches},$$
$$PB = 36 \cos. 50\ 00 = 36 \times \cdot 64279 = 23 \cdot 14044 \text{ inches},$$
$$PE = 36 \cos. 54\ 31 = 36 \times \cdot 58038 = 20 \cdot 89368 \text{ inches}.$$

127. EXAMPLE 2. The resultant of three forces acting simultaneously at the same point of a body, is equal to a pressure of 120 tons, and the inclinations of its direction to each of the composants are respectively 60°, 75°, and 34° 15′; what is the magnitude of each composant, their inclinations to each other being mutually 90°?

Draw the straight line PE (fig. 1) of any convenient length, and at the point P make the angles FPE, FPA, and FPB equal respectively to the given angles 34° 15′, 60° and 75°. Make PF equal to 120, the measure of the given resultant, and from the point F let fall the perpendiculars FE, FA, and FB, meeting the straight lines PE, PA, and PB, in the points E, A, and B; then are PA, PB, and PE the magnitudes of the required composants. Fig. 2 represents the solid constructed on those lines under the specified conditions, and the calculation is effected by the property deduced from the diagram to equation (L), viz.





That the edges of a rectangular parallelopipedon are respectively equal to the product of the diagonal by the natural cosine of its inclination to the edges.

Therefore, we have

$$\begin{aligned} PA &= PF \cos. 60^\circ\ 0' = 120 \times \cdot 5 \text{———} = 60 \text{ tons,} \\ PB &= PF \cos. 75\ \ 0 = 120 \times \cdot 25882 = 31 \cdot 058 \text{ —} \\ PE &= PF \cos. 34\ 16 = 120 \times \cdot 82643 = 99 \cdot 171 \text{ —} \end{aligned}$$

Such is the method of procedure when the different planes in which the component and resultant forces may be situated, are at right angles to one another.

In the second member of the general proposition we asserted, and have since proved, that "any number of forces acting together on one particle of matter must be balanced by a force which is equal and opposite to their resulting force;" and "in most of our practical discussions we know, or at least attend to, a part only of the forces which are acting on a material particle; and in such cases we reason as if we saw the whole: yet is our mathematical reasoning good with respect to the equivalent of all the parcels which we are contemplating, and the equivalents of the smaller parcels of which it consists; and the neglected force, or parcel of forces, induces no error on our conclusions."

"In the spontaneous phenomena of nature, the investigation of our ultimate object of search is frequently very difficult, on account of the multiplicity of directions and intensities of the operating forces. We may generally facilitate this process, by substituting equivalent motions or forces, acting in convenient directions. It is in this way that the navigator computes the ship's place with very little trouble, by substituting equivalent motions in the meridional and equatorial directions for the real oblique courses of the ship. Instead of setting down ten miles on a course S. 36° 52′ W., he supposes that the ship has sailed eight miles due south, and six miles due west, which brings her near to the same place. Then, instead of fourteen miles south-west, he sets down ten miles south and ten miles west; and he proceeds in the same way for every other course and distance. Having done this for the course of a whole day, he adds all the southings into one sum, and all the westings into another; he considers them as forming the sides of a right-angled triangle; he looks for them, paired together, in his traverse table, and then notices what angle and what distance corresponds to this pair. This gives him the position and magnitude of the straight line joining the beginning and end of his day's work."

"The miner proceeds in the same way when he takes the plan of subterraneous workings; measuring, as he goes along, and noticing the bearing of each line by the compass, and setting down from his traverse table the northing or southing, and the easting or westing,

for each oblique line. But there is another circumstance which he must attend to, namely, the slope of the various drifts, galleries, and other workings. This he does by noting the rise or dip of each sloping line. He adds all these into one sum, and, taking the risings from the dips, he obtains the whole dip. Thus he learns how far the workings proceed to the north, how far to the east, and how far to the dip."

"The reflecting reader will perceive, that the line joining the two extremities of this progression, will form the diagonal of a regular parallelopiped; one of whose sides lies north and south, the other lies east and west, and the third is right up and down."

"The mechanician proceeds in the very same way in the investigation of the very complicated phenomena which frequently engage his attention. He considers every motion as compounded of three motions in some convenient directions, at right angles to each other. He also considers every force as resulting from the joint action of three forces, at right angles to each other, and takes the sum or difference of these in the same or opposite directions. From this process he obtains the three sides of a parallelopiped, and from these computes the position and magnitude of the diagonal. This is the motion or force resulting from the composition of all the partial ones."

"This procedure is called the Estimation or Reduction of forces; and the most usual and useful method of reduction, is to estimate all forces in the directions of three lines drawn from one point, at right angles to each other, like the three plane angles of a rectangular chest. These are commonly called the three co-ordinates. The resulting force will be the diagonal of this parallelopiped. This process occurs in all disquisitions in which the mutual action of solids and fluids is considered, and where the oscillation or rotation of detached free bodies is the subject of discussion." *

Section Second.

WHEN THE THREE FORCES ACTING AT THE SAME POINT, ARE INCLINED TO EACH OTHER IN ANY ANGLE WHATEVER.

What we have hitherto considered, is applicable to that state of the parallelopipedon where one of the edges at least is perpendicular to the plane of the other two, and where the three edges are supposed to be mutually perpendicular to one another; we shall therefore extend our inquiries to the development of the principles, when the parallelopipedon is wholly oblique, or when its edges, which represent the direction of the forces, are inclined to one

* Dr. Robison's Mechanical Philosophy, Vol. i. pp. 72--74.

another in any angles whatever; that is to say, when the three forces *in different planes act upon a material point, but in oblique directions,* as in the obtuse or acute rhomboid.

128. PROBLEM *To determine the magnitude of the resultant when the three forces in different planes act upon a material point, but in oblique directions, or when the parallelopipedon is wholly oblique.*

For which purpose, let the inclination of the direction of the force a to that of the force b, be represented by the letter φ; the inclination of the direction of the force c to the resultant of the forces a and b by φ'; and in the first place, suppose that the direction of the force c is inclined equally to the directions of the forces a and b.

Then, according to equation (a), the resultant of the two forces a and b is expressed generally, as follows:

$$r=\sqrt{a^2+b^2\pm 2\,a\,b\cos.\varphi};$$

consequently, the expression for the resultant of the three forces a, b, and c, is

$$\text{R}=\sqrt{a^2+b^2+c^2\pm 2\,a\,b\cos.\varphi\pm 2\,c\cos.\varphi'\sqrt{a^2+b^2\pm 2\,a\,b\cos.\varphi}}.\quad(1)$$

129. This equation is identical with that marked (l), which we obtained for three forces situated in the same plane, and directed to the same point; but in the present instance it is not general, being applicable only when the inclination of the third force c to the resultant of a and b is given; now, it is easy to perceive that this is a datum which must be computed from the respective inclinations and magnitudes of the forces themselves, and the method of determination is as follows:

Find the angle x by equation (k), such, that

$$\cot.x=\frac{a}{b}\operatorname{cosec}.\varphi\pm\cot.\varphi;$$

and by Spherical Trigonometry find an angle ϖ, such, that

$$\cos.\varpi=\cos.\varphi''\operatorname{cosec}.\varphi'\operatorname{cosec}.\varphi-\cot.\varphi'\cot.\varphi;$$

where we have to observe, that φ, φ', and φ'', are the respective inclinations which the directions of the forces a, b, and c, make with one another; that is,

φ is the angle which the direction of the force a makes with b,
φ' is the angle which the direction of the force a makes with c,
and φ'' is the angle which the direction of the force b makes with a,

And these three inclinations are here considered as constituting the sides of a spherical triangle, the vertical angle of which is ϖ just determined.

Again, by Spherical Trigonometry, find an angle θ, such that

$$\cos.\theta=\sin.\varphi' x \cos.\varpi+\cos.\varphi' \cos.x.^{*}$$

If instead of cos. ϖ in this equation, we substitute its value in *terms of φ, φ', and φ'' from the equation immediately preceding*, we shall obtain

$$\cos.\theta=\sin.x\,(\cos.\varphi'' \operatorname{cosec}.\varphi-\cos.\varphi' \cot.\varphi)+\cos.\varphi' \cos.x.$$

If instead of cosec. φ, and cot. φ, we substitute

$\dfrac{1}{\sin.\varphi}$, and $\dfrac{\cos.\varphi}{\sin.\varphi}$; we further get

$$\cos.\theta=\sin.x.\left(\frac{\cos.\varphi''}{\sin.\varphi}-\frac{\cos.\varphi' \cos.\varphi}{\sin.\varphi}\right)+\cos.\varphi' \cos.x;$$

and moreover, if we destroy the parenthesis, and reduce the terms to a common denominator, we shall obtain

$$\cos.\theta=\frac{\sin.x.\cos.\varphi''-\sin.x.\cos.\varphi' \cos.\varphi+\cos.\varphi' \cos.x.\sin.\varphi,}{\sin.\varphi}$$

or, which is the same thing,

$$\cos.\theta=\frac{\sin.x.\cos.\varphi''+(\sin.\varphi \cos.x-\cos.\varphi \sin.x)\cos.\varphi';}{\sin.\varphi}$$

but we have shown (in note c), that

$$\sin.\varphi \cos.x-\cos.\varphi \sin.x=\sin.(\varphi-x);$$

consequently, by substitution, we have

$$\cos.\theta=\frac{\sin.x \cos.\varphi''+\sin,(\varphi-x)\cos.\varphi'.}{\sin.\varphi}$$

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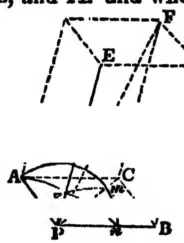
* *Note* F. Let ABFE be the solid parallelopipedon, involving the conditions that regulate the actions of the three forces a, b, and c, whose magnitudes are represented by the edges of the solid PA, PB, and PE and whose inclinations are respectively APB$=\phi$; APE$=\phi'$ and BPE$=\phi''$. Draw the diagonal PC, of the oblique parallelogram APBC, and PO shall be the resultant of the two forces a and b, whose magnitudes are represented by the two edges of the solid PA and PB. Let θ=FPC, the angle which the force c, whose magnitude is represented by PE, makes with the diagonal PC, the resultant of the two forces a and b; then find the value of θ.



Suppose the point P, or angle of the parallelopipedon at which the forces act, to be the centre of a sphere, of which the radii are PA, Pz, Pm, and Pn; and let the arcs Amn, Az, zn, and zm be drawn; then shall zAn and zAm be two spherical triangles, of which the sides An, Az, and zn, are respectively the measures of ϕ, ϕ', and ϕ'', the given inclinations; and the sides Am and zm are the measures of the angles x and θ to be determined by calculation.

Now, we have shown in equation (k), that

$$\cot.x=\frac{a}{b}\operatorname{cosec}.\phi\pm\cot.\quad;$$

consequently, there remains to be determined the arc zm, which, as we have shown above, is the measure of θ, the angle for which the calculation is solely made. And to find the value of θ, we must observe, that in the spherical triangle zAn, there are given the three sides zA, An, and zn, to find the angle zAn opposite the side zn; then, in the spherical triangle zAm, there are given the two sides zA, mA, of which mA is found by calculation, and the contained angle mAz; to find the side zm opposite the angle mAz, which is also found by calculation; then is the side zm the measure of θ the required angle.

We have now very nearly obtained the object of our inquiry, for we have only to find expressions for sin. x, and sin. $(\varphi-x)$, in terms of the angle φ, and its containing lines PA and PB, (see the figure to the note p. 73), which expressions being substituted in the above equation, will determine the angle which the direction of the force $\text{PE}=c$, makes with $\text{PC}=r$, the resultant of the two forces a and b.

But by Plane Trigonometry, we have

$$r : \sin.\varphi :: b : \sin.x = \frac{b \sin.\varphi}{r}$$

$$r : \sin.\varphi :: a : \sin.(\varphi-x) = \frac{a \sin.\varphi}{r};$$

Now, r by equation (a), is expressed by $\sqrt{a^2+b^2\pm 2ab\cos.\varphi}$; consequently we have

$$\sin.x = \frac{b \sin.\varphi}{\sqrt{a^2+b^2\pm 2ab\cos.\varphi}}$$

$$\text{and } \sin.(\varphi-x) = \frac{a \sin.\varphi}{\sqrt{a^2+b^2\pm 2ab\cos.\varphi}}.$$

Hence, ultimately, by substitution, we obtain

$$\cos.\theta = \frac{a\cos.\varphi'+b\cos.\varphi''}{\sqrt{a^2+b^2\pm 2ab\cos.\varphi}}.$$

This is a very elegant expression for the inclination of the direction of the third force, to the resultant of the other two, and in its application it is general, provided that the algebraic signs of the quantities be duly attended to, which of course must rest with the reader, not the author.

130. Let this value of cos. θ be substituted for cos. φ', its representative in equation (1), and we obtain generally, for the resultant of three forces, whose directions are anyhow inclined to one another,

$$\text{R} = \sqrt{a^2+b^2+c^2\pm 2ab\cos.\varphi \pm 2ac\cos.\varphi' \pm 2bc\cos.\varphi''}. \quad (2)$$

If any of the angles φ, φ' or φ'' be acute, the upper or positive sign must be used in that term where the acute angle occurs; but when any of the angles are obtuse, the lower or negative sign will then have place.

131. If φ' and φ'', the inclinations of the force c to the forces a and b are both right angles; then cos. $\varphi'=o$, and cos. $\varphi''=o$; consequently, the two last terms under the radical sign in equation (2) disappear, and we get

$$\text{R} = \sqrt{a^2+b^2+c^2\pm 2ab\cos.\varphi}.$$

This equation is the same as that which we originally deduced from the diagram for equation (G), on the supposition that the force PE acted at right angles to each of the other two.

132. If φ, φ' and φ'', the three angles of inclination, be each equal to a right angle, then cos. φ, cos. φ' and cos. φ'' are each equal

to zero, and the three terms where they enter in equation (2) disappear, and we get

$$R=\sqrt{a^2+b^2+c^2};$$

this accords with equation (H), where the three forces a, b, and c are mutually inclined to each other in an angle of 90 degrees.

133. If the forces a, b, and c are equal among themselves, and mutually inclined in angles of 90 degrees; then

$$R = a\sqrt{3};$$

and this again, is the same as equation (I), where the forces are supposed to be represented, in magnitude and direction, by the three contiguous edges of a cube.

134. If the forces a, b, and c are equal, and the inclinations φ, φ' and φ'' of any magnitude whatever, then equation (2) becomes

$$R=\sqrt{3+2\,(\pm\cos.\varphi\pm\cos.\varphi'\pm\cos.\varphi'')} \qquad (3)$$

135. If the forces a, b and c, and the angles of their inclinations, are equal, then equation (2) becomes

$$R=a\sqrt{3\pm 6\cos.\varphi}. \qquad (4)$$

This equation applies to the case of an oblique cube or *obtuse rhomboid*, and if the forces a, b, and c are equal, while the angles of inclination φ, φ', and φ'' are each equal to 60 degrees, then equation (2) becomes

$$R = a\sqrt{6}. \qquad (5)$$

This equation applies to the *acute rhomboid*; and if the angles of inclination φ, φ', and φ'' are each equal to 120°, then equation (2) becomes

$$R = a\sqrt{3-3}=o.$$

This equation indicates that the directions of the forces are all in the same plane, and the energy of any one of them balances the joint energies of the other two; consequently the forces sustain one another in equilibrio; this property has been alluded to in equation (d).

136. If the angles of inclination φ, φ', and φ'' are equal to one another, while the forces a, b, and c attain any unequal magnitudes whatever, then equation (2) becomes

$$R=\sqrt{a^2+b^2+c^2\pm 2\cos.\varphi\,(ab+ac+bc)}. \qquad (6)$$

Various other cases might be proposed, and the appropriate formulæ deduced from the general equation (2), but those which we have given are the most useful, and for this reason we think it needless to pursue the derivation farther, as it is presumed, from what has already been done, that the attentive reader will find no difficulty in reducing any other case that is likely to occur.

For the practical illustration of this subject, we shall now propose a few numerical examples, as follow:

137. Example 1. The magnitudes or intensities of three forces, a, b, and c, which are disposed in different planes, and act simultaneously at the same point of a body, are respectively equal to

weights or pressures of 20, 30, and 40 tons; what is the magnitude of the resultant, supposing the inclinations of their directions to be as under, viz.

$$\left.\begin{array}{l}\text{The inclination of } a \text{ to } b, \text{ or } \varphi\ = 75 \text{ degrees,}\\ \text{——————— } a - c, \text{ or } \varphi' = 80 \text{ ———,}\\ \text{——————— } b - c, \text{ or } \varphi'' = 85 \text{ ———.}\end{array}\right\}?$$

Here we have given $a=20$; $b=30$; $c=40$; $\varphi=75°$; $\varphi'=80°$ and $\varphi''=85°$; where the inclinations are all less than $90°$; consequently, their cosines are all affirmative; and since both the forces and their inclinations are unequal, the question falls under the general form in equation (2); whence we get

$$R=\sqrt{20^2+30^2+40^2+1200 \cos. 75°+1600 \cos. 80°+2400 \cos. 85°};$$

but $\cos. 75°=\cdot25882$; $\cos. 80°=\cdot17365$, and $\cos. 85°=\cdot08716$; consequently,

$$R=\sqrt{400+900+1600+310\cdot584+277\cdot04+209\cdot184};$$

$$\text{that is, } R=\sqrt{3697\cdot608}=60\cdot81 \text{ tons.}$$

138. EXAMPLE 2. The magnitudes or intensities of the forces remaining as in the last example; query, the magnitude of the resultant, supposing the composants to be inclined to one another as follows, viz.

$$\left.\begin{array}{l}\text{The inclination of } a \text{ to } b, \text{ or } \varphi\ = 100 \text{ degrees,}\\ \text{——————— } a - c, \text{ or } \varphi' = 120 \text{ ———,}\\ \text{——————— } b - c, \text{ or } \varphi'' = 130 \text{ ———.}\end{array}\right\}?$$

Here, we have given $a=20$; $b=30$; $c=40$; $\varphi=100°$; $\varphi'=120°$ and $\varphi''=130°$; where the inclinations are all greater than $90°$; consequently, their cosines are all negative; and since both the forces and their inclinations are unequal, the question falls under the general form in equation (2), whence we have

$$R=\sqrt{20^2+30^2+40^2-(1200 \cos. 100°+1600 \cos. 120°+2400 \cos. 130°)};$$

but $\cos. 100°=-\cdot17365$; $\cos. 120°=-\cdot5$; and $\cos. 130°=-\cdot64279$; consequently we get

$$R=\sqrt{400+900+1600-(208\cdot38+800+1562\cdot696)};$$

$$\text{that is, } R=\sqrt{348\cdot924}=18\cdot68 \text{ tons nearly.}$$

139. EXAMPLE 3. Suppose that three equal forces, situated in different planes but acting simultaneously at the same point of a body, have their inclinations respectively equal to 50, 60, and 75 degrees; what is the magnitude of the resultant or single equivalent, that of the composant being equal to a pressure of 120 tons?

Here we have given $a=120$; $\varphi=50°$; $\varphi'=60°$ and $\varphi''=75°$; where the angles of inclination are all less than $90°$; consequently, their cosines are additive; and since the forces are equal, but their inclinations unequal, the question falls under the particular form in equation (3); whence we have

$$R=120\sqrt{3+2(\cos. 50°+\cos. 60°+\cos. 75°)};$$

but cos. $50° = \cdot 64279$; cos. $60° = \cdot 5$, and cos. $75° = \cdot 25882$; consequently, by substitution, we get

$$R = 120\sqrt{3+2\,(\cdot 64279 + \cdot 5 + \cdot 25882)};$$

that is, $R = 120\sqrt{5 \cdot 80322} = 120 \times 2 \cdot 409 = 289 \cdot 08$ tons.

140. Example 4. The magnitudes or intensities of the composants remaining; what is the magnitude of their resultant; supposing the inclinations to be respectively supplemental to those in example 3?

Here we have given, $a = 120$; $\varphi = 130°$; $\varphi' = 120°$, and $\varphi'' = 105°$; where the angles of inclination being greater than 90°, their cosines are taken negatively; but as they are supplemental to those in the third example, their numerical values are the same; hence the necessity of referring to the table of natural cosines is here avoided, and we have by equation (3)

$$R = 120\sqrt{3+2\,(-\cdot 64279 - \cdot 5 - \cdot 25882)};$$

that is, $R = 120\sqrt{0 \cdot 19678} = 53 \cdot 232$ tons.

141. Example 5. If three forces, each equal to 180 lbs., act simultaneously at the same point of a body, but have their directions in different planes; what is the magnitude of the resultant, supposing the directions of the composants to be mutually inclined to each other in angles of 80° 30′?

Here we have given, $a = 180$, and $\varphi = \varphi' = \varphi'' = 80°\ 30'$; where the inclinations being less than 90°, the cosines are positive, and since the forces and the inclinations are both equal, the question falls under the form in equation (4); whence we have

$$R = 180\sqrt{3 + 6 \text{ cos. } 80°\ 30'};$$

but cos. $80°\ 30' = \cdot 16505$; consequently, by substitution, we obtain

$$R = 180\sqrt{3 + 6 \times \cdot 16505};$$

that is, $R = 180\sqrt{3 \cdot 9903} = 359 \cdot 56$ lbs.

142. Example 6. Let the forces remain, as in the fifth example, and suppose their inclinations to be respectively 99° 30′; what then is the magnitude of the resultant?

Here we have given, $a = 180$, and $\varphi = \varphi' = \varphi'' = 99°\ 30'$; where the inclinations being greater than 90°, the cosines are negative; consequently by equation (4), we get

$$R = 180\sqrt{3 - 6 \text{ cos. } 99°\ 30'};$$

but cos. $99°\ 30' = -\cdot 16505$; consequently, by substitution, we obtain

$$R = 180\sqrt{3 - 6 \times \cdot 16505};$$

that is, $R = 180\sqrt{2 \cdot 0097} = 255 \cdot 168$ lbs.

143. Example 7. Suppose that three forces, each equal to 256lbs. be applied to the acute angle of an obtuse rhomboid, and have their directions coincident with the three contiguous edges; what is the magnitude of the resultant?

Here we have given, $a=256$, and $\varphi=60^\circ$; consequently, by equation (5), we have

$$R=256\sqrt{6}=627\cdot 1 \text{ lbs.}$$

If we compare the results of the examples 2, 4, and 6, with those of the examples 1, 3, and 5, the immense influence that a change in the angles of inclination has upon the effect of the forces will become manifest, from which we infer the great importance of attending to the proper directions of forces that are intended to produce any useful practical results.

144. EXAMPLE 8. Suppose that the magnitudes or intensities of three forces a, b, and c, situated in different planes, but acting at the same point of a body, are respectively represented by weights of 18, 22, and 26 tons; what is the magnitude of the resultant, supposing the inclination of the forces to one another to be mutually 56°?

Here, we have given, $a=18$; $b=22$; $c=26$; $\varphi=\varphi'=\varphi''=56^\circ$; where the inclinations being less than 90°, their cosines are positive, and since the forces a, b, and c are unequal among themselves, the question falls under the particular form in equation (6); consequently, we have

$$R=\sqrt{18^2+22^2+26^2+2\,(\overline{18\times 22}+\overline{18\times 26}+\overline{22\times 26})\cos, 56^\circ};$$

but cos. $56^\circ=\cdot 55919$; therefore, by substitution, we get

$$R=\sqrt{324+484+676+4872\times\cdot 55919};$$

that is, $R=\sqrt{4208\cdot 37368}=64\cdot 88$ lbs.

145. EXAMPLE 9. Let the magnitudes of the forces remain as in the 8th example, but suppose their directions to be mutually inclined to each other in angles of 124°; what then is the magnitude of the resultant?

Here, the angles of inclination being each supplemental to what they were in the foregoing example, the cosines will have the same numerical value, but they will be affected with a contrary sign; consequently, by equation (6), we have

$$R=\sqrt{324+484+676-4872\times\cdot 55919};$$

that is, $R=\sqrt{-1240\cdot 37368}=35\cdot 218\sqrt{-1}$;

an imaginary quantity, which indicates, that the question under this last form is impossible, as indeed we know from other principles; for a solid angle cannot be constituted under three plane angles, whose sum exceeds 360°. Now, in the present case, the sum of the three plane angles is 372°, exceeding the maximum limit by 12°; the example, however, is valuable for pointing out the limit of possibility.

It may not perhaps be improper here to remark, that in examples of this nature the angles are not to be chosen indiscriminately when the magnitude of the resultant is given; for, as we have already stated, there is a necessary connection between the angles; and it is such, that the magnitude of any one of them must always be

dependent on the other two, and must in consequence be determined by equation (L); two of them may, however, be taken at random, provided that their sum exceeds 90°, and the third being calculated accordingly, the edges of the solid, or the magnitudes of the composant forces, can then be ascertained.

SECTION THIRD.

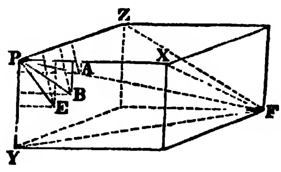
OF RECTANGULAR CO-ORDINATES, OR THE METHOD OF DETERMINING THE DIRECTION OF FORCES, ANYHOW SITUATED IN SPACE.

146. It will be perceived, that in treating of the rectangular parallelopipedon, and the examples allied to it, in so far as regards the action of three forces situated in different planes, but directed to the same point; we have in a manner anticipated the principle of the rectangular co-ordinates, but since this method of determining the direction of forces anyhow posited in space, is of the greatest importance, as is abundantly shown in numerous physical inquiries, we shall now proceed to treat the subject a little more in detail, and for which purpose it will be necessary to extend our notation, as in the following general problem:

147. PROBLEM. *To determine the intensity and direction of the resultant, when the magnitudes or intensities of three forces acting in different planes are applied to the same point of a rectangular parallelopipedon.*

Let $a=$ PA, $b=$ PB, $c=$ PE, be three different forces whose common point of application is P, but which do not all exist in the same plane.

Through the point P draw the three rectangular co-ordinates PX, PY, and PZ, of which PX and PY may be supposed to exist in the same plane, and PZ perpendicular thereto, concurring in the same point P.



Through the points A, B, and E, the extremities of the lines which represent the given forces, draw other lines respectively perpendicular to the three co-ordinates, and each force will thereby be reduced or projected into other forces in those directions, concurring also in the point P.

Let the angles which the directions of the forces form with each of the co-ordinates be respectively represented as below, viz.

PA with PX, PY, and PZ, the angles A, A′, A″,
PB ———————————, the angles B, B′, B″,
PE ———————————, the angles C, C′, C″.

Then by plane trigonometry, we shall have for the projected composants, or reductions of the given forces on each of the coordinates, the following expressions or values, viz.

The projections of the force
PA on the co-ordinates PX, PY and PZ are, PA cos. A; PA cos. A′, and PA cos. A″,
PB ———————————, PB cos. B; PB cos. B′, and PB cos. B″,
PE ———————————, PE cos. C; PE cos. C′, and PE cos. C″.

If we now assume x, x', and x'', to denote the angles which the resultant PF = R makes with the co-ordinates PX, PY, and PZ; then the projected composants, or reductions of the resultant, will be respectively R cos. x, R cos. x', and R cos. x''.

But by construction,

$$\left.\begin{aligned} R\cos. x &= PA\cos. A + PB\cos. B + PE\cos. C, \\ R\cos. x' &= PA\cos. A' + PB\cos. B' + PE\cos. C', \\ R\cos. x'' &= PA\cos. A'' + PB\cos. B'' + PE\cos. C''. \end{aligned}\right\} \quad (M)$$

Now, the magnitude or intensity of the resultant PF = R, is obviously the diagonal of the rectangular parallelopipedon constructed on the lines expressed by these equations, and consequently its square is equal to the sum of their squares, as has been already shown in the diagram under equation (H), that is

$$R^2(\cos^2 x + \cos^2 x' + \cos^2 x'') = \left\{\begin{aligned} &(PA\cos.A + PB\cos.B + PE\cos.C)^2 + \\ &(PA\cos.A' + PB\cos.B' + PE\cos.C')^2 + \\ &(PA\cos.A'' + PB\cos.B'' + PE\cos.C'')^2 \end{aligned}\right\}; (N)$$

but since x, x', and x'', are the angles which the resultant makes with the co-ordinates, or, which is the same thing, the angles which the diagonal of a rectangular parallelopipedon makes with the three edges, contiguous to that angle of the solid where it originates, we have by equation (L)

$$\cos^2. x + \cos^2 x' + \cos^2. x'' = 1;$$

consequently, by substitution and evolution, equation (N) becomes

$$R = \left\{\begin{aligned} &(PA\cos. A + PB\cos. B + PE\cos. C)^2 + \\ &(PA\cos. A' + PB\cos. B' + PE\cos. C')^2 + \\ &(PA\cos. A'' + PB\cos. B'' + PE\cos. C'')^2 \end{aligned}\right\}^{\frac{1}{2}} \quad (O)$$

148. If the parenthetical terms of equation (N) were actually expanded and arranged in the most commodious manner possible, the expression for the value of R would be extremely prolix, and in consequence tedious to reduce; it will therefore be better to determine the magnitudes of the projected composants of R at once from equation (M); then, the square root of the sum of the squares of the co-ordinates or composants thus determined, will give the magnitude or intensity of the resultant required, from which the direction will easily be made known.

Numerical examples will render the whole of this problem plain; and we shall take the trouble of working them out at full length, which will thereby assist the reader in similar attempts.

149. EXAMPLE. Suppose the magnitudes or intensities of three forces PA, PB, and PE, acting in different planes, but applied to the same point, to be represented respectively by the numbers 20, 30, and 40; what is the intensity and direction of the resultant, the

angles which the directions of the forces make with two of the rectangular co-ordinates being respectively as below, viz.

PA with PX and PY, the angles 50° and 60°,
PB ———————————— 47 —— 58,
PE ———————————— 66 —— 72?

In the first place, then, the angles which the lines PA, PB, and PE, make with the third rectangular co-ordinate PZ, must be computed from each pair of the given angles, by means of equation (L), in the following manner:

$$\text{nat. cos. } 50^\circ = \cdot 64279, \text{ its square is } \cdot 4131789841$$
$$\text{nat. cos. } 60 = \cdot 50000, \text{ its square is } \cdot 2500000000$$
$$\cos^2. 50^\circ + \cos^2. 60^\circ = \cdot 6631789841;$$
$$\text{then } \sqrt{1 - \cdot 6631789841} = \cdot 58036 = \text{nat. cos. } 54^\circ\ 31'.$$

Hence, the angle which the direction of PA makes with the co-ordinate PZ, is $54^\circ\ 31'$.

$$\text{nat. cos. } 47^\circ = \cdot 682\ldots, \text{ its square is } \cdot 465724\ldots.$$
$$\text{nat. cos. } 58 = \cdot 52992, \text{ its square is } \cdot 2808152064$$
$$\cos^2. 47^\circ + \cos^2. 58^\circ = \cdot 7459392064;$$
$$\text{then } \sqrt{1 - \cdot 7459392064} = \cdot 50404 = \text{nat. cos. } 59^\circ\ 44'.$$

Hence, the angle which the direction of PB makes with the co-ordinate PZ, is $59^\circ\ 44'$.

$$\text{nat. cos. } 66^\circ = \cdot 40674, \text{ its square is } \cdot 1654374276$$
$$\text{nat. cos. } 72 = \cdot 30902, \text{ its square is } \cdot 0954933604$$
$$\cos^2. 66^\circ + \cos^2. 72^\circ = \cdot 2609307880;$$
$$\text{then } \sqrt{1 - \cdot 260930788} = \cdot 85969 = \text{nat. cos. } 30^\circ\ 43'.$$

Hence, the angle which the direction of PE makes with the co-ordinate PZ, is $30^\circ\ 43'$.

150. Having thus computed the angles which the co-ordinate PZ makes with the direction of each of the forces PA, PB, and PE, we have next to fulfil the conditions of the question; that is, to determine the magnitude or intensity and direction of the resultant PF. Now, if each of the data be substituted respectively in equation (M) for the symbol that represents it, we shall obtain

$$\text{R cos. } x = 20 \cos. 50^\circ\ 00' + 30 \cos. 47^\circ\ 00' + 40 \cos. 66^\circ\ 00',$$
$$\text{R cos. } x' = 20 \cos. 60\ \ 00 + 30 \cos. 58\ \ 00 + 40 \cos. 72\ \ 00,$$
$$\text{R cos. } x'' = 20 \cos. 54\ \ 31 + 30 \cos. 59\ \ 44 + 40 \cos. 30\ \ 43.$$

Which expressions actually become

$$\left.\begin{aligned} \text{R cos. } x &= 49 \cdot 5854, \\ \text{R cos. } x' &= 38 \cdot 2584, \\ \text{R cos. } x'' &= 61 \cdot 116\ . \end{aligned}\right\} \qquad (\text{P})$$

Then, by squaring each of these, and taking the sum on both sides, we get

$$\text{R}^2 (\cos^2. x + \cos^2. x' + \cos^2. x'') = 7657 \cdot 57.$$

but $\cos^2. x + \cos^2. x' + \cos^2. x'' = 1$ (see equation (L)), consequently, we shall have

$$\text{R}^2 = 7657 \cdot 57,$$

and by extracting the square root of both sides, we finally obtain for the magnitude of the resultant

$$R=\sqrt{7657\cdot57}=87\cdot5.$$

Now, if each of the equations (P), be divided by the resultant just found, we shall get

$$\cos. x=\frac{49\cdot5854}{87\cdot5}\ \cdot56680=\text{nat. cos. } 55^\circ\ 28',$$

$$\cos. x'=\frac{38\cdot2584}{87\cdot5}=\cdot43724=\text{nat. cos. } 64^\circ\ 04',$$

$$\cos. x''=\frac{61\cdot116}{87\cdot5}=\cdot69847=\text{nat. cos. } 45^\circ\ 42'$$

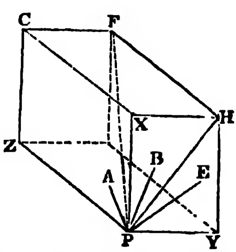
Hence the direction of the resultant is known, for it makes with the co-ordinates respectively the following angles, viz. :—

With the co-ordinate PX, the angle 55° 28′,
———————— PY, ———— 64° 04′,
· PZ, -45° 42′.

And moreover, the co-ordinates themselves, or the sides of the rectangular parallelopipedon, whose diagonal forms the resultant, are, as we have already shown in equation (P), respectively

$$PX=49\cdot5854$$
$$PY=38\cdot2584$$
$$PZ=61\cdot116.$$

151. If, therefore, we construct a rectangular parallelopipedon upon these sides, such construction will exhibit the entire conditions of the question. Thus,



Upon the straight line PY equal to 38·2584, taken from a scale of equal parts of any convenient dimensions, construct the rectangular parallelogram PXHY, having the side PX equal to 49·5854, taken from the same scale as PY; join PH, and at the point P therein, erect the perpendicular PZ, making it equal to 61·116 taken from the same scale as PX and PY; complete the rectangular parallelopipedon PZFY, and join PF; then is PF the resultant of the three forces PA, PB, and PE, whose positions in the figure are merely assumed.

Now, if PF be taken in the compasses, and applied to the scale from which the values of PX, PY, and PZ were taken, it will be found to measure 87·5, exactly the same as was found by calculation.

Furthermore, the angles FPX$=55^\circ\ 28'$; FPY$=64^\circ\ 04'$, and FPZ$=45^\circ\ 42'$, are the inclinations of the resultant PF, to each of the co-ordinates PX, PY, and PZ; but which, on account of the oblique position of the figure, must be measured from a scale of chords, and by the application of a particular projection, which it shall now be our purpose to explain in the following examples.

152. EXAMPLE 2. Three forces, a, b, and c, whose directions are situated in different planes, but concur in the same point, have their magnitudes or intensities respectively represented by weights or pressures of 50, 60, and 70 tons; what is the magnitude and direction of their resultant, the inclinations of their directions to the rectangular co-ordinates being as follows?

PA $= a$ makes with PX and PY, the angles 40 and 60 degrees,
PB $= b$ ——————————————— 45 —— 70 ———,
PE $= c$ ——————————————— 50 —— 80 ———:

Now the angles which the direction of the forces a, b, and c, make with the third co-ordinate PZ, must be determined from equation (L), as follows.

nat. cos. $40° = \cdot 76604$, its square is $\cdot 58682$,
nat. cos. $60 \;\; = \cdot 5$——, its square is $\cdot 25$,

$$\cos.^2 40° + \cos.^2 60° = \cdot 83682;$$

then, $\sqrt{1 - \cdot 83682} = \sqrt{0 \cdot 16318} = \cdot 40395 =$ nat. cos. 66° 10′.

Hence, the angle which the direction of the force PA makes with the co-ordinate PZ is, 66° 10′.

nat. cos. $45° = \cdot 70711$, its square is $\cdot 5$,
nat. cos. $70 \;\; = \cdot 34202$, its square is $\cdot 11698$,

$$\cos.^2 45° + \cos.^2 70° = \cdot 61698;$$

then, $\sqrt{1 - \cdot 61698} = \sqrt{0 \cdot 38302} = \cdot 61889 =$ nat. cos. 51° 46′.

Hence, the angle which the direction of the force PB makes with the co-ordinate PZ is, 51° 46′.

nat. cos. $50° = \cdot 64279$, its square is $\cdot 41316$,
nat. cos. $80 \;\; = \cdot 17365$, its square is $\cdot 03015$,

$$\cos^2 50° + \cos.^2 80° = \cdot 44331;$$

then $\sqrt{1 - \cdot 44331} = \sqrt{0 \cdot 55669} = \cdot 74611 =$ nat. cos. 41° 45′.

Hence, the angle which the direction of the force PE makes with the co-ordinate PZ, is 41° 45′.

Therefore, by substituting the given and computed data in equation (M), we get

$$R \cos. x = 50 \cos. 40° \; 00' + 60 \cos. 45° \; 00' + 70 \cos. 50° \; 00',$$
$$R \cos. x' = 50 \cos. 60 \;\; 00 + 60 \cos. 70 \;\; 00 + 70 \cos. 80 \;\; 00,$$
$$R \cos. x'' = 50 \cos. 66 \;\; 10 + 60 \cos. 51 \;\; 46 + 70 \cos. 41 \;\; 45.$$

In which equations, R is the required resultant, and x, x' and x'' the angles which the resultant makes with each of the co-ordinates PX, PY, and PZ.

153. If we substitute the numerical values of the cosines of the angles of inclination in the several terms of the foregoing equations, and perform the operations, the collected results will become

$$\left.\begin{aligned} R \cos. x &= 125 \cdot 7239, \\ R \cos. x' &= \;\; 57 \cdot 6767, \\ R \cos. x'' &= 109 \cdot 5586. \end{aligned}\right\} \qquad (M')$$

By squaring these reduced equations, and taking the sum of both sides, we get

$$R^2 (\cos^2. x + \cos^2. x' + \cos^2. x'') = 31125 \cdot 24;$$

but by equation (L), it is shown that

$$\cos^2.x+\cos^2.x'+\cos^2.x''=1;$$

consequently $R^2=31125{\cdot}24$,

and by evolution, we get

$$R=\sqrt{31125{\cdot}24}=176{\cdot}42 \text{ tons},$$

the magnitude or intensity of the required resultant; consequently, the angles which the resultant makes with each of the co-ordinates can easily be found, for we have only to divide both sides of the expressions in equation (M′), by R, and we obtain

$$\cos.x=\frac{125{\cdot}7239}{176{\cdot}42}={\cdot}71264=\text{nat. cos. } 44^\circ\ 33',$$

$$\cos.x'=\frac{57{\cdot}6767}{176{\cdot}42}={\cdot}32692=\text{nat. cos. } 70\ \ 55,$$

$$\cos.x''=\frac{109{\cdot}5586}{176{\cdot}42}={\cdot}62101=\text{nat. cos. } 51\ \ 37,$$

Hence the direction of the resultant has been determined, for it makes with the rectangular co-ordinates the following angles, viz.

With the ordinate PX, the angle 44° 33′,
———————— PY, ———— 70 55,
———————— PZ, ———— 51 37.

And the magnitudes of the ordinates themselves are respectively as under, viz.:—

$$PX=125{\cdot}7239$$
$$PY=57{\cdot}6767$$
$$PZ=109{\cdot}5586.$$

We shall, for the sake of a little variety, propose another example, which will, in its nature, approach to the converse of that given above; it is as follows:—

154. EXAMPLE 3. Suppose a weight of 176·42 tons to be suspended from the common summit of three props, whose inclinations to the rectangular co-ordinates are respectively as below, viz.

PA to PX, PY, and PZ, in the angles 40° 60°, and 66° 10′,
PB———————————— 45 70 ——— 51 46,
PE———————————— 50 80 ——— 41 45;

and, moreover, the inclinations of the direction in which the weight hangs with each of the co-ordinates, are

With PX the angle 44°33′,
PY———— 70 55,
PZ———— 51 37;

what pressure is exerted upon the ground in the direction of each of the props?

Let w, y, and z represent the three pressures required by the question, then by equation (M), we get

$$176{\cdot}42 \cos. 44^\circ\ 33'=w \cos. 40^\circ\ 0'+y \cos. 45^\circ\ 0'+z \cos. 50^\circ\ 0',$$
$$176{\cdot}42 \cos. 70\ \ 55=w \cos\ 60\ \ 0+y \cos. 70\ \ 0+z \cos. 80\ \ 0,$$
$$176{\cdot}42 \cos. 51\ \ 37=w \cos.\ 66\ 10+y \cos. 51\ 46+z \cos. 41\ 45.$$

From these three simple equations, in which every thing is known but w, y, and z, these quantities may be eliminated as follows, viz.

Find the natural cosines of all the angles proposed or given in the question, from a table of sines, tangents, &c.; then, if those natural cosines be substituted for the angular expressions above employed, there will arise a class of numerical equations involving only the first powers of the required quantities.

Now, the natural cosines of the angles as taken from the table above alluded to, are respectively as under, viz.:

nat. cos. 44° 33′ = ·71264;	nat. cos. 40° 00′ = ·76604;
nat. cos. 45 00 = ·70711;	nat. cos. 50 00 = ·64279;
nat. cos. 70 55 = ·32694;	nat. cos. 60 00 = ·50000;
nat. cos. 70 00 = ·34202;	nat. cos. 80 00 = ·17365;
nat. cos. 51 37 = ·62092;	nat. cos. 66 10 = ·40408;
nat. cos. 51 46 = ·61887;	nat. cos. 41 45 = ·74606;

Hence, we obtain

$$\begin{aligned}
\cdot 76604\,w + \cdot 70711\,y + \cdot 64279\,z &= 125\cdot 72395,\\
\cdot 50000\,w + \cdot 34202\,y + \cdot 17365\,z &= 57\cdot 67876,\\
\cdot 40408\,w + \cdot 61887\,y + \cdot 74606\,z &= 109\cdot 31918.
\end{aligned}$$

From the first of these we have $w = \dfrac{125\cdot 72395 - \cdot 70711\,y - \cdot 64279\,z}{\cdot 76604}$

From the second we have $w = \dfrac{57\cdot 67876 - \cdot 34202\,y - \cdot 17365\,z}{\cdot 5}$

From the third we have $w = \dfrac{109\cdot 31918 - \cdot 61887\,y - \cdot 74606\,z}{\cdot 40408}$

Comparing the first of these with the second, and the second with the third, we have

$$y = \frac{18\cdot 67775 - \cdot 18838\,z}{\cdot 09156},$$

$$y = \frac{31\cdot 35277 - \cdot 30287\,z}{\cdot 17123}.$$

And again, by comparing these two values of y, we get

$$\begin{aligned}
z &= 72\cdot 3 \text{ tons},\\
y &= 55\cdot 2 \text{ —},\\
w &= 52\cdot 5 \text{ —}.
\end{aligned}$$

It may be considered that these numbers ought to correspond to those given in the previous example, (art. 152.) but our readers will have the goodness to observe that we have not taken the cosines of the angles, as there determined, true to the nearest unit; consequently the above results must partake of the deficiency.

155. If any of the angles which the direction of the forces make with the rectangular co-ordinates, should be greater than a right angle, their cosines will enter equation (M) negatively, and consequently the negative effects must be attended to in the calculation, otherwise we would be led to very erroneous conclusions. For, when any of the forces are inclined to the co-ordinates in angles

greater than 90 degrees, it is evident that those forces so situated have their directions entirely without the angle of the solid where they concur, and therefore the reduction of the rectangular co-ordinates must be taken in opposition to the reductions of the remaining forces, for it is obvious that their projections fall on the co-ordinates produced beyond P in the opposite direction.

It may be useful, before we proceed further, to propose an example illustrative of this case, and to construct a diagram from the computed results, in order to show in what manner the effects of the obtuse forces are to be estimated.

156. Example. Let the magnitudes of three forces a, b, and c, whose intensities and directions are represented by the straight lines, PA, PB, and PE, be respectively equal to the numbers 20, 30, and 40, as in the foregoing example; what will be the magnitude and direction of the resultant, the angles which the directions of the forces make with two of the rectangular co-ordinates being as below, viz.

$$\left.\begin{array}{llll} \text{PA with PX and PY, the angles} & 67^\circ\,32' & \text{and} & 123^\circ\,08', \\ \text{PB} \text{————————} & 24\ \ 15 & \text{——} & 78\ \ 36, \\ \text{PE} \text{————————} & 38\ \ 46 & \text{——} & 57\ \ 58, \end{array}\right\}?$$

Compute, by means of equation (L), the angles which the directions of each of the forces make with the third co-ordinate PZ, and they will be as follows, viz.

$$\begin{array}{ll} \text{PA with PZ makes the angle} & 41^\circ\,48', \\ \text{PB} \text{————————} & 68\ \ 54, \\ \text{PE} \text{————————} & 70\ \ 34. \end{array}$$

And these angles being substituted according to their terms, together with the magnitudes of the forces, equation (M) becomes

$$\begin{aligned} R\cos. x &= 20\cos.\ 67^\circ 32' + 30\cos. 24^\circ 15' + 40\cos. 38^\circ 46', \\ R\cos. x' &= -20\cos. 123\ \ 8 + 30\cos. 78\ \ 36 + 40\cos. 57\ \ 58, \\ R\cos. x'' &= 20\cos.\ 41\ \ 48 + 30\cos. 68\ \ 54 + 40\cos. 70\ \ 34. \end{aligned}$$

Hence, then, it appears that the first term on the right-hand side of the middle equation is to be taken negatively; consequently, the opposite projection of the force PA falls on PY, produced or extended on the other side of P, the concurrent point, and the reduced equations become

$$\left.\begin{aligned} R\cos. x &= 66\cdot1838, \\ R\cos. x' &= 16\cdot2148, \\ R\cos. x'' &= 36\cdot3544. \end{aligned}\right\}. \qquad (Q)$$

Then, by squaring and taking the sum of the terms on each side, we obtain

$$R^2(\cos^2. x + \cos^2. x' + \cos^2. x'') = 5964\cdot86,$$

but $\cos^2. x + \cos^2. x' + \cos^2. x'' = 1$, see equation (L); consequently we shall have

$$R^2 = 5964\cdot86,$$

and, by extracting the square root of both sides of this equation, we finally obtain, for the magnitude or intensity of the resultant,

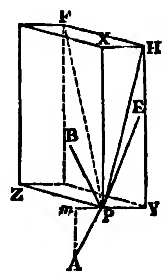
$$R = \sqrt{5964 \cdot 86} = 77 \cdot 23.$$

Now, the three co-ordinates, or the sides of the rectangular parallelopipedon whose diagonal we have just found, are respectively

$$\left.\begin{array}{l} PX = 66 \cdot 1838, \\ PY = 16 \cdot 2148, \\ PZ = 36 \cdot 3544. \end{array}\right\} \text{ see equation (Q)}$$

If, therefore, we construct a rectangular solid upon these sides, such a solid, or the lines which it contains, will resolve the question. The geometrical construction is as below.

157. Let the straight lines PA, PB, and PE, represent the three given forces, whose magnitudes or intensities are respectively expressed by the numbers 20, 30, and 40, measured from a scale of equal parts.



Through P, the point of application, draw the straight lines PX and PY at right angles to each other, making PX equal to the number 66·1838, and PY equal to 16·2148, each taken from the same scale with PA, PB, and PE;—complete the rectangular parallelogram PXHY, and join PH. At the point P in PH, erect the perpendicular PZ, making it equal to the number 36·3544, measured as above, and complete the rectangular solid PZFY; draw the diagonal PF; then is PF the resultant of the three forces PA, PB, and PE, which, if applied to the same scale on which the other lines were measured, will indicate 77·23, the same as was obtained by calculation.

Make the angle YPA equal to 123° 8′, the given inclination of PA to the co-ordinate PY; make PA equal to 20, the number which expresses the magnitude of the force PA; from the extremity A, draw Am perpendicularly to YP produced; then are Am and Pm, the composants of PA, of which Pm represents the effect of PA in the direction of the co-ordinate PY; but because it falls on the contrary side of the point P, it must be accounted negative; and consequently it counteracts the effects of PB and PE, when reduced to the same co-ordinate, by a quantity equal to its magnitude.

158. Let each side of the equations (Q), be divided by the resultant, as we have above determined it, and we shall have

$$\cos. x = \frac{66 \cdot 1838}{77 \cdot 23} = \cdot 85697 = \text{nat. cos. } 31° \ 1',$$

$$\cos. x' = \frac{16 \cdot 2148}{77 \cdot 23} = \cdot 20996 = \text{nat. cos. } 77 \ 53,$$

$$\cos. x'' = \frac{36 \cdot 3544}{77 \cdot 23} = \cdot 47073 = \text{nat. cos. } 61 \ 55.$$

Consequently the direction of the resultant PF is known, for it makes with the rectangular co-ordinates respectively the following angles, viz.

With the co-ordinate PX, the angle FPX $= 31^\circ\ 01'$
——————— PY, ——— FPY $= 77\ \ 53$
——————— PZ, ——— FPZ $= 61\ \ 55$

The angle YPA is $123^\circ\ 8'$, and consequently its supplement mPA is $56^\circ\ 52'$; therefore, by Plane Trigonometry, we have

$$\text{rad} : \cos\ 56^\circ\ 52' :: 20 : \text{P}m,$$

or by assuming radius equal to unity, and equating the products of the mean and extreme terms,

$$\text{P}m = \cdot 54659 \times 20 = 10\cdot 9318,$$

the very same as it is found to be, when measured from the figure on a scale of equal parts, of the same dimensions as that by which the other lines were compared.

159. If we restore the letters a, b, and c, the algebraic representatives of the forces, and return to equation (N), which we left unexpanded, we shall have, after expansion,

$$\text{R}^2(\cos^2. x+\cos^2. x'+\cos^2. x'')=$$
$$\left\{\begin{array}{l} a^2(\cos^2.\text{A}+\cos^2.\text{A}'+\cos^2.\text{A}'')+2ab(\cos.\text{A}\cos.\text{B}+\cos.\text{A}'\cos.\text{B}'+\cos.\text{A}''\cos.\text{B}'')+ \\ b^2(\cos^2.\text{B}+\cos^2.\text{B}'+\cos^2.\text{B}'')+2ac(\cos.\text{A}\cos.\text{C}+\cos.\text{A}'\cos.\text{C}'+\cos.\text{A}''\cos.\text{C}'')+ \\ c^2(\cos^2.\text{C}+\cos^2.\text{C}'+\cos^2.\text{C}'')+2bc(\cos.\text{B}\cos.\text{C}+\cos.\text{B}'\cos.\text{C}'+\cos.\text{B}''\cos.\text{C}''). \end{array}\right\}$$

Now, it is a well-known truth, and we have demonstrated the same in equation (L), that when a straight line is any how inclined to three rectangular co-ordinates, the sum of the squares of the cosines of its three inclinations is equal to unity; consequently we have

$$\cos^2. x+\cos^2. x'+\cos^2. x''=1,$$
$$\cos^2. \text{A}+\cos^2. \text{A}'+\cos^2. \text{A}''=1,$$
$$\cos^2. \text{B}+\cos^2. \text{B}'+\cos^2. \text{B}''=1,$$
$$\cos^2. \text{C}+\cos^2. \text{C}'+\cos^2. \text{C}''=1.$$

Therefore, the expanded equation condenses into the following more elegant and commodious form, viz.

$$\text{R}^2=\left\{\begin{array}{l} a^2+2ab\ (\cos.\text{A}\cos.\text{B}+\cos.\text{A}'\cos.\text{B}'+\cos.\text{A}''\cos.\text{B}'')+ \\ b^2+2ac\ (\cos.\text{A}\cos.\text{C}+\cos.\text{A}'\cos.\text{C}'+\cos.\text{A}''\cos.\text{C}'')+ \\ c^2+2bc\ (\cos.\text{B}\cos.\text{C}+\cos.\text{B}'\cos.\text{C}'+\cos.\text{B}''\cos.\text{C}''). \end{array}\right\}\ (\text{R})$$

The equation in its present form can be applied without much trouble, although it may still be preferable to employ equation (M), but if a proper arrangement of the steps be here adopted, it would be more consistent with strict theory to determine the resultant from equation (R); we shall, therefore, resolve the first of the two foregoing examples by this method, in order that the reader may be enabled to trace the process, and make a comparison between the two methods here presented to his view.

Since the computation of the third angles, dependent on the equation $\cos.^2\text{A}+\cos^2.\text{B}+\cos^2.\text{C}=1$, would be the same in both cases, it would only be a waste of time and labour to re-compute

them, but it must be observed, that in every independent question, these angles are to be determined after the manner here alluded to.

160. EXAMPLE. The forces in the example which we propose to resolve, are expressed by the numbers 20, 30, and 40, and the angles which their directions make with the three rectangular co-ordinates, are respectively as beneath, viz.

PA with PX, PY, and PZ, the angles	50°, 60°	and	54° 31′
PB ———————————	47, 58	—	59 44
PE ———————————	66, 72	—	30 43

Let these data be compared with the respective symbols in equation (R), and the numerical operation will then be more distinctly indicated; thus,

$$\begin{array}{lll} a = 20, & b = 30, & c = 40, \\ A = 50^\circ 00' & B = 47^\circ 00' & C = 66^\circ 00' \\ A' = 60\ 00 & B' = 58\ 00 & C' = 72\ 00 \\ A'' = 54\ 31 & B'' = 59\ 44 & C'' = 30\ 43. \end{array}$$

Here follows the operation.

$2ab = 1200$	log.	3·079181;	$2ab = 1200$	log.	3·079181
A = 50°	log. cos.*	9·808067;	A′ = 60°	log. cos.	9·698970
B = 47	log. cos.	9·833783;	B′ = 58	log. cos.	9·724210
nat. num. = 526·055	log.	2·721031	nat. num. = 317·952	log.	2·502361

$2ab = 1200$	log.	3·079181
A″ = 54° 31′	log. cos.	9·763777
B″ = 59 44	log. cos.	9·702452
nat. num. = 351·083	log.	2·545410.

Hence we have $a^2 + 2ab$ (cos. A cos. B + cos. A′ cos. B′ + cos. A″ cos. B″) = 400 + 526·055 + 317·952 + 351·083 = 1595·09.

$2ac = 1600$	log.	3·204120;	$2ac = 1600$	log.	3·204120
A = 50°	log. cos.	9·808067;	A′ = 60°	log. cos.	9·698970
C = 66	log. cos.	9·609313;	C′ = 72	log. cos.	9·489982
nat. num. = 418·311	log.	2·621500	nat. num. = 247·213	log.	2·393072

$2ac = 1600$	log.	3·204120
A″ = 54° 31′	log. cos.	9·763777
C″ = 30 43	log. cos.	9·934349
nat. num. = 798·446	log.	2·902246.

Hence we have $b^2 + 2ac$ (cos. A cos. C + cos. A′ cos. C′ + cos. A″ cos. C″) = 900 + 418·311 + 247·213 + 798·446 = 2363·97.

$2bc = 2400$	log.	3·380211;	$2bc = 2400$	log.	3·380211
B = 47°	log. cos.	9·833783;	B′ = 58°	log. cos.	9·724210
C = 66	log. cos.	9·609313;	C′ = 72	log. cos.	9·489982
nat. num. = 665·743	log.	2·823307	nat. num. = 393·009	log.	2·594403

$2bc = 2400$	log.	3·380211
B″ = 59° 44′	log. cos.	9·702452
C″ = 30 43	log. cos.	9·934349
nat. num. = 1039·945	log.	3·017012.

Hence we have $c^2 + 2bc$ (cos. B cos. C + cos. B′ cos. C′ + cos. B″ cos. C″) = 1600 + 665·743 + 393·009 + 1039·945 = 3698·697.

* To save the labour of multiplying together two decimal numbers of five figures each, we prefer employing the logarithmic cosines, and taking out the natural numbers to the extent of three decimal places only.

Therefore, collecting those results $(1595{\cdot}09+2363{\cdot}97+3698{\cdot}697)$, and extracting the square root, we obtain

$\text{R}=\sqrt{7657\cdot757}=87\cdot5$ very nearly, the same as by the former method; the extraction of the root in the last step is indicated in equation (O).

161. Whence it appears, that the determination of the resultant by means of equation (M), is much simpler than it is by equation (R), but then, as we have already observed, it is not so consonant with rigorous theory; for since the angles which the resultant makes with each of the co-ordinates, are not supposed to be known till the resultant itself has been found, it follows, that the resultant is combined with an unknown quantity in each expression of equation (M), and is therefore indeterminable, until by squaring and summing the terms on each side of the equation, we discover that the aggregate of the quantities multiplied into R^2 is equal to unity, and consequently, that R^2 is equal to the sum of the squares of the other terms; this is indicated in equation (N), and by restoring the Algebraic symbols for the forces, we arrive at the modified and ultimate form represented in equation (R).

162. Other writers, in treating of this subject, have generally adopted a single character to denote the triple compound expressions on the right hand side of equation (M), such as

$$n=\text{PA}\cos.\text{A}+\text{PB}\cos.\text{B}+\text{PE}\cos.\text{C},$$
$$n'=\text{PA}\cos.\text{A}'+\text{PB}\cos.\text{B}'+\text{PE}\cos.\text{C}',$$
$$n''=\text{PA}\cos.\text{A}''+\text{PB}\cos.\text{B}''+\text{PE}\cos.\text{C}'',$$

and by this artifice, have succeeded in giving to the subject an air of simplicity which it does not possess; for in the actual solution of a question, it becomes necessary to resolve the single characters so adopted into their constituent elements, and to trace back the arrangement till we arrive at the identical expressions for which the substitution was originally made.

But although this method of condensation, gains nothing in point of utility in actual practice, yet it is very commodious for the purpose of investigation, and for this reason we shall now employ it.

163. Returning then to equation (M), and adopting the abbreviated characters, we shall have

$$\left.\begin{aligned}\text{R}\cos.x&=n,\\ \text{R}\cos.x'&=n',\\ \text{R}\cos.x''&=n''\end{aligned}\right\}\qquad(\text{S})$$

then, by squaring and taking the sum of the terms on both sides, we obtain

$$\text{R}^2(\cos^2.x+\cos^2.x'+\cos^2x'')=n^2+n'^2+n''^2;$$

but $(\cos^2.x+\cos^2.x'+\cos^2.x'')=1$; see equation (L).

consequently, by substitution, we get

$$\text{R}^2=n^2+n'^2+n''^2,$$

therefore, by extracting the square root, we arrive immediately at an expression for the magnitude or intensity of the resultant; that is,

$$R=\sqrt{n^2+n'^2+n''^2} \qquad (T)$$

Again, from equation (S), we have

$$\cos. x=\frac{n}{R}; \quad \cos. x'=\frac{n'}{R}; \quad \cos. x''=\frac{n''}{R};$$

hence also, the direction of the resultant is known, for x, x' and x'' are, as has been previously shown, the angles which the resultant $PF=R$, makes with the rectangular co-ordinates PX, PY, and PZ.

164. If an equilibrium obtains between the forces, it is obvious that R the resultant is equal to *zero*;

$$\text{consequently, } R\cos. x=o;$$
$$R\cos. x'=o;$$
$$R\cos. x''=o;$$

and the conditions of equilibrium are expressed generally, by causing each member in equation (M) to vanish, that is, by putting

$$a\cos. A+b\cos. B+c\cos. C=o;$$
$$a\cos. A'+b\cos. B'+c\cos. C'=o;$$
$$a\cos. A''+b\cos. B''+c\cos. C''=o;$$

These are the equations of equilibrium for three bodies, anyhow situated in space and applied to the same point, and this brings us to the third division of our subject, viz.

CASE III.

WHEN THE COMPOSANT AND RESULTANT FORCES ARE SITUATED IN THE SAME PLANE, BUT DIRECTED TO DIFFERENT PARTS OF A BODY.

165. This case obviously branches itself into two varieties, viz.

1. *When the directions of the forces are at right angles to the line of application.**
2. *When the directions of the forces are not at right angles to the line of application, but oblique to it.*

In each of these varieties, the forces may be conceived to act either in the same or in different directions, and the principles which develope the conditions of intensity or magnitude, are as follows, viz.

* *The line of application*, is the straight line passing through the points where the forces act.

For the first variety.

If two parallel forces act perpendicularly upon a straight line, either in the same or in different directions, their resultant is parallel to them, equal to their sum or difference, and divides the line of application into two parts, which are to each other reciprocally as the magnitudes or intensities of the composant forces.

For the second variety.

If two parallel forces act obliquely upon a straight line, either in the same or in different directions, their resultant is parallel to them, equal to their sum or difference, and divides the straight line drawn through its point of application perpendicularly to the directions of the forces, into two parts, which are to each other reciprocally as the magnitudes or intensities of the composant forces.

166. If the two composant forces be supposed to act in the same direction, the point where the resultant is applied will fall somewhere in the line between them; but if they are supposed to act in contrary directions, the resultant will meet the production of the straight line connecting the points where the forces act.

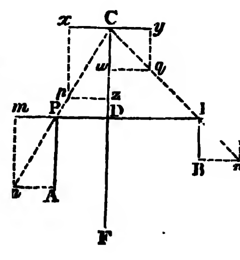
This being premised, we shall now proceed to show, in what manner the magnitude and direction of the resultant are to be ascertained; and first,

SECTION FIRST.

WHEN TWO PARALLEL FORCES ACT AT RIGHT ANGLES TO THE EXTREMITIES OF A STRAIGHT LINE, AND IN THE SAME DIRECTION.

167. Let P and P′, the extremities of the straight line, PP′ be the points where the forces are applied, whose magnitudes and directions are represented by PA and P′B, at right angles to PP′.

Produce PP′ both ways to m amd m'; make Pm of any magnitude whatever, equal to Pm'; then, upon the straight lines PA, Pm, and P′B, Pm', construct the rectangular parallelograms PAnm and P′B$n'm'$; join nP, n'P′, and produce them till they meet in C; through C draw CF parallel to PA or P′B, cutting the line of application PP′ perpendicularly in D; make DF equal to the sum of PA and P′B, then is DF the magnitude or intensity of the resultant sought.



This is evident, for the two forces Pn and P′n', which by composition, are equivalent to the four forces PA, Pm, and P′B, Pm', may be transferred to the point C, and being estimated in the direction of the lines CP and CP′, may be resolved or decomposed into the four rectangular forces, Cz, Cx, and Cw, Cy, which are respectively equal to the components

of Pn and P$'n'$. But by construction, Pm is equal and directly opposite to P$'m'$; consequently Cx is equal and directly opposite to Cy; now, we have elsewhere stated, that when two equal forces are applied to the same point, and act in contrary directions, they destroy one another's effects; therefore, abstracting the equal forces Cx and Cy, we have remaining at the point C, the two forces Cz and Cw, which are respectively equal to the given forces PA and P$'$B; and since they act in the same direction, their united intensities, must be equal to the intensity of a single force whose magnitude is expressed by their sum: that is, according to our principle,

$$\text{DF} = r = \text{PA} + \text{P}'\text{B} \qquad (\text{U})$$

168. Hence then it appears, *that when two parallel forces of given magnitudes, act at right angles to the extremities of a straight line, and in the same direction,* the magnitude of the resultant is determined by simply taking the sum of the given forces; and, because a force of any magnitude, produces the same effect at whatever point of its direction it may be applied, we have supposed the resultant in this case to be applied at the point D, where CF, the line of its direction, cuts PP$'$ the line of application. It is therefore, only requisite to assign the distance of the point D from P and P$'$, the points where the forces PA and P$'$B are supposed to be applied; and, for this purpose,

Let $a=$ PA, one of the composant forces,
 $b=$ P$'$B, the other composant,
 $d=$ PD, the distance of the force PA from the application of the resultant at D,
 $\delta=$ P$'$D, the distance of the force P$'$B,
 $x=$ Pm $=$ P$'m'$, the added force,
and $r=$ DF, the resultant, or single equivalent of the given composants PA and P$'$B.

Then, by reason of the similar triangles, PCD, pCz, we have, by employing the above notation,

$$\text{CD} : d :: a : x;$$

and again, by reason of the similar triangles, P$'$CD, qCw, we have

$$\text{CD} : \delta :: b : x;$$

therefore, by expunging the common terms from each analogy, and equating the products of the remaining terms, we obtain

$$ad = b\delta. \qquad (\text{V})$$

This equation obviously implies, that D, the point of application of the resultant, divides the straight line PP$'$, the distance between the points of application of the composants, into two parts P$'$D and PD, which are reciprocally proportional to the straight lines PA and P$'$B, that represent the magnitudes or intensities of the forces a and b

For if by the rules of algebra, we transform the equation into an expression of ratio, it becomes

$$\frac{a}{\delta} = \frac{b}{d}$$

and if this expression of ratio be again expanded into an analogy, we obtain

$$a : b :: \delta : d; \text{ that is}$$

As the magnitude or intensity of the force a,
is to the magnitude or intensity of the force b
so is the distance of the point of application of the force b,
from the point of application of the resultant r,
to the distance of the point of application of the force a from the same point.

169. This is a true and accurate statement of a reciprocal or inverse proportion, which we have exhibited at length for the express purpose of defining the phrase.

Compounding the preceding analogy, we get

$$a+b : b :: d+\delta : d,$$
$$a+b : a :: d+\delta : \delta;$$

but by our principle or equation (U), $a+b=r$, and if we put D to denote $d+\delta$, or the distance between the points of application of the composants, of which d and δ are the two parts, determined by the position of the point where the resultant is applied, these analogies become

$$r : b :: \text{D} : d,$$
$$r : a :: \text{D} : \delta;$$

and by equating the products of the extreme and mean terms, we have

$$\left.\begin{array}{ll} 1. & rd=b\text{D}, \\ 2. & r\delta=a\text{D}; \end{array}\right\} \qquad (\text{W})$$

These two simple equations are sufficient for determining every particular respecting the composition and resolution of two component forces, acting perpendicularly in the same direction at the extremities of a given straight line.

170. By comparing the terms of the two analogies from which these equations are derived, we get

$$b : a : r :: d : \delta : \text{D};$$

from which we infer *that the two composant and resultant forces, are respectively to each other as the distances included between the points of application of the other two;* and from which inference, as well as from equation (W), it is evident, that if any three of the six terms be given, the others can easily be found, provided always, that the three given terms are not of one kind.

We proceed forthwith to develope the equations, and to illustrate their application by a few appropriate numerical examples; and in order to avoid repetition, it may be proper to remark, that in all the problems and examples immediately succeeding, the forces are supposed to act in the same direction, perpendicularly to the extremities of a straight line, equal in length to the distance between the points of application, the situation of the resultant occurring between these points.

171. PROBLEM 1. *Given the whole distance* D, *between the points of application of the composants; the distance d, between the points of application of the resultant r, and the composant a; with b, the magnitude or intensity of the other composant; to find r, the magnitude or intensity of the resultant.*

No. 1. equation (W) involves these conditions; that is,

$$rd = b\text{D},$$

divide both sides of the equation by d, and we get

$$r = \frac{b\text{D}}{d},$$

this is the equation expressing the value of the resultant r, in terms of the given quantities, b, D, and d; but had the composant a, the the distance δ and the whole distance D been given, the expression would have been

$$r = \frac{a\text{D}}{\delta},$$

an equation identical in form with the foregoing, but involving different quantities; hence, the general practical rule for determining the resultant from these data, is as follows.

172. Rule.—*Multiply the whole distance between the component forces, by the magnitude or intensity of the given force, then divide the product by the distance between the resultant, and the point where the unknown force is applied; and the quotient will give the magnitude or intensity of the resultant required.*

173. EXAMPLE I. Suppose two forces to act simultaneously on two material points in a straight line, situated from each other at the distance of 18 feet; what is the magnitude of the resultant, supposing its point of application to be 3 feet distant from that of one force, whose magnitude is expressed by a weight of 9 tons, and 15 feet from that of the other force, whose magnitude is unknown?

The process as indicated by the equation, and expressed by the rule, is

$$r = \frac{9 \times 18}{15} = 10 \cdot 8 \text{ tons};$$

now, it has been shown in equation (U), that the resultant is equivalent to the sum of the composants; consequently, having determined the magnitude of the resultant, that of the other composant becomes known, that is,

$$10 \cdot 8 - 9 = 1 \cdot 8 \text{ tons.}$$

hence, the three forces are 9, 1·8 and 10·8, and the three distances are 15, 3, and 18 feet; consequently, by the preceding inference, we have

$$9 : 1 \cdot 8 : 10 \cdot 8 :: 15 : 3 : 18;$$

and these by expansion become

$$9 : 1 \cdot 8 :: 15 : 3,$$
$$9 : 10 \cdot 8 :: 15 : 18,$$
$$1 \cdot 8 : 10 \cdot 8 :: 3 : 18.$$

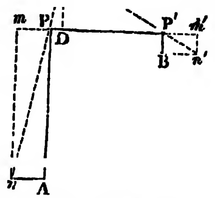
174. Example 2. Suppose a weight of a certain magnitude to be suspended from a straight inflexible bar of wood or iron, at the distance of 10 inches from one extremity, and 90 inches from the other; what is the magnitude of the suspended weight, when it can just be sustained by two men acting at the ends of the bar, supposing the one who acts at the greatest distance to exert a force equal to 15 lbs.

Here we have given $b=15$ lbs ; $d=10$ inches; $\delta=90$ inches, and consequently $\text{D}=100$ inches; therefore, by the rule we have

$$\frac{100\times 15}{10}=150 \text{ lbs. for the resultant, but}$$

it is shown, that $r=a+b$; that is, the resultant is equal to the sum of the forces, consequently, $a=r-b$, that is, $a=150-15=135$ lbs., the force exerted by the man who acts at the least distance from the point of suspension. The geometrical construction of this example may be effected in the following manner :

Make PP′ equal to $d+\delta$, or D equal to 100 inches, from a scale of equal parts; produce PP′ both ways to m and m,′ making Pm and P′m′ equal to one another, and of any convenient magnitude at pleasure. Let P′B represent the given force acting at the point P′, perpendicularly to the bar PP′; on the lines P′B and P′m′ construct the rectangular parallelogram P′Bn′m′ join n′P′, which produce to meet DC the straight line passing through the point of suspension D in the point C; join CP, which produce to n; then through the point m draw mn parallel to P′B or CD, meeting CP produced in the point n, complete the rectangular parallelogram PAnm, then is PA, the magnitude of the other force, applied at the point P, and PA + P′B is the magnitude of the resultant.



175. Problem 2.—*Given the whole distance* D, *between the points of applicatian of the composants ; the distance d, between the points of application of the resultant r and the composant a ; with r, the magnitude of the resultant ; to find the magnitude of b, one of the composants.*

The conditions of this problem are involved in No. 1, equation (W); that is

$$b\text{D}=rd,$$

divide both sides of this equation by D, and we get

$$b=\frac{rd}{\text{D}},$$

this equation determines the value of the composant b, in terms of the given quantities d, D, and r; and the expression for the value of the other composant a, in terms of the remaining quantities δ, D, and r, is

$$a=\frac{rd}{\text{D}}$$

These equations being identical in form, the general practical rule derived from them is as follows:

176. Rule.—*Multiply the magnitude, or intensity of the resultant, by the distance between the point of application, and that of the composant opposite to the one whose magnitude is sought; then, divide the product by the whole distance between the points of application of the composants, and the quotient will be the magnitude of the composant required.*

177. Example 1.—Suppose two forces to act simultaneously on two material points in a straight line, distant from each other 56 inches; what are the magnitudes of those forces, supposing the resultant to be 34 cwt., and its point of application distant from one composant 22, and from the other 34, inches?

The process indicated by the equations and expressed by the rule is

$$\frac{34 \times 22}{56} = 13 \cdot 357 \text{ cwt.}$$

$$\frac{34 \times 34}{56} = 20 \cdot 643 \text{ cwt.}$$

Hence the composant and resultant forces, with the respective distances, exhibited according to the foregoing inference, are as below, viz.

$$20 \cdot 643 : 13 \cdot 357 :: 34 : 22,$$
$$20 \cdot 643 : 34 \cdots\cdots :: 34 : 56,$$
$$13 \cdot 357 : 34 \cdots\cdots :: 22 : 56.$$

178. Example 2. What weights must be attached to the extremities of a straight uniform bar of iron, 36 feet long, to support a weight of 120 tons, suspended at the distance of 6 feet from one extremity and 30 feet from the other?

Here we have given $d=6$ feet; $\delta=30$ feet, and $r=120$ tons; consequently, by the rule to find the force a, we have

$$a = \frac{120 \times 30}{36} = 100 \text{ tons};$$

and to find the force b, we have

$$b = \frac{120 \times 6}{36} = 20 \text{ tons};$$

therefore, in order that the tensions on each side of the beam may be the same, a weight of 20 tons must be suspended from the extremity of the longer arm, and a eight of 100 tons from the extremity of the shorter.

179. Problem 3. *Given the magnitudes of the two composant forces a and b, with* D, *the distance between their points of application, to find the distances d and δ, between each composant and the point where the resultant is applied.*

It has already been shown that the resultant is equal in magnitude to the sum of the two composants; that is, $r=a+b$. Let this

value of r be substituted for it in the first and second of equation (W), and we have

$$(a+b)\,d = b\text{D},$$
$$(a+b)\,\delta = a\text{D}.$$

These two equations involve the conditions of the problem, and if both sides of each be divided by $a+b$, we shall obtain

$$d=\frac{b\,\text{D}}{a+b},$$

for the distance between the point of application of the composant a, and that of the resultant r, and

$$\delta=\frac{a\,\text{D}}{a+b},$$

for the distance between the point of application of the composant b, and that of the resultant r.

Now, these equations being identical in form, the general practical rule derived from them is as follows:

180. Rule.—*Multiply either composant by the distance between them, and divide the product by their sum, for the distance between the resultant, and the composant opposite to that by which the whole distance is multiplied.*

181. Example 1. Two forces, whose magnitudes are respectively represented by weights of 125 and 328 tons, act simultaneously on two material points in a straight line at the distance of 48 feet from each other: at what distance from each must another force be applied, to produce the same effect?

The operation indicated by the equations, and expressed by the rule, is

$$\frac{328\times 48}{125+328}=34\cdot755 \text{ feet},$$

the distance between the composant a and the resultant r, and

$$\frac{125\times 48}{125+328}=13\cdot245 \text{ feet},$$

the distance between the composant b and the resultant r.

Consequently, the composant and resultant forces, with their respective distances, exhibited according to our inference, are as under viz.

$$328 \quad 125::34\cdot755 : 13\cdot245,$$
$$328 \quad 453::34\cdot755 : 48,$$
$$125 \quad 453::13\cdot245 : 48.$$

182. Example 2. Two weights, the one of 20, and the other of 50 tons, are attached to the extremities of a straight inflexible uniform bar of iron, 60 feet long; at what point in the length of the bar ought a weight of 70 tons to be attached, so that the whole may remain at rest?

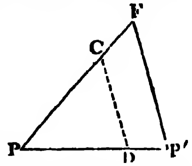
Here, we have given $a=20$ tons; $b=50$ tons, and D, or $d+\delta=$ 60 feet, then by the rule, we have

$$d=\frac{60\times50}{70}=42\tfrac{6}{7}\text{ feet};$$

consequently $\delta=\text{D}-d$; that is, $60-42\tfrac{6}{7}=17\tfrac{1}{7}$ feet; wherefore, it appears that the opposing load must be applied $42\frac{1}{7}$ feet distant from the lesser force, and $17\frac{1}{2}$ feet distant from the greater.

The geometrical construction of this example, may be effected as follows:

Make PP′ equal to 60 feet, the whole length of the bar, or distance between the points of application of the composant forces, and through the point P, draw the straight line PF, making with PP′ any angle whatever; take PF equal to 70 and PC equal to 50 tons; join FP′ and through the point C draw CD parallel to FP′, then is D the point required, and PD, P′D are its distances from the points P and P′, the exremities of the bar at which the forces a and b are supposed to be applied.



183. PROBLEM 4.—*Given the magnitudes of the two composant forces a and b, with d, the distance between the point of application of the resultant r and that of the composant a; to find δ, the distance between the resultant and the other composant b.*

Here, as in the last problem, if we take $a+b=r$, and substitute the sum in the first and second of equation (W), we get, as before

$$(a+b)\,d=b\text{D},$$
$$(a+b)\,\delta=a\text{D}.$$

Now, it is obvious that in the first of these expressions all the quantities are given except D, and in the second, they are all given except D and δ; therefore, if both sides of the first expression be divided by b, we have

$$\text{D}=\frac{(a+b)\,d}{b},$$

and if this value of D be substituted for it in the second expression, it becomes

$$(a+b)\,\delta=\frac{a\,(a+b)\,d}{b},$$

consequently, by expunging the common factor $(a+b)$, we obtain

$$\delta=\frac{ad}{b};$$

if, however, δ had been given instead of d, the expression for the value of d would have been

$$d=\frac{b\,\delta}{a},$$

and as these equations are identical in form, the general practical rule derived from them is as follows:

184. Rule.—*Multiply one of the given composants by the distance of its point of application from that of the resultant, and divide the product by the other composant for the distance sought.*

185. Example 1. The magnitude of two forces are respectively represented by weights of 79 and 112 tons, and the distance of one of them from the point where the resultant is applied, is 27 feet; what is the distance of the other composant from the same point?

Here we have given $a=79$ tons; $b=112$ tons, and $d=27$ feet; therefore, the operation as indicated by the equations and expressed by the rule, is

$$\frac{79\times 27}{112}=19\cdot 0446 \text{ feet},$$

consequently, D, or the whole distance between the composants, is $27+19\cdot 0446=46\cdot 0446$ feet.

But if δ had been given in the example equal to 27 feet, then we should have had for the value of d

$$\frac{112\times 27}{79}=38\cdot 278 \text{ feet};$$

and consequently the whole distance between the composants, is $27+38\cdot 278=65\cdot 278$ feet.

The composant and resultant forces, with their respective distances, exhibited according to our inference for both these cases, are as below, viz.

For the first case, we have

$$112 : 79 :: 27 : 19\cdot 0446,$$
$$112 : 191 :: 27 : 46\cdot 0446,$$
$$79 : 191 :: 19\cdot 0446 : 46\cdot 0446.$$

For the second case, we have

$$112 : 79 :: 38\cdot 278 : 27,$$
$$112 : 191 :: 38\cdot 278 : 65\cdot 278,$$
$$79 : 191 :: 27 : 65\cdot 278.$$

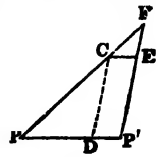
186. Example 2. There are two weights, the one equal to 20, and the other equal to 50 tons, suspended from the extremities of a beam, which rests upon an axis 12 feet distant from the point of application of the greater weight; what must be the distance between the axis and the other weight, so that the system shall remain at rest?

Here, we have given $a=20$ tons; $b=50$ tons, and $\delta=12$ feet; then, by the rule, we have

$$d=\frac{50\times 12}{20}=30 \text{ feet}.$$

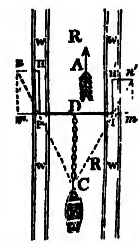
The geometrical construction of this example may be effected as follows:

Draw the straight line PF *at pleasure, make* PC *equal to the greater* force, and CF equal to the lesser; through the point C, draw the straight line CE, making any angle whatever with PF, and through P draw PP′ parallel to CE; make CE equal to twelve feet, the distance between the axis and the greater weight; join FE, which produce to meet PP′ in the point P′; through C, draw CD parallel to P′F; then is the point D the place of the axis, and PD, P′D are its distances from the points P and P′, where the weights are suspended; if the distance PD be taken in the compasses and applied to a scale of equal parts, it will be found to measure exactly 30 feet.



187. The following practical example, in which the solution is conducted graphically as well as numerically, will show the application of the subject on which we have been treating, and tend to establish the reader's confidence in the truth of the conclusions at which we have arrived.

188. EXAMPLE 3. Two horses, one of which is capable of sustaining a dead weight of 650lbs., while the other cannot sustain more than 420lbs.; are applied to a boat for the purpose of maintaining it at rest against the current of a river, whose velocity is such as to give the boat an effort of motion equal to 1070lbs.; now, supposing the horses to operate perpendicularly at the extremities of a pole 60 feet long, stretching directly across the stream; at what point of the pole must the boat be applied, so that the horses may resist a portion of its effort according to their strength?



Let R, R represent the river, running against the direction of the little arrow A; W, W, the towing pathway on each side of the stream, and PP′ the pole to which the horses are attached, by the ropes PH and PH′. Make PP′ equal to 60 feet from a scale of equal parts of any convenient dimensions, and produce it both ways to m and m', making Pm, of any length whatever, and equal to Pm'. At the points P and P′, erect the perpendiculars PH and P′H′, making them respectively proportional to 650 and 420, and complete the rectangular parallelograms PmnH and P′$m'n'$H′; join nP, n'P′ and produce them till they meet in C; from C demit the perpendicular CD, then is D the point in the pole PP″ where the boat B must be applied, and if the distances PD and P′D, be taken in the compasses, and applied to the same scale as that on which PP′, PH and P′H′ were measured, they will be found to measure respectively 23.55, and 36.45 feet.

The numerical solution is effected by the rule to the third of the foregoing problems, and the operation is expressed in the following manner:

$$\frac{420 \times 60}{1070} = 23 \cdot 55 \text{ feet, from P,}$$

$$\frac{650 \times 60}{1070} = 36 \cdot 45 \text{ feet from P}'.$$

And these distances with the corresponding forces, being exhibited according to our inference will stand as below.

$$420 : 650 :: 23 \cdot 55 : 36 \cdot 45,$$
$$420 : 1070 :: 23 \cdot 55 : 60,$$
$$650 : 1070 :: 36 \cdot 45 : 60.$$

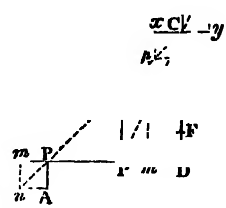
189. What we have hitherto done respecting the composition of parallel forces, applies to the case in which the composants are supposed to act perpendicularly to the extremities of a straight line, and in the same direction; but when they act in opposite directions, the mode of investigation being different, will therefore require a separate section.

Section Second.

When the parallel forces act at right angles to the extremities of a straight line, but in opposite directions to one another.

190. Problem. *Let* P *and* P′, *the extremities of the straight line* PP′, *be the points where the forces are supposed to be applied, whose magnitudes and directions are represented by the straight lines* PA *and* P′B, *at right angles to* PP′, *but lying in opposite directions with respect to it; to determine the magnitude of the resultant, and the particulars of distance with respect to each of the composants.*

Produce PP′ both ways to m and m', and make Pm of any magnitude whatever equal to P′m'; then, upon the straight lines PA, PM and P′B, P′m', construct the rectangular parallelograms PAnm and P′B$n'm'$; join the diagonals nP, P′n', and produce them till they meet in the point C. From C, on PP′ produced, let fall the perpendicular CD, cutting the line PP′ in the point D;— then is D the point of application of the resultant; take DF equal to the difference between P′B, and PA, or the difference between PA and P′B, according as the one or the other is the greater, which in the present figure is P′B; then will DF represent the magnitude or intensity of the resultant, and PD, P′D are its respective distances from the points of application of the composants.



The two forces Pn and P′n', which by composition are equivalent to the four forces PA, Pm and P′B, P′m', may be transferred to the positions Cp, Cq concurring at the point C, and again resolved into the rectangular composants Cz, Cx and Cw, Cy, which are respectively equal to the original composants of Pm and P′n'. But by construction Pm is equal and directly opposed to P′m'; consequently Cx is equal and directly opposed to Cy, therefore they anihilate each other's effects and there remain at the point C, the two forces Cz and Cw, which are respectively equal to the original forces PA and P′B; and since they act in different directions, the effect produced by their united energies, must be equal to the effect of a single force applied at D, whose magnitude is expressed by their difference; that is, according to our principle for the first variety, Art. 165.

$$\mathrm{DF} = r = \mathrm{PA} \backsim \mathrm{P'B},$$

or by using the proper notation, we have

$$r = a \backsim b. \qquad \text{(x)}$$

191. From which it appears, *that when two parallel forces of given magnitudes, act at right angles to the extremities of a straight line, and in different directions*, the magnitude of the resultant is determined by simply taking the difference of the given forces; and because a force of any magnitude, produces the same effect, at whatever point in the line of its direction it may be applied, we have supposed the resultant to be applied to the point D, where CF the line of its direction intersects the production of PP′. It therefore remains to determine the distances of the point D, from P and P′, the points where the composants a and b are supposed to act.

From the similar triangles PCD, pcz, and P′CD, Cqw, we have the following analogies, viz.

$$\mathrm{CD} : d :: a : x,$$
$$\mathrm{CD} : \delta :: b : x;$$

hence, by expunging the common terms CD and x, and equating the products of the means, we have

$$ad = b\delta.$$

This is the identical equation which we obtained for the preceding case, where the composant forces a and b are supposed to act in the same direction; consequently, the remarks which we made under equation (V) are equally applicable here, and therefore they need not be repeated.

192. The expression of ratio deduced from the above equation is,

$$\frac{a}{\delta} = \frac{b}{d},$$

and the expanded analogy is as follows, viz.

$$a : b :: d : \delta.$$

By division of ratios we obtain

$$a \backsim b : b :: d \backsim \delta : d,$$
$$a \backsim b : a :: d \backsim \delta : \delta;$$

but, according to our principle, or by equation (x), we have $a \backsim b = r$, and if we put $\mathrm{D} = d \backsim \delta$; these analogies become

$$r : b :: \mathrm{D} : d,$$
$$r : a :: \mathrm{D} : \delta.$$

Equating the products of the extreme and mean terms, we get

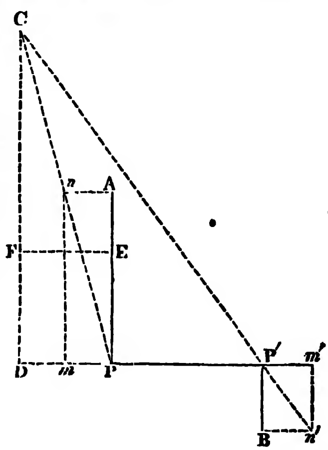
$$rd = b\mathrm{D},$$
$$r\delta = a\mathrm{D}.$$

Here again, the equations coincide exactly in form and symbols with those which we derived for the case in which the forces are supposed to act in the same direction; consequently, the problems, rules, and examples, employed in elucidating equation (W), may be referred to in this case also, and therefore a repetition would be quite superfluous.

It may however be useful to propose a practical example, and resolve it both by construction and calculation, as the coincidence of the results obtained by the two methods, will prove the truth of both.

193. Example 1. Two forces whose magnitudes or intensities are represented respectively by the numbers 42 and 16, are supposed to act perpendicularly at the extremities of a straight lever 36 feet in length, but in opposite directions; required the magnitude of the resultant, and the particulars of distance with respect to each of the composants?

Construction. From a scale of equal parts, take PP′ equal to 36 feet, the distance between the points of application of the forces a and b; at the points P, P′, erect the perpendiculars PA and P′B, making PA equal to 42 and P′B equal to 16, the numbers which represent the magnitudes or intensities of the composant forces, taken from the same scale as the intermediate line PP′. Produce PP′ both ways to m and m', making the parts produced of any convenient length whatever, and equal to each other; complete the rectangular parallelograms PAnm and P′B$n'm'$, and produce their diagonals Pn and n'P′ to meet in the point C; from C, demit the perpendicular CD to meet PP′, produced in the point D; then are PD and P′D, the distances of the points where the forces act from D, the point where the resultant is applied. Take AE equal to P′B, and through E draw EF parallel to PD; then is DF the magnitude or intensity of the resultant; and the lines PD, P′D, and DF being applied to the assumed scale, will be found to measure as follows, viz. PD$=22\cdot154$; P′D$=58\cdot154$; and DF$=26$.



The calculation is effected nearly after the manner exhibited in the third problem preceding, and is as follows:

$$\left.\begin{aligned}(a \backsim b)\,d &= b\mathrm{D}, \text{ or } d=\frac{b\mathrm{D}}{a\backsim b}\\ (a \backsim b)\,\delta &= a\mathrm{D}, \text{ or } \delta=\frac{a\mathrm{D}}{a\backsim b}\end{aligned}\right\} \qquad (\mathrm{Y})$$

These equations give $\mathrm{PD}=\dfrac{16\times 36}{42-16}=22\cdot154$,

and $\mathrm{P'D}=\dfrac{42\times 36}{42-16}=58\cdot154$,

and by equation (X) we have $\mathrm{DF}=42-16=26$.

The calculation may however be conducted otherwise, if we assume a numerical value for the added line $Pm=P'm'$; suppose 14, and put the letter x to denote the whole distance $P'D$, for then we have $PD=x-36$, and by similar triangles, it is

$$14 : 16 :: x : CD,$$
$$14 : 42 :: x-36 : CD;$$

consequently, we have by comparison

$$16x=42x-1512;$$

that is, $42x-16x=26x=1512$, or $x=58\cdot154$, the same as before.

194. EXAMPLE 2. A straight inflexible bar of iron 120 inches in length, is acted upon at the extremities by two parallel forces, one of which exerts a pressure equal to 30 tons, and the other a pressure equal to 20 tons; at what distance from the points where the forces act must the resultant be applied, and what is its magnitude, supposing the forces to act at right angles to the bar, and in opposite directions?

Here we have given $a=30$ tons, $b=20$ tons, and $D=120$ inches, or 10 feet; consequently, by equation (Y), we have

$$d=\frac{20\times120}{30-20}=240 \text{ inches, or } 20 \text{ feet},$$

$$\text{and } \delta=\frac{30\times120}{30-20}=360 \text{ inches, or } 30 \text{ feet};$$

therefore, by equation (X), the value of the resultant is

$$r=30-20=10 \text{ tons}.$$

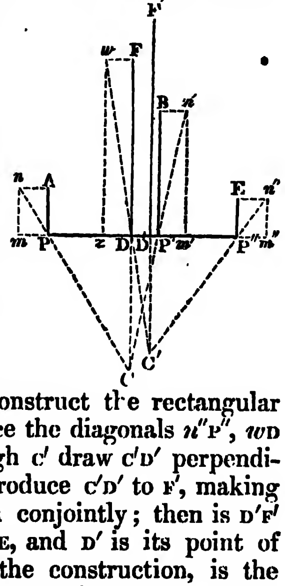
195. What we have hitherto done in this third division of our subject, has reference only to the composition of two forces, which are supposed to act at right angles to the straight line connecting their points of application, whether the directions of those forces lie on the same, or on different sides of that straight line. It is however easy to perceive, that the same principle extends to any number of forces, acting perpendicularly at different points in a straight line, for the resultant is always equal to their sum when they act all in the same direction, or it is equal to the excess of the sum of those which act in one direction, above the sum of those which act in a contrary direction, when it so happens that some of them lie on one side, and some of them on the other side of the straight line which passes through their points of application. This being the case, the magnitude or intensity of the resultant is easily assigned, when the magnitudes of the composants and the circumstances of their actions and directions are known; the chief difficulty then, consists in assigning the point where the resultant ought to be applied, so as to be capable of producing the same effect at that point, as all the given composants can produce, when their effects are considered conjointly, under the proposed circumstunces of action and direction. The method of determining the point of application of the resultant, is by compounding the given forces two and two, after the manner heretofore exhibited.

Section Third.

When from different points of a straight line any number of parallel forces act at right angles to that line.

196. Problem. *Let the straight lines* PA, P′B, *and* P″E, *represent the magnitudes or intensities of the three forces a, b, and c, proposed for composition, and acting perpendicularly at the points* P, P′ *and* P″, *in the straight line* PP″, *and all in the same direction, to determine the point at which the resultant ought to be applied, and also its magnitude or intensity.*

Produce the straight line PP″, the distance between the extreme forces, both ways to m and m'', making Pm equal to P″m'', and of any convenient magnitude whatever; take also P′m' equal to Pm or P″m'', and construct the rectangular parallelograms PAnm, and P′B$n'm'$ and produce the diagonals nP, n'P′ to meet each other in the point C; through C, draw CD perpendicular to PP″, the line of application, and produce CD to F, making DF equal to the sum of PA and P′B taken conjointly; then is DF the resultant of the composants a and b, and D is its point of application.



Again, take Dz equal to Pm, P′m', or P″m'', but directed the same way as Pm, and construct the rectangular parallelograms P″E$n''m''$, DFwz, and produce the diagonals n''P″, wD to meet each other in the point C′; through C′ draw C′D′ perpendicular to PP″, the line of application, and produce C′D′ to F′, making D′F′ equal to the sum of DF and P″E taken conjointly; then is D′F′ the resultant of the two forces DF and P″E, and D′ is its point of application; but DF, by the first step of the construction, is the resultant of a and b, the two forces whose magnitudes are represented by the straight lines PA and P′B; consequently D′F′ is the resultant of the three forces a, b, and c, whose magnitudes are represented by the straight lines PA, P′B, and P″E

197. The above is the general method of finding, by construction, the point of application of the resultant of three parallel forces, acting perpendicularly at three given points on one side of a straight line; but it is persumed that a particular example resolved numerically, will greatly facilitate the reader's conception of the subject, and expand his views with respect to the importance of its application.

198. Example 1. Suppose that the magnitudes or intensities of three forces, acting perpendicularly at three points on the same side of a straight line, are respectively represented by the numbers 12, 22, and 8, while the distances between the middle force and each extreme, are respectively 24 and 16 feet; at what point of the line must the resultant of the three forces be applied?

We have stated already, in the case of two parallel forces acting under the proposed conditions, that

The resultant divides the line of application into two parts, which are to each other, reciprocally as the magnitudes or intensities of those forces; (see our principle.)

Consequently, according to this principle, we shall have

$$12+22 : 24 :: 12 : 8 \cdot 47,$$

$$\text{and } 12+22 : 24 :: 22 : 15 \cdot 53;$$

hence, the resultant of the two forces, whose magnitudes or intensities are represented by the numbers 12 and 22, divides the distance between their points of application into two parts, which are respectively equal to 15·53 and 8·47 feet; that is, the point where the resultant is applied, is 15·53 feet distant from the force whose magnitude is represented by the number 12, and 8·47 feet distant from the middle force, agreeably to the example, whose magnitude is represented by the number 22; the respective distances according to the principle being reciprocally as the magnitudes of the forces.

Again, the resultant of these two forces is equal to their sum, (see equation U); that is, $r = 12 + 22 = 34$; and the distance between its point of application and that of the third composant force is, by the nature of the question, equal to 8·47+16; that is, 24·47 feet; therefore, if we consider the resultant just found as an independent force, and compound it in the same manner with the third composant, our principle gives the following operation, viz.

$$12+22+8 : 8 \cdot 47+16 :: 12+22 : 19 \cdot 81,$$

$$\text{and } 12+22+8 : 8 \cdot 47+16 :: 8 \quad : 4 \cdot 66;$$

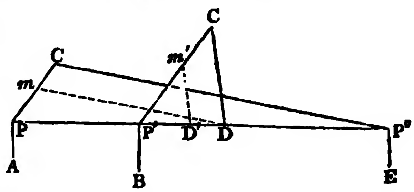
hence it appears, that the point of application of the resultant of the three forces, is distant from that of each force as follows, viz.

20·19 feet from the first extreme, or that whose magnitude is represented by the number 12
3·81 feet from the middle force, or that whose magnitude is represented by the number 22
19·81 feet from the second extreme, or that whose magnitude is represented by the number 8

and since this last distance is greater than that which occurs between the second extreme and the middle force, it indicates that the place of the resultant falls between the middle force and the first extreme; but its situation generally, will obviously depend on the relative magnitudes of the given forces, and the positions which they occupy, but in all practical cases these particulars will be determined from circumstances entirely independent of theory, and consequently no ambiguity can ever enter the solution, for the point of application will always occur, where that power ought to act, which would maintain the system at rest.

199. Example 2. A straight inflexible beam of uniform thickness, and 60 feet in length, has two weights attached to its extremities, the one of 5 and the other of 6 tons; and at the distance of 20 feet from the least weight, is attached another of 8 tons; at what point of its length ought the beam to be supported, in order that the system shall remain at rest?

Let the three forces a, b, and c, be represented in magnitude and direction by the straight lines PA, P′B, and P″E at right angles to PP″ in the points P, P′, and P″, and equal respectively to the numbers 5, 8, and 6, as taken from a scale of equal parts. Make the whole line PP″ equal to 60 feet, the whole distance between the points of application of the extreme weights PA and P″E; also, make PP′ equal to 20 feet, the distance between the place of the least extreme force PA, and that of the middle force P′B.



Through the points of application P and P, draw the straight lines PC and P′C′, making any angles whatever with the line PP″; make Pm equal to P″E, and mC equal to PA; join CP″, and through the point m, draw mD parallel to CP″; then is D the place of the resultant of the two forces PA and P″E.

Again, make P′M′ equal to PC, and m'C′ equal to P′B; join C′D, and through the point m' draw m'D′ parallel to C′D; then is D the place of the resultant of the three forces a, b, and c, whose magnitudes are represented by the straight lines PA, P′B, and P″E, or D is the point in the length of the beam PP″ where it ought to be supported, in order that the system may remain at rest.

200. Put D$=$PP″, the whole distance between the extreme forces; $d=$PP′, the distance between the first extreme and middle forces; and assume x to denote the distance PD, while $x'=$D′D, the distance between the places of the resultants.

Then, by the similar triangles CPP″, and mPD, we have

$$\text{PC} : \text{PP}'' :: \text{P}m : \text{PD}; \text{ that is,}$$

$$(a+c) : \text{D} :: c : x = \frac{c\text{D}}{(a+c)},$$

$$\text{but } \text{P}'\text{D} = x - d = \text{PD} - \text{PP}';$$

$$\text{therefore, } \text{P}'\text{D} = \frac{c\text{D}}{(a+c)} - d = \frac{c(\text{D}-d)-ad}{(a+c)}$$

Again, by the similar triangles C′P′D, and m'P′D′, we have

$$\text{P}'\text{C}' : \text{P}'\text{D} :: m'\text{C}' : \text{D}'\text{D}; \text{ that is,}$$

$$(a+b+c) : \frac{c(\text{D}-d)-ad}{(a+c)} :: b : x' = \frac{bc(\text{D}-d)-abd}{(a+b+c)(a+c)}.$$

Now, $x-x'=$PD$-$D′D$=$PD′; that is

$$\text{PD}' = \frac{c\text{D}+bd}{(a+b+c)}; \qquad (\text{Z})$$

consequently, P″D′$=$D$-$PD′$=$D$-(x-x')$; that is,

$$\text{P}''\text{D}' = \frac{a\text{D}+b(\text{D}-d)}{(a+b+c)} \qquad (\text{A}')$$

201. Let now the numbers proposed in the example be substituted for their appropriate symbols, in the equations (Z) and (A′), and we shall have

$$\text{PD}' = \frac{6\times60+8\times20}{5+8+6} = 27\tfrac{7}{19}\text{ feet,}$$

$$\text{and P}''\text{D}' = \frac{5\times60+8(60-20)}{5+8+6} = 32\tfrac{12}{19}\text{ feet.}$$

We may however deduce a numerical solution, directly from the figure, without any previous investigation, as follows:

$$5+6 : 60 :: 6 : 32\tfrac{8}{11}\text{ feet,}$$

the distance of the resultant of the two extreme forces, from the point of application of the least composant force; therefore,

$$60-32\tfrac{8}{11}=27\tfrac{3}{11}\text{ feet,}$$

the distance of the resultant of the two extreme forces, from the point of application of the other composant force;

$$\text{but } 32\tfrac{8}{11}-20=12\tfrac{8}{11}\text{ feet,}$$

the distance between the place of the resultant of the extreme forces, and the point of application of the middle force; therefore,

$$5+8+6 : 12\tfrac{8}{11} :: 8 : 5\tfrac{75}{209}\text{ feet,}$$

from the place of the first resultant toward the middle force;

$$\text{then } 27\tfrac{3}{11}+5\tfrac{75}{209}=32\tfrac{12}{19}\text{ feet,}$$

from one extremity of the line,

$$\text{and } 60-32\tfrac{12}{19}=27\tfrac{7}{19}\text{ feet, from the other.}$$

202. We shall in the next place give examples of the composition of three forces, when the action of one of them is opposed to the joint action of the other two; and, in order to avoid the introduction of any inconsistency, or any reference to principles that have not previously been expounded, we shall consider the middle force as the opposing one, and those which act in unison will consequently exert themselves at the extremities of the line of application.

Our reason for this limitation is, that we have not as yet made any reference to a centre of resistance, which must of necessity be admitted under any other condition than that which we have specified; that is, when the middle force is made the opposing one.

In the case of two forces, however, this condition cannot have place; consequently a centre of resistance must be implied, otherwise it would be difficult to conceive how two forces could act at different points of a straight line in parallel and opposite directions. Similar remarks will apply to forces acting on a straight inflexible line, considered to be void of gravity or weight, when their directions are parallel, and situated on the same side of the line which joins their points of application, or that on which they are supposed to act.

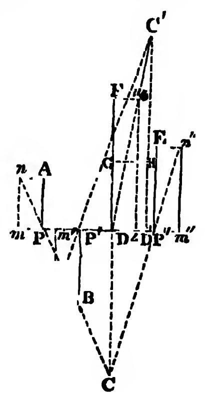
Examples constructed on the principles just expressed will have this good effect; they will shew the reader the *method of continuation*, when *any* number of parallel forces may be proposed to be compounded, and give more weight to the subject we discuss, than scores of marginal authorities could confer on a dry analytical investigation alone.

SECTION FOURTH.

WHEN THE ACTION OF ONE OF THE THREE PARALLEL FORCES IS OPPOSED TO THE JOINT ACTION OF THE OTHER TWO.

203. PROBLEM. *Let the straight lines* PA, P′B, *and* P″E, *represent the magnitudes or intensities of the three composant forces* a, b, *and* c, *which are supposed to act perpendicularly at the points* P, P′, *and* P″, *in the straight line* PP″ *passing through their points of application; and let the forces* a *and* c, *whose magnitudes are represented by* PA *and* P″E, *be supposed to exert themselves in one direction, while the middle force* b, *whose magnitude is represented by the straight line* P′B, *is supposed to exert itself in opposition to the united energies of the extreme forces* a *and* c, *represented in magnitude and direction by the straight lines* PA *and* P″E; *to determine at what point in the straight line the resultant of the three forces must be applied, supposing the middle force to act in opposition to the other two.*

Produce the straight line PP″, which measures the distance between the points of application of the extreme forces, both ways to m and m'', making Pm equal to P″m'', and of any length that is found convenient for the purpose of construction: complete the rectangular parallelograms PAnm, P″E$n''m''$, and produce the diagonals nP and n''P″ to meet each other in the point C; through C, draw CD perpendicular to PP″, the line of application, and produce CD to F, making DF equal to the sum of PA and P″E taken conjointly; then is DF the resultant of the two extreme composants a and c, and D is its point of application.



Again, make P′m' and Dz each of them equal to Pm or P″m'', and complete the rectangular parallelograms P′B$n'm'$, DFwz, and produce the diagonals n'P′ and Dw to meet each other in the point C′; through C′ draw C′D′ meeting PP″, the line of application perpendicularly, in D′; then are PD′ and P″D′, the distances of the points where the extreme forces a and c act, from D′ the point where the resultant is applied, and P′D′ is the distance of b the middle force from the same point.

Take FG equal to P′B, and through G draw GH parallel to PP″; then is D′H the resultant of the three forces a, b, and c, whose magnitudes and directions are represented by the straight lines PA, P′B, and P″E.

The following numerical examples will, it is presumed, suffice to illustrate the preceding construction.

204. EXAMPLE 1. Suppose the magnitudes or intensities of three forces, acting perpendicularly at three successive points in a straight line, are, when taken in order, respectively represented by the numbers 5, 6, and 7, while the distances between the middle force and each of the extreme ones, are respectively 4 and 8 feet; at what point in the straight line must the resultant of the three forces be applied, supposing the middle one to act in a contrary direction, or in opposition to the other two?

Here it is obvious that we must first find the point of application of the resultant of the two extreme forces, or those that act in the same direction; either by the proportion stated in our principle, or by the rule given under the third of the preceding problems, (Art. 180,) and then, consider the resultant as a single force acting at this point, in opposition to the middle one, or that which was supposed to resist the efforts of the other two. Thus will the problem be reduced to the determination of the point where the resultant of two parallel and opposite forces is to be applied.

First then, by the proportion stated in our principle, we have

$$5+7 : 4+8 :: 7 : 7 \text{ feet},$$
$$5+7 : 4+8 :: 5 : 5 \text{ feet};$$

from which it appears, that the point where the resultant of the two extreme forces is applied, is 7 feet from the first and 5 feet from the second; consequently, its distance from the middle opposing force, is $7-4=3$ feet; here then, our question is reduced to the following, viz.

Having the magnitudes of two parallel opposing forces given, respectively equal, or proportional to, the numbers 6 and 12, and the distance between their points of application equal to 3 feet; to find the point of application of the resultant.

This case has been previously resolved, (Art. 181,) and if we express the respective distances in our present diagram, according to the manner there exhibited, we shall have

$$\text{P}'\text{D}' = \frac{12\times3}{5+7-7} = 6,$$
$$\text{DD}' = \frac{6\times3}{5+7-6} = 3;$$

or if we assume any number or symbol, such as unity or f, to represent the equal opposing forces $\text{P}'m'$ and $\text{D}z$, and put x for the whole distance $\text{P}'\text{D}'$, we shall have $\text{DD}'=x-3$, and by similar triangles, it is

$$f : 6 :: x : \text{C}'\text{D}',$$
$$f : 12 :: x-3 : \text{C}'\text{D}';$$

consequently, by comparison, we have

$$12\,x-36=6\,x;$$

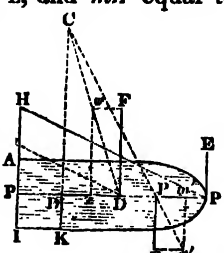
that is, by transposition $12\,x-6x=36$, or $x=6$, and $x-3=3$;

hence we have P′D′=6 feet, and DD′=3 feet, the very same as they were found to measure by the former method. The respective distances of the point where the resultant is applied, from each of the composants, are consequently as follows, viz.

The distance of the point D′ from P	is 7+ 3=10	feet,
———————————— P′	is 3+ 3= 6	ditto,
———————————— P″	is 12−10= 2	ditto.

EXAMPLE 2. At right angles to the extremities of the mast line of a ship, 200 feet in length, are attached ropes, which acting in the same direction, urge the ship towards one side of the bason, with forces of 40 and 50 tons respectively. To one of the masts, which is 50 feet distant from the head of the ship, where the greater force is supposed to act, is attached another rope, which urges her towards the opposite side of the basin with a force of 55 tons, acting in a direction parallel to the other two; at what point in the line of the masts ought a fourth force to be applied, to sustain the ship at rest, and what is the intensity of that force?

Let the shaded figure AP″I represent the upper deck of a ship, of which the line PP″, which divides it into two equal and similar parts is the mast line; P′ the place of the mast, and PA, P′B, and P″E, the given forces acting at right angles to the line PP″ in the points P, P′ and P″.



Produce PA to H, and make P*m* equal to P″E, and *m*H equal to PA; join HP″, and through the point *m*, draw *m*D parallel to HP″, meeting the line of the masts PP″ in the point D; then is D the place of the resultant of the two forces PA and P″E, which act in the same direction at the stern and head of the vessel; through D draw DF parallel and equal to PH, then is DF the resultant of PA and P″E.

Take D*z* equal to P′*m*′ of any convenient magnitude at pleasure, and complete the rectangular parallelograms DF*wz* and P′B*n*′*m*′; join D*w* and *n*′P′, which produce to meet each other in the point C; through C draw CD′ parallel to DF; then is D′ the place of the resultant of the three forces PA, P′B and P″E, and, consequently, D′ is the point at which a fourth force ought to be applied to keep the ship at rest; produce CD′ to K, and make D′K equal to the difference between DF and P′B; then is D′K the intensity of the force required; therefore, the forces P′B and D′K, acting on one side of the line PP″, at the points P′ and D′, will just balance the forces PA and P″E, acting on the other side at the points P and P″.

Put D=PP″, the whole length of the mast line, or distance between the extreme forces; *d*=PP′, the distance between the first extreme and middle forces; and assume x to denote the distance PD; while x'=DD′, the distance between the places of the resultants.

Then by the similar triangles HPP″ and mPD, we have

PH : PP″ :: Pm : PD; that is,

$$(a+c) : \text{D} :: c : x = \frac{c\text{D}}{a+c},$$

but P′D $= d - x =$ PP′ − PD;

therefore P′D $= d - \frac{c\text{D}}{(a+c)} = \frac{d(a+c)-c\text{D}}{(a+c)}$.

Now P′D′ = P′D + DD′; that is,

$$\text{P D}' = \frac{d(a+c)-c\text{D}}{(a+c)} + x' = \frac{(a+c)(d+x')-c\text{D}}{(a+c)};$$

therefore, by the similar triangles CD′P′, and P′Bn', we have

Bn' : BP′ :: D′P′ : D′C; that is

$$\text{B}n' : b :: \frac{(a+c)(d+x')-c\text{D}}{(a+c)} : \text{CD}' = \frac{b\{(a+c)(d+x')-c\text{D}\}}{\text{B}n'(a+c)}.$$

and by the similar triangles CD′D, and wzD, we have

Dz = Bn' : wz :: DD′ : CD′; that is,

$$\text{B}n' : (a+c) :: x' : \text{CD}' = \frac{x(a+c)}{\text{B}n'};$$

consequently, by comparison, we have

$$\frac{x(a+c)}{\text{B}n'} = \frac{b\{(a+c)(d+x)-c\text{D}\}}{\text{B}n'(a+c)}$$

from which equation, by reduction we get

$$\text{DD}' = x' = \frac{b\{d(a+c)-c\text{D}\}}{(a+c)(a+c-b)},$$

to which add the value of P′D, and we have

$$\text{P}'\text{D}' = \frac{d(a+c)-c\text{D}}{(a+c-b)}. \qquad (\text{B}')$$

But PD′ = PP′ − P′D′; that is PD′ $= \frac{c\text{D}-bd}{(a+c-b)}$. (C′)

Therefore, substituting the numbers proposed in the example, for their appropriate symbols, in the equations (B′) and (C′), and we shall have

P′D′ $= \frac{150(40+50)-50\times200}{40+50-55} = 100$ feet from the place of the mast,

PD′ $= \frac{50\times200-55\times150}{40+50-55} = 50$ feet, from the stern of the ship; and consequently P″D′ = 100 + 50 = 150 feet from the head.

205. The magnitudes of the forces, and their distances from one another may be such, as to place the point where the resultant is applied, entirely beyond either extremity of the line connecting the extreme forces, but in practice, the circumstance of the case, aided by an accurate construction, will, in every instance, be sufficient to preclude the admission of error.

We merely mention this, to account for the circumstance of omitting to illustrate the several varieties that may occur, in respect of the equality of magnitude, the equality of distance, and the various other conditions that influence the position of the point where the resultant ought to act, in order that its single effort may be equal to the united efforts of all the composants, or if opposed to them, may be sufficient to maintain the system in a state of quiescence.

206. This suggests to us another construction, and an additional principle, which if properly attended to, will become the means of leading to some very important results in practical mechanics. We anticipated the principle in our remarks at the end of the solution to the example for the composition of three parallel forces, acting in the same direction. It is there stated, "that the point of application of the resultant will always occur where that power ought to act that would maintain the system at rest;" and we now find the statement corroborated by the construction and solution of the case immediately preceding, where the action of one of the composants is opposed to the joint action of the other two. It there appears, that if the resultant of the three forces be produced, and made to act on that side of the line of application where the opposing force acts, and if it be compounded with that force, the resultant will occur in the same line with the resultant of the other two forces, directly opposite, and equal to it in magnitude; which moreover establishes *the principle of equilibrium.**

Section Fifth.

WHEN THE DIRECTIONS OF THE FORCES ARE NOT AT RIGHT ANGLES TO THE LINE OF APPLICATION, BUT OBLIQUE TO IT.

207. We stated at the outset of this division of our subject, the general principle which developes the several conditions for the second variety. That principle we now adopt as a distinct

Proposition. *If two parallel forces act obliquely at two points in a straight line, either in the same or in different directions, their resultant is parallel to them, equal to their sum or difference, and divides the straight line drawn through its point of application, perpendicularly to the*

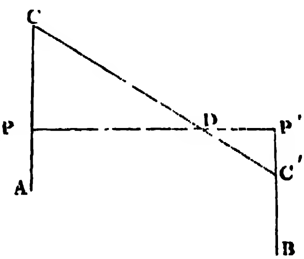
* **Equilibrium** is an equality of weights, powers, or forces of any sort. Bodies at rest are said to be in a state of equilibrium, when they are so balanced by opposing forces as to have no tendency to motion in any direction.

directions of the forces, into two parts, which are to each other, inversely, as the the magnitudes or intensities of the forces.

From the above principle it may be inferred, that the several conclusions at which we have arrived, in establishing the theory of parallel forces, acting at right angles to material points in a straight line, will equally apply in the case of parallel forces acting obliquely; for no element or principle of construction can be at all affected by reason of the obliquity, as will be made manifest by the following demonstration:

208. Let the straight lines PA and P′B, represent the magnitudes or intensities of two forces, acting perpendicularly to the straight line PP′ at the points P and P′. Find D, a point in the line of application PP′, such, that

$$PA : P'B :: P'D : PD;$$



then is D the point where the resultant of the forces PA and P′B is applied.

Through the point D, draw the straight line CC′, anyhow inclined to the line of application, PP′, and produce AP to meet the line CC′ in C; then, by similar triangles, we have

$$C'D : CD :: P'D : PD.$$

Here it is obvious, that the two last terms of each analogy are the same; expunging therefore the common terms, and comparing those that remain, we have

$$PA : P'B :: C'D : CD.$$

Hence it appears, that the point D divides the line CC′, exactly in the same proportion as it divides the line PP′; that is, into parts, which are to each other reciprocally as the magnitudes of the forces PA and P′B. Now, it is evident, that the force PA which is applied at P, in the line PP′, would produce exactly the same effect, if it were transferred to the point C in the line of its direction; and it would be the same with regard to the force P′B, if it were transferred to the point C′; it consequently follows, that D is the point of application of the resultant of the two forces PA and P′B as referred to the line CC′.

Corol. From this it is manifest, that the obliquity of direction has no influence whatever on the principle and method of construction, provided that the composant and resultant forces are always supposed to act in directions that are parallel to one another.

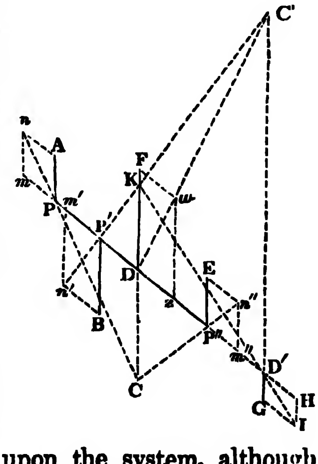
But although, as we have said, there is no difference in the method or principles of construction, occasioned by the obliquity of direction, yet we think it of importance to propose an example in corroboration of our assertion, and, in order that one example may suffice, we shall propose it in such a manner as to embrace

both cases of the problem; that is, when the forces are supposed to act on the same, and on different sides of the line of application.

209. EXAMPLE. Suppose the magnitudes or intensities of three component forces, acting in directions parallel to one another, at three material points in a straight line, are respectively represented by the numbers 15, 24, and 17; the distances between their points of application, reckoned on the oblique line passing through them, being respectively 18 and 44 feet; at what point in that line must the resultant be applied, supposing the middle or greatest force to act in opposition to the other two, and the line of application to cross their directions in an angle of 52 degrees?

This example, except in the value of the data, and the obliquity of the line of application to the directions of the forces, is the same as that immediately preceding, and the intelligent reader will perceive, that the data have been selected in such a manner, as to cause the place of the resultant to fall wholly without the forces. We have purposely made this selection, in order to shew that the determination of the point where the resultant or its opposite equivalent ought to be applied, is of far greater importance, in a practical point of view, than the determination of its magnitude: indeed, in the case of parallel forces, the magnitude of the resultant is known, when the magnitudes of the composants, and the several conditions that modify their energies, are given; consequently, no specific operation is requisite for the determination of its magnitude; the chief object then, is to find the point of application, and for this purpose

Let PA = 15, P′B = 24, and P″E = 17, represent the linear magnitudes of the three composant forces *a*, *b*, and *c*, which are supposed to act obliquely at the points P, P′, and P″, in the straight line PP″, crossing their directions in an angle of 52 degrees; and let the extreme forces *a* and *c*, whose magnitudes are represented by PA and P″E, be supposed to exert themselves in one direction, while the middle force *b*, whose magnitude is represented by P′B, is supposed to exert itself in an opposite direction, at a point 18 feet distant from *a*, and 44 feet distant from *c*.



Produce the straight line PP″ both ways to *m* and *m*″, making P*m* equal to P″*m*″ of any convenient length whatever; then it is evident, that since P*m* and P″*m*″ are equal, and directly opposed to each other, they can have no effect upon the system, although they serve admirably for the purpose of construction.

Complete the oblique parallelograms PA nm, P″E $n''m''$, draw the diagonals Pn, P″n'' and produce them till they meet each other in the point C; through the point of concourse C, draw CD parallel to PA or P″E; then is D the point where the resultant of the two forces PA and PE″ ought to be applied. Produce CD to F, and make DF equal to the sum of PA and P″E taken conjointly; make P′m' and Dz each equal to Pm or P″m'', and complete the oblique parallelograms P′B $n'm'$, DFwz, and produce the diagonals n'P′, Dw, to meet each other in the point C′; through C′, the point of concourse, draw C′D′ parallel to PA, P′B, or P″E, meeting PP″ produced in D′; then is D′ the point where the resultant of the three forces PA, P′B, and P″E ought to be applied, and PD′ $=77\cdot72$, P′D′ $=59\cdot72$, and P″D′ $=15\cdot72$, are its respective distances from the points where the several forces act.

Produce C′D′ to G, and PD′ to H, making D′H equal to P″m'', and D′G equal to the difference between DF and P′B; complete the oblique parallelogram D′GIH, and produce the diagonal ID′ to meet n'P′ produced in K; then will K be a point in the line DF; consequently, the resultant of the two forces P′B and D′G is equal and directly opposite to the resultant of the two forces PA and P″E; therefore, if a force D′G, equal to the resultant of the three forces PA, P′B, and P″E, be applied at the point D′, as determined by the construction, and all the four forces act simultaneously at their respective points, the whole system will remain at rest.

210. The numerical operation is precisely the same as that employed in the preceding example, and is as follows:

$$15+17 : 18+44 :: 17 : 32\cdot93 \text{ feet},$$
$$15+17 : 18+44 :: 15 : 29\cdot07 \text{ feet};$$

hence it appears, that the point where the resultant of the extreme forces PA and P″E is applied, is 32·93 feet distant from the first, and 29·07 feet distant from the second.

Let f represent the subsidiary force Pm, P′m', or P″m'', and put x for the distance P′D′, between the point of application for the middle force, and the resultant of all the three; then, we shall have DD′ $=x-14\cdot93$; therefore, by similar triangles, we get

$$f : 24 :: x : \text{C′D′},$$
$$f : 32 :: x-14\cdot93 : \text{C′D′};$$

consequently, by comparison, we have

$$24\,x = 32\,x - 477\cdot76;$$

that is, by transposition, $32\,x - 24\,x = 477\cdot76$, or $x = 59\cdot72$; hence, the several distances of the point D′ from the forces are as below, viz.

The distance of the point D′ from P is, $59\cdot72+18=77\cdot72$ feet,
———————————————P′——————— $=59\cdot72$ —,
———————————————P″— $59\cdot72-44=15\cdot72$ —.

211. Such, then, is the method of assigning the point of application of the resultant of three parallel forces, acting in the same plane, whether their directions are perpendicular or oblique to the line of application, or whether they act all in one way or in different ways; and it is easy to perceive, that the same principle of composition will apply to any number of parallel forces, but the operation becomes more tedious, and the diagrams more complex, as the number increases; for which reason, we have thought proper to limit our inquiries to three composants only, presuming that the reader will be enabled to extend the principle which we have employed to any number of forces whatever, whether they act on the same, or on different sides of the connecting line; the following precept, however, may be useful, when there are several forces acting in opposition to one another, on each side of the line of application.

212. GENERAL RULE FOR THE SOLUTION OF SEVERAL FORCES ACTING IN OPPOSITION TO ONE ANOTHER, ON EACH SIDE OF THE LINE OF APPLICATION.

RULE.—*Let all the forces acting in unison, on one side of the line of application, be compounded; then all those acting in opposition on the other side; and, lastly, compound the resultants arising from the composition of the opposite forces, and the point where this resultant meets the line of application, will be the point required in the question.*

213. We shall here anticipate an objection that may be made to our manner of treating the latter part of this case; that is, in having abandoned the algebraic mode of developement for that which is exclusively geometrical. This objection, although just, is very easily answered. In the first place, we were unwilling to crowd our pages throughout with algebraical formulæ; and, in the second place, the varieties and examples bear so near a resemblance to one another, that the formulæ and rules derived from the preceding cases, would, with very little or no alteration, apply here; in fact, the numerical operations which we have exhibited in the solutions, are the direct appliances of the algebraic indications, or they rather shew the method by which such expressions were obtained. This being the case, we choose rather to depart a little from our original plan, than to bring forward theorems and rules that have been exhibited and drawn up in words at length in another place.

We are now arrived at the last, and least important, division of our subject, which we shall endeavour to discuss as briefly as possible, by reason of its small importance in the theory of elementary mechanics.

CASE IV.

WHEN THE COMPOSANT AND RESULTANT FORCES ARE SITUATED IN DIFFERENT PLANES, AND DIRECTED TO DIFFERENT POINTS OF A BODY.

214. In this case, which is the fourth and last branch of our subject, according to the order of division, there are, as in the preceding, two varieties, viz.

1. *When the forces act all in one direction.*
2. *When some of the forces act in one direction, and some of them in a contrary direction.*

Here, also, the forces may be conceived to act either at right angles, or obliquely to the lines connecting their points of application; but, to avoid prolixity in the solution, we shall suppose the directions of all the forces to be parallel to one another, whether they act at right angles or obliquely to the connecting lines, and whether they act all in one or in different directions.

In this case it is obvious, that there cannot be less than three composant forces concerned in the inquiry, for, in whatsoever manner two forces are applied, a plane can always be made to pass along their directions, and the resultant of any two forces is always in the same plane with themselves; therefore, in order to have the forces situated in different planes, it becomes necessary to add a third force to the system, and then they may be considered as being situated two and two in the same plane, and the three planes in which they act will constitute the sides of a triangular prism, of which one end is the plane passing through the points of application, and the three edges are respectively the directions of the forces.

SECTION FIRST.

WHEN THE FORCES ACT ALL IN ONE DIRECTION.

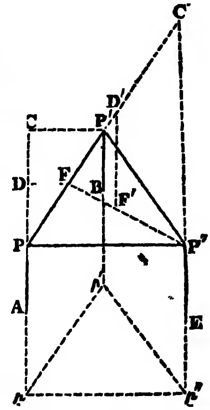
215. When the forces act at right angles to their connecting lines, the figure formed by planes passing along their directions is a right prism; but when they act obliquely, the figure so formed is an oblique prism; in both cases, however, the magnitude or intensity of the resultant, and its point of application, will be the same; the only effect of the obliquity being to change the direction of the resultant into that of the general direction of the composants.

The annexed diagram will explain the nature of the forces, acting in the manner implied above, and the principle on which the solution depends is the same as that given for the preceding case, and which we have expressed in the following

216. PROPOSITION. *The magnitude or intensity of the resultant is equal to the sum of the composants, when they act all in one direction, or to the excess of the sum of those acting on one side of the plane passing through their points of application, above the sum of those acting on the other side*

of it, when they act in contrary directions; and the point of application of the resultant of any two of the forces, divides their connecting line into two parts, which are to each other reciprocally as the magnitudes or intensities of the forces.

Let the parallel straight lines PA, P′B, and P″E, anyhow posited in space, represent the magnitudes of the three composant forces a, b, and c, whose resultant is required.



Connect the points P, P′, and P″, by the straight lines PP′, P′P″, and PP″; parallel to which, at any convenient distance, draw their equivalents pp', $p'p''$, and pp'', and produce PA, P′B, and P″E, to meet the plane $pp'p''$ in the points p, p', and p''; then does the prism PP′P″$pp'p''$ unfold the action of the parallel forces a, b, and c, whose magnitudes are respectively represented by the straight lines PA, P′B, and P″E.

If the straight lines PA, P′B, and P″E, are each of them perpendicular to the plane PP′P″, in the points of application P, P′, and P″, they shall also be perpendicular to the plane $pp'p''$, in the points p, p', and p'', and therefore the figure is a right prism, the edges of which Pp, P′p', and P″p'', represent the directions of the forces, and the end or base PP′P″ is the plane of application, the boundaries of which are the connecting lines PP′, P′P″, and PP″.

If, however, the lines PA, P′B, and P″E, instead of being perpendicular to the plane PP′P″, are oblique to it, they are also oblique to the opposite plane $pp'p''$, and the figure is consequently an oblique prism; but its edges do nevertheless indicate the directions of the forces, and its end or base PP′P″ is the plane of application; consequently, whether the forces act at right angles or obliquely to the connecting lines, the figure which unfolds the circumstances of action is a prism, in whose end or base the position of the resultant must occur. This being premised, we shall now proceed to shew the method of construction.

Produce AP to C, make PD equal to P′B, and DC equal to PA; join CP′, and through the point D draw DF parallel to CP′; then is F the point where the resultant of the two forces a and b, whose magnitudes are represented by PA and P′B, ought to be applied.

Again, produce PP′ to C′, make FD′ equal to P″E and D′C′ equal PC, or equal to the sum of PA, P′B taken conjointly; join C′P″, and through the point D′ draw D′F′ parallel to C′P″; then is F′ the point where the resultant of the three forces a, b, and c, whose magnitudes are represented by the straight lines PA, P′B, and P″E, ought to be applied; and the point F′ is called

The centre of the parallel forces.

The three composant forces, being, as heretofore, denoted by the three letters of the alphabet, a, b, and c,

Put $d=\text{PP}'$, the straight line connecting the points of the forces a and b,

$\delta=\text{PP}''$, the straight line connecting the points of the forces a and c;

$n=\text{PF}$, the distance of the point of the first resultant* from the point of the force a,

$x=\text{FF}'$ the distance between the points of the first and second resultants;

$\varphi=\text{P}'\text{PP}''$, the angle contained by the straight lines PP' and PP'', connecting the point of the force a with the points of the forces b and c.

Then, by the proportion specified in our principle, or by the similar triangles PCP' and PDF, we have

$$a+b : b :: d : n;$$

or, by equating the products of the mean and extreme terms, and dividing by the coefficient of n, we obtain

$$n=\frac{bd}{(a+b)}. \qquad (\text{D}')$$

And by Plane Trigonometry, we have

$$\text{FP}''=\sqrt{n^2+\delta^2\pm 2\,n\delta\cos.\varphi};$$

and, again, by the proportion specified in our principle, or by the similar triangles $\text{FC}'\text{P}''$ and $\text{FD}'\text{F}'$, we have

$$(a+b+c) : c :: \sqrt{n^2+\delta^2\pm 2\,n\delta\cos.\varphi} : x;$$

or by equating the products of the mean and extreme terms, and dividing by the coefficient of x, we get

$$x=\frac{c\sqrt{n^2+\delta^2\pm 2\,n\delta\cos.\varphi}}{(a+b+c)};$$

if, for n and n^2 in this equation, we substitute $\frac{bd}{(a+b)}$ and $\frac{b^2d^2}{(a+b)^2}$ as computed above, we finally obtain for the value of x,

$$x=\left\{\frac{c\sqrt{\frac{b^2d^2}{(a+b)^2}+\delta^2\pm\frac{2\,bd\delta}{(a+b)}\cos.\varphi}}{(a+b+c)}\right\}. \qquad (\text{E}')$$

Where we have to observe, that when φ is obtuse, the upper sign of the term involving $\cos.\varphi$ must be employed; but when φ is acute, we must make use of the lower or negative sign.

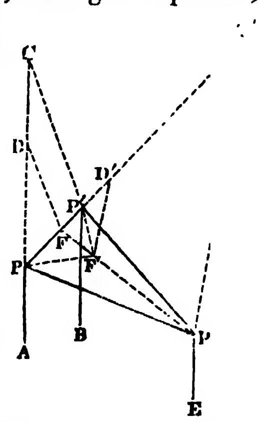
* *The first resultant* is that which is determined from the two composants a and b, and *the second resultant* is that which is determined from the three composants a, b, and c.

The equation marked (E′) expresses generally the distance between the points of the first and second resultant; but since the expression is a little complicated, it may perhaps be useful to resolve an example, both by construction and numerically, in order that the reader may perceive the relation of the two methods, and how the one serves to check and verify the other.

217. EXAMPLE 1. Suppose that the magnitudes or intensities of three parallel forces, acting simultaneously at right angles to three material points in the same plane, but not in the same straight line with one another, are represented respectively by the numbers 12, 19, and 10; at what point in the plane will the resultant, or centre of the parallel forces, occur, supposing the distance of the force a from the forces b and c, to be respectively 13 and 29 feet, and that those distances are inclined to each other in an angle of 67° 56′?

Let the three parallel straight lines PA, P′B, and P″E, which represent the magnitudes or intensities of the forces a, b, and c, be made respectively equal to 12, 19, and 10, and let PP′ equal to 13, and PP″ equal to 29, be inclined to each other in an angle of 67 degrees 56 minutes.

Produce the straight line AP to the point C, making PD equal P′B, and DC equal to PA; that is, equal respectively to the numbers 19 and 12; join CP′, and through D draw DF parallel to CP′, and join FP″. Produce FP′ to C′, making FD′ equal to P″E and D′C′ equal PC; that is, equal respectively to the numbers 10 and 12+19=31; join C′P″, and through D′ draw D′F′ parallel to C′P″ meeting P″F in the point F′; then is the point F′ the place of the resultant, or the centre of the parallel forces; join PF′ and P′F′, then will PF′, P′F′, and P″F′, be the distances of the three forces, respectively, from the point F′, or place of the resultant.



218. The following is the mode of calculation from equation (E′), viz.

$$2\,bd\delta = 2\times 19\times 13\times 29 = 14326,$$

$$2\,bd\delta\ \cos.\varphi = 14326\times \cdot 37569 = \ 5382\cdot 13494,$$

$$\left(\frac{2\,bd\delta}{a+b}\right)\cos.\varphi = \frac{5382\cdot 13494}{12+19} = 173\cdot 29467;$$

$$\delta^2 = \ 29\times 29 \ = 841,$$

$$b^2\,d^2 = (19\times 13)^2 = 61009,$$

$$(a+b)^2 = (12+19)^2 = 961,$$

$$\frac{b^2\,d^2}{(a+b)^2} = \frac{61009}{961} = 63\cdot 4745;$$

$$\frac{b^2 d^2}{(a+b)^2}+\delta^2=63\cdot4745+841=904\cdot4745,$$

$$c\sqrt{\left(\frac{b^2 a^2}{(a+b)^2}\right)+\delta^2-\frac{2bd\delta}{a+b}\cos.\varphi}=10\sqrt{904\cdot4745-173\cdot29467}=270\cdot4;$$

consequently we have $x=\frac{270\cdot4}{41}=6\cdot6$ feet, very nearly.

The whole length of the line FP″ is expressed by the combination under the radical sign in the numerator of the fraction in equation (E′), and is as follows, viz. :—

$$\text{FP}''=\sqrt{\frac{b^2d^2}{(a+b)^2}+\delta^2-\left(\frac{2bd\delta}{a+b}\right)\cos.\varphi.}=\sqrt{904\cdot4745-173\cdot29467}=27\cdot04 \text{ feet;}$$

consequently, F′P″ $=27\cdot04-6\cdot6=20\cdot44$ feet;
and moreover, the distance PF, by equation (D′), is

$$\frac{bd}{(a+b)}=\frac{19\times13}{12+19}=7\cdot97 \text{ feet;}$$

hence, the distances PF′ and P′F′ can easily be found.

If the lines PA, P′B, and P″E, instead of being at right angles to the plane PP′P″, were anyhow inclined to it, but still retaining their parallelism, the magnitude of the resultant FC′, and its point of application at F′, would, notwithstanding, be the same; for it is obvious, since the angles which the directions of the forces make with the plane of application, do not enter the general formula, the obliquity can have no influence whatever on the point of the resultant, and its magnitude must be equal to the aggregate of the composants, whether they act at right angles or obliquely to the plane; it is, therefore, unnecessary to give an example of oblique forces, because its solution would present us with no variety, it being effected by the same equation and in the same manner as that immediately preceding.

The distances PF and FF′, can be deduced immediately from the figure, without reference to the equation which expresses those distances; thus,

By construction, PC is equal to the sum of PA and P′B taken conjointly; the part PD being equal to P′B and DC equal to PA; then, by similar triangles, we have

$$19+12 : 13 :: 19 : 7\cdot97=\text{PF};$$

then, in the triangle FPP″, we have given the two sides PF, PP″, and the contained angle, to find the side FP″ opposite to the given angle.

By Plane Trigonometry, we have

$$\text{FP}''=\sqrt{7\cdot97^2+29^2-2\times7\cdot97\times29\times\cdot37569}=27\cdot04;$$

and again, by similar triangles, we have

$$10+19+12 : 27\cdot04 :: 10 : 6\cdot6=\text{FF}', \text{ the same as before.}$$

219. In the preceding example, we have supposed the angle in the plane of application, contained by the lines PP' and PP'' as being acute, in which case, the term involving the cosine of that angle is subtractive; if, however, the value of φ had been obtuse, the steps of the operation would have been the same as above, only the term involving cos. φ, would in that case be additive; thus,

Let the data remain with the exception of the angle φ, which take equal to the supplement of $67° 56'$, or $112° 4'$· then its cosine is $-\cdot 37569$, and

$$FP' = \sqrt{7\cdot 97^2 + 29^2 + 2 \times 7\cdot 97 \times 29 \times \cdot 37569} = 32\cdot 83;$$

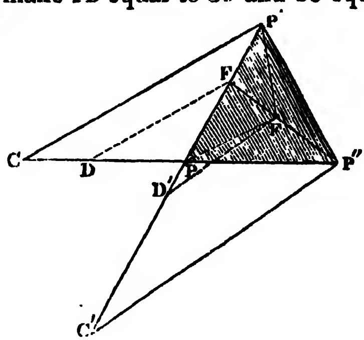
then, by similar triangles,

$$10+19+12 : 32\cdot 83 :: 10 : 8\cdot 01 \text{ feet nearly.}$$

220. Example 2.—Suppose that to the angular points of a board, in the form of an equilateral triangle, each of whose sides is 12 feet, there are attached weights of 20, 30, and 40lbs., at what distance from the angular points must the board be suspended, in order that its plane may be horizontal?

Let $PP'P''$ be the equilateral triangular board, at whose angles the weights P, P' and P'' are applied, respectively equal to 20, 30, and 40lbs., and of which it is required to find the point F, where, by being suspended, the plane of the board shall be horizontal.

Produce the side $P''P$ to C, and make PD equal to 30 and DC equal to 20, taken from any scale of equal parts, either the same or different from that on which the sides of the triangle are measured; join CP', and through the point D, draw DF parallel to CP', meeting the side of the board in the point F, then is the point F the place of the resultant of the two forces P and P'; join $P''F$ and produce the side $P'P$ to C'; make FD' equal to 40 and $D'C'$ equal to $20+30=50$, and join $C'P''$; through the point D', draw $D'F'$ parallel to $C'P''$; then is F' the point in the plane of the board, by which it ought to be suspended, in order that the weights P, P' and P'' may sustain it in a horizontal position.



Put $D = PP'$, the side of the equilateral triangle,

$d = PF$, the distance of the force at P from F, the place of the resultant of the two forces P and P'.

and $\delta = P'F$, the distance of the force at P' from the place of the resultant, and let a, b, and c, represent the forces or weights applied at the points P, P', and P'' respectively.

Then by the third problem to the equation marked (w, at p. 98), we obtain

$$d=\frac{b\text{D}}{(a+b)}, \text{ and } \delta=\frac{a\text{D}}{(a+b)};$$

but, it was shewn in the last example that the line P″F, is expressed by the following equation, viz.

$$\text{P}''\text{F}=\sqrt{\frac{a^2\,\text{D}^2}{(a+b)^2}+\text{D}^2-\frac{2\,a\text{D}^2}{(a+b)}\cos.\ \varphi},$$

and because the triangle is equilateral, $\varphi=60^\circ$, and its cosine $=\frac{1}{2}$; consequently, by substitution and reduction, we get

$$\text{P F}=\frac{\text{D}}{(a+b)}\sqrt{\left(\frac{a^3 \backsim b^3}{a \backsim b}\right)};$$

then, if we conceive the forces a and b, to be a single force applied at the point F, the same problem to equation (w), gives

$$\text{P}''\text{F}'=\frac{\text{D}}{(a+b+c)}\sqrt{\left(\frac{a \backsim b^3}{a \backsim b}\right)}.$$

Therefore, if the numbers proposed in the example, be substituted for their appropriate symbols in the above equation, we have

$$\text{P}''\text{F}'=\frac{12}{20+30+40}\sqrt{\frac{30^3-20^3}{30\ -20}}=5\cdot 812 \text{ feet};$$

and the place of the point is consequently determined; and if the board be anyhow supported at the point F′, the whole will remain at rest in a horizontal position.

221. If the forces a, b, and c are equal to one another, and φ less than a right angle, then equation (E′), becomes

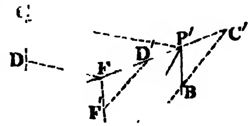
$$x=\frac{1}{6}\sqrt{d^2+4\delta^2-4d\delta\cos.\ \varphi}. \qquad (\text{F}')$$

In which equation no force occurs; consequently, the magnitudes of the lines connecting the force a with the forces b and c, (or rather the points where the forces are applied), together with the angle of their inclination, are sufficient to determine the place of the resultant; for F, the point of the first resultant, obviously falls in the middle of the line PP′, and F′, the point of the second resultant, must, for the same reason, occur at one third of the line FP″ distant from F. This will become manifest from the following construction.

222. Let the straight lines PA, P′B, and P″E, which represent the magnitudes of the forces a, b, and c, be equal and parallel to one another, and let the angle P′PP″ contained by the straight lines PP′ and PP″, be less than a right angle.

Produce the straight line AP to C, making PD equal to DC, and each of them equal to PA, P′B, or P″E, which are by hypothesis all equal to each other; join CP, and through D, draw DF parallel to

CP′, meeting PP′ in F; then is F the point of the first resultant, or that of the two forces PA and P′B; and because PD is by construction equal to DC, it follows, that PF is equal to FP′; consequently F is the middle of the line PP′.



Join FP″, and produce FP′ to C′. making FD′ equal to P″E, and D′C′ equal to PC; that is, FC′ is equal to PA, P′B, and P″E taken conjointly, or FD′ is one third of FC′; join C′I and through D′ draw D′F′ parallel to C′P″; then is F′ the point of the second resultant, and FF′ is equal to one third of the line FP″.

Hence this simple construction, without taking into the account any of the forces: viz.

Bisect the straight line PP′ in the point F; join FP″, which trisect in the point F′; then is F′ the place of the resultant sought.

This is the construction indicated in equation (F′), which the following example shews the method of reducing.

223. EXAMPLE. Suppose the distance of the point of application of the force a, from the points of application of the forces b and c, to be respectively 35 and 32 feet, and the angle contained by the straight lines connecting those points, to be $57°$; at what point in the plane of application must the resultant occur?

Here we have $d=35$ feet, $\delta=22$ feet, and $\varphi=57°$, its natural cosine is $\cdot 54464$; consequently, by substituting these numbers in equation (F′), we get

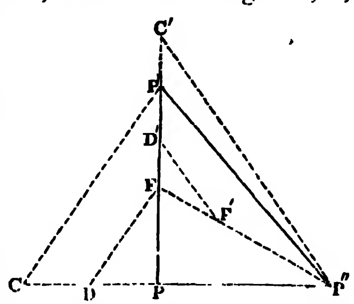
$$FF'=x=\tfrac{1}{6}\sqrt{35^2+4\times 22^2-4\times 35\times 22\times \cdot 54464}=6\cdot 42 \text{ feet}.$$

224. If the forces a, b, and c, are equal to one another, and φ a right angle; then cos. $\varphi=o$, and equation (F′), becomes

$$x=\tfrac{1}{6}\sqrt{d^2+4\delta^2} \quad . \qquad (G')$$

The construction of this case is exceedingly simple, and is as follows:

Let P′PP″ be the plane of application, of which the angles P, P′, and P″, are the points where the forces act; PP′ and PP″ perpendicular to each other, being respectively the distances of the point P from the points P′ and P″, or the distances of the force a from the forces b and c.



Produce P″P to the point C, making PD equal to DC; join CP′, and through the point D, draw DF parallel to CP′, meeting PP′ in F; or, which is the same thing, bisect PP′ in F and join FP″: produce FP′ to the point C′, making F′D equal to PD,

and D′C′ equal to PC; that is, make FC′ equal to three times FD′; join C′P″, and through the point D′ draw D′F′ parallel to C′P,″ meeting FP″ in F′; then is F′ the point in the plane, where the resultant of any three equal forces, acting in directions parallel to one another at the points P, P′, and P″, ought to be applied.

This is the construction indicated in equation (G′), and the numerical operation deduced from the same source, is as under; supposing PP $=30\frac{1}{2}$ feet, and PP″ $=27\frac{1}{2}$ feet; then we get

$$\text{FF}'=x=\tfrac{1}{6}\sqrt{30\cdot5^2+4\times27\cdot5^2}=10\cdot48 \text{ feet, nearly.}$$

225. If the forces a, b, and c, are equal to one another, and φ greater than a right angle, then equation (F′), becomes

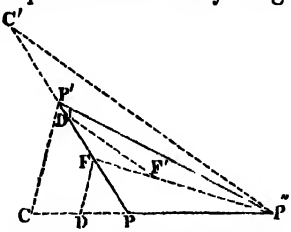
$$x=\tfrac{1}{6}\sqrt{d^2+4\delta^2+4d\delta\cos.\varphi.} \qquad (\text{H}')$$

This is the same as equation (F′) foregoing, with the exception of the sign of the last term under the radical expression, which, on account of φ the angle of inclination being obtuse, becomes positive, whereas, when φ is acute, the same term is negative.

The construction of this case is as follows.

Let P′PP″ be the plane passing through the points of application P, P′, and P″, of the three forces a, b, and c; and let PP′ and PP″ be the distances of the force a from the forces b and c; the angle P′PP″ being greater than a right angle.

Produce P″P to the point C, making PD equal to DC of any magnitude whatever, and join CP′; through D, draw DF parallel to CP′ meeting PP′ in F; join FP″, and produce FP′ to the point C′ making FD′ equal to PD, and D′C′ to PC; or, which is the same thing, make FC′ equal to three times FD′; join C′P″, and through D′, draw D′F′ parallel to C′P″ meeting FP″ in the point F′; then is F′ the point in the plane of application, where the resultant of any three equal forces, acting in parallel directions at the points P, P′ and P″, ought to be applied, in order to produce, by its single effort, the same effect as the three composants acting simultaneously at their respective points.



226. The preceding is the construction indicated by equation (H′) and the calculation is simply as follows: viz.

Suppose the lines PP′ and PP″, which connect the point of application of the force a, with the points of application of the forces b and c, to be respectively 21 and 23 feet, and the angle P′PP″ contained between them 126° 30′; at what point in the plane of application will the resultant obtain?

Here we have $d=21$ feet, $\delta=23$ feet, and $\varphi=126° 30'$; its natural cosine is, $-\cdot 59482$; consequently, by substituting these numbers in equation (H′), we get

$$\text{FF}'=x=\tfrac{1}{6}\sqrt{21^2+4\times 23^2+4\times 21\times 23\times\cdot 59482}=10\cdot 15 \text{ feet,}$$

very nearly.

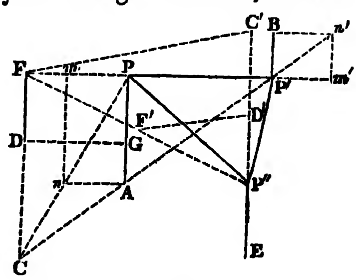
227. What we have hitherto done, has respect only to the case in which the forces are supposed to act all in one direction, on the same side of the plane passing through their points of application; this is, of course, the most important case, and consequently that which requires the greatest share of the reader's attention; but in order that nothing may be wanting to elucidate the subject, we shall, before bringing it to a conclusion, point out the method of assigning the point of the resultant, when the composants are supposed to act in contrary directions, on the opposite sides of the plane of application.

Section Second.

WHEN SOME OF THE FORCES ACT IN ONE DIRECTION, AND SOME OF THEM IN A CONTRARY DIRECTION.

228. Problem. Let the parallel straight lines PA, P′B, and P″E, represent the magnitudes or intensities of the forces a, b, and c, acting at the points P, P′, and P″, in the plane PP′P″, and let the two forces a and c, represented by the straight lines PA and P″E, be supposed to act in one direction on the same side of the plane, while the force b, represented by the straight line P′B, is supposed to act in a contrary direction on the opposite side; the directions of all the forces being parallel to each other, and either at right angles or oblique to the plane of application, to find the point of the resultant.

Connect the points P, P″, and P′, by the straight lines PP′, PP″ and P′P″, and produce the straight line PP′ both ways to m, m', and make Pm equal to P′m', of any magnitude whatever, that may be found convenient for the purpose of construction; complete the parallelograms PAnm, P′B$n'm'$, and produce the diagonals Pn, n'P′ to meet each other in the point C; through the point C draw CF parallel to PA or P′B, meeting Pm produced in F; then is F the place of the first resultant, or that of the two composants a and b, acting in opposition to each other at the points P and P′.



Make AG equal to P′B, and through G draw GD parallel to P′P meeting CF in D; then does DF represent the magnitude of the resultant of the two forces PA and P′B; which, because PA is by assumption greater than P′B, falls on the same side of the plane of application as the remaining force P″E.

Produce EP″ to C′, and make P″D′ equal to DF and D′C′ equal to P″E; join C′F, and through D′, draw D′F′ parallel to C′F meeting FP′ in the point F′; then, is F′ the place of the second resultant, or the resultant of the three forces a, b, and c, whose magnitudes or intensities are represented by the straight lines PA, P′B, and P″E.

229. EXAMPLE. If PP′ $=23\cdot3$ feet, PP″ $=25$ feet, and the angle P′PP″ $=\varphi=40°$; then the numerical calculation of the point F′ will stand as under, the magnitudes of the forces being respectively equal to the numbers 17, 7, and 12.

Let f be taken to represent the subsidiary force $\text{P}m=\text{P}'m'$, and assume x equal to the distance P′F; then will PF $=x-23\cdot3$, and by the similar triangles P′Bn', P′FC and Pmn, PFC, we have

$$f:\ 7::x:\text{CF},$$
$$f:17::x-23\cdot3:\text{CF};$$

expunge the common terms, and by comparison we obtain

$$7x=17x-396\cdot1;$$

or by transposition, we have

$$10x=396\cdot1,\text{ or } x=39\cdot61\text{ feet};$$

consequently $x-23\cdot3=39\cdot61-23\cdot3=16\cdot31$ feet, for the produced distance from P to F, the place of the first resultant.

By Plane Trigonometry, we have

$$\text{FP}''=\sqrt{16\cdot31^2+25^2+2\times16\cdot31\times25\times\cdot766}=38\cdot93\text{ feet.}$$

But by construction

$$\text{P}''\text{C}'=\text{DF}+\text{P}''\text{E};\text{ that is, P}''\text{C}'=17-7+12=22\text{ feet};$$

then by the similar triangles P″FC′ and P″F′D′, we have

$$22:38\cdot93::17-7:17\cdot7=\text{P}''\text{F}',$$

the distance of the place of the second resultant, or that of the three given composants, from the point P″; consequently,

$$38\cdot93-17\cdot7=21\cdot23=\text{FF}',$$

the distance between the place of the first and second resultants; the distances of the point F from P and P′, the places of the two remaining forces a and b, are easily found.

Such then is the DOCTRINE OF THE PARALLELOGRAM OF FORCES; and although we have treated the subject very diffusely, yet we may not have given it all the variety of which it is susceptible, but it is presumed that very little of great practical importance has been omitted; and we have only to add, that if the reader study with attention all that has been laid before him, he will find no difficulty of resolving other cases that may occur in the course of his practice.

In order to facilitate the process of reference, it is convenient for the purpose, to collect and arrange the several formulæ of which the theory is constituted; it being of the greatest importance in all practical cases, to lay the different theorems of computation immediately under the eye of the reader, as he is then enabled to select from the mass, the particular form which applies to the subject in hand, without being put to the trouble of searching after his object amongst the scattered materials of the previous pages.

The following is a tabular view of the equations connected with the doctrine of the Composition and Resolution of Forces:—

Relation of the Forces a and b.	Values of the angle ϕ.	*Of the Composition of Two Forces; a and b the composants, r the resultant, and ϕ the angle of inclination.*	Page where found.
		$r=\sqrt{a^2+b^2\pm 2\,ab\cos.\phi}$	13
	$\phi=0$	$r=\sqrt{a^2+b^2+2\,ab}=a+b$	13
	$\phi=180°$	$r=\sqrt{a^2+b^2-2\,ab}=a-b$	14
$a=b$	$\phi=180°$	$r=\sqrt{2\,a^2-2\,a^2}=o$	14
$a=b$	$\phi=120°$	$r=\sqrt{2\,a^2-a^2}=a$	14
$a : b$*	$\phi=90°$	$r=\sqrt{a^2+b^2}$	15
$a=b$	$\phi=90°$	$r=a\sqrt{2}$	15
$a=b$	$\phi :$	$r=2\,a\cos.\frac{1}{2}\phi$	15
		The formulæ which follow refer to the portions of the angle ϕ, made by the direction of the resultant.	
		$\sin.x=\dfrac{b\sin.\phi}{\sqrt{a^2+b^2\pm 2\,ab\cos.\phi}}$	20
		$\cot.x=\dfrac{a}{b}\operatorname{cosec}.\phi\pm\cot.\phi$	22
		$\cot.x=\dfrac{b}{a}\operatorname{cosec}.\phi\pm\cot.\phi$	22

Relation of the Forces a, b and c.	Values of the angles ϕ and ϕ'.	*Of the Composition of Three Forces; a, b, and c the composants, R the resultant, and ϕ, ϕ' the angles of inclination.*	Page
		$R=\sqrt{a^2+b^2+c^2\pm 2ab\cos.\phi\pm 2c\cos.(\phi'+x)\sqrt{a^2+b^2\pm 2ab\cos.\phi}}$	30
	ϕ and $\phi'=0$	$R=\sqrt{a^2+b^2+c^2+2ab+2ac+2bc}=a+b+c$	32
	$\phi=\phi'=180°$	$R=\sqrt{a^2+b^2+c^2-2ab+2ac+2bc}=a+c-b$	32

* We employ the two dots to denote any relation whatever between the quantities.

Relation of the Forces a, b and c.	Values of the angles ϕ and ϕ'.	*Of the Composition of Three Forces; a, b and c the composants,* R *the resultant, and ϕ, ϕ' the angles of inclination.*	Page where found.
	$\phi=\phi'=90^\circ$	$R=\sqrt{a^2+b^2+c^2-2c\sin. x\sqrt{a^2+b^2}}$ . .	32
$a=b=c$	$\phi=\phi'=90$	$R=\sqrt{3a^2-2a^2}=a$	32
$a=b=c$	$\phi=\phi'$ less than 90°	$R=a\sqrt{3+2\cos.\phi+4\cos.\frac{1}{2}\phi\cos.(\phi'+\frac{1}{2}\phi)}$.	33
$a=b=c$	$\phi=90$, $\phi'>90$	$R=a\sqrt{3\pm2\cos.(\phi'+45^\circ)\sqrt{2}}$ . . .	34
$a=b=c$	ϕ and $\phi'<90$	$R=a\sqrt{3-2\cos.\phi-4\cos.\frac{1}{2}\phi\cos.(\phi'+\frac{1}{2}\phi)}$	36
$a=b=c$	$\phi=\phi'<120$	$R=a(2\cos.\phi-1)$	37
		For the direction of the resultant of the three forces a, b and c.	
		$\cot. x'=\dfrac{c\,\mathrm{cosec.}(\phi'+x)}{\sqrt{a^2+b^2\pm2ab\cos.\phi}}\pm\cot.(\phi'+x)$*	43
$a=b=c$		$\cot. x'=\dfrac{\mathrm{cosec.}(\phi'+\frac{1}{2}\phi)}{\sqrt{2\pm2\cos.\phi}}\pm\cot.(\phi'+x)$	44
		Of the Resolution and Reduction of Two Forces; a and b the composants, r the resultant, and ϕ the angle of inclination.	
	ϕ acute	$a=\pm\sqrt{r^2+b^2(\cos^2.\phi-1}-b^2\cos.\phi$ . .	49
	ϕ obtuse	$a=\pm\sqrt{r^2+b^2(\cos^2.\phi-1}+b^2\cos.\phi$ . .	49
		$a=r\sin. x\,\mathrm{cosec.}\phi$	51
		$b=r\sin.(\phi-x)\cos.\phi$	51
	$\phi=90^\circ$	$a=r\sin. x$	51
	$\phi=90^\circ$	$b=r\cos. x$	52
		Of the Composition of Two Forces by the method of rectangular co-ordinates; a, b the composants, r the resultant, and ϕ, π the inclinations to the axes.	
		$r\cos. x=b\cos.\pi+a\cos.(\phi+\pi)$ . . .	54
		$r\sin. x=b\sin.\pi+a\sin.(\phi+\pi)$ . . .	54
$a=b$		$r\cos. x=2a\cos.(\frac{1}{2}\phi+\pi)\cos.\frac{1}{2}\phi$ . .	54
		$r\sin. x=2a\sin.(\frac{1}{2}\phi-\pi)\cos.\frac{1}{2}\phi$ . .	54
		$\tan. x=\dfrac{b\tan.\pi+a\sec.\pi\sin.(\phi+\pi)}{b+a\sec.\pi\cos.(\phi+\pi)}$ . .	55
		$r=\{b\cos.\pi+a\cos.(\phi+\pi)\}\sec. x$ . .	56
		$r=\{b\sin.\pi+a\sin.(\phi+\pi)\}\cos. x$ . .	56
		$br\sin.(x-\pi)=ab\sin.\phi$	60
		$r=a\sin.\phi\,\mathrm{cosec.}(x-\pi)$	60

* The reader must here refer to the diagrams, pages 34—38.

Relation of the Forces a, b and c.	Values of the inclination.	*Of the Composition of Three Forces by the method of rectangular co-ordinates, when the planes in which the forces act are at right angles to one another; a, b and c the composants,* R *the resultant, and* ϕ *the inclination.*	Page where found.
		$R=\sqrt{a^2+b^2+c^2\pm 2ab\cos.\phi}$	64
	$\phi=90°$	$R=\sqrt{a^2+b^2+c^2}$.	64
$a=b=c$	$\phi=90°$	$R=a\sqrt{3}$	65
$a=b=c$	$\phi \angle$ or 7 than 90°	$R=a\sqrt{3\pm 2\cos.\phi}$	67
$a=b=c$	$\phi=30°$	$R=a\sqrt{4\cos^2.\frac{1}{2}\phi+1}$	68
		$\cos^2.A+\cos^2.B+\cos^2.C=1$*	68
	Values of the inclinations ϕ, ϕ' & ϕ''.	*Of the composition of Three Forces by the method of rectangular co-ordinates, when the planes in which the forces act are oblique to one another; a, b and c the composants,* R *the resultant, and* ϕ,ϕ',ϕ'' *the inclination.*	
		$R=\sqrt{a^2+b^2+c^2\pm 2ab\cos.\phi\pm 2ac\cos\phi'\pm 2bc\cos.\phi''}$	74
$a=b=c$	$\phi:\phi':\phi''$ of any mag.	$R=\sqrt{3+2(\pm\cos.\phi\pm\cos.\phi'\pm\cos.\phi'')}$.	75
$a=b=c$	$\phi=\phi'=\phi''$	$R=a\sqrt{3\pm 6\cos.\phi}$	75
$a=b=c$	$\phi=\phi'=\phi''=60°$	$R=a\sqrt{6}$	75
$a=b=c$	$\phi=\phi'=\phi''=120°$	$R=a\sqrt{3-3}=o$	75
$a:b:c$	$\phi=\phi'=\phi''$	$R=\sqrt{a^2+b^2+c^2\pm 2\cos.\phi\,(ab+ac+bc)}$.	75

There are several other equations to be found in the text, besides those which we have registered above; but because they are of less practical importance, and not so well adapted for tabular arrangement, we have thought proper to omit them altogether, those which we have collected being sufficient for every purpose to which the composition and resolution of forces become applicable. The formulæ that we refer to having been omitted, will be found at pages 80, 88, 90, 91, 94, 103, 108, 113, 121, 125 and 127.

The phrase *Arithmetic of Sines* which occurs in the text, is that department of science which has for its object the tracing out and explaining the various relations of angular magnitudes.

The *Trigonometrical Values of Sines*, a phrase which also occurs; means the numerical values, as arranged in the tables employed in all angular calculations.

* The reader must here refer to the diagram, art. 107.

THE END OF THE COMPOSITION AND RESOLUTION OF FORCES.

Second Treatise.

OF THE CENTRE OF GRAVITY.

INTRODUCTION.

It clearly appears from Plutarch, that the *Pythagoreans* were aware of the tendency of all material points or particles to a common centre, which, they maintained, was the sun; and Aristotle, who combats this doctrine, argues that the mass of the earth must be heavier than the mass of the sun, which was believed to be a body of fire; and that, therefore, the earth, and not the sun, as the *Pythagoreans* asserted, ought to occupy the centre of the universe. But this only proves that the doctrine of a centre of pressure or of gravity, in a system of bodies, was known long before the age of Archimedes; and, if of a system so vast as the sun and his attendant planets, shall we be so ungenerous as to deny to the remotest antiquity the principles of that science, and the use of those machines, which have suspended the name of the Syracusan philosopher in the temple of Fame.

Archimedes discovered and demonstrated, that "if one of the "arms of a balance be increased, and the equilibrium still con-"tinue, the weight appended to that arm must be proportionably "diminished."

This important discovery is said to have conducted the Syracusan philosopher to another fact equally useful in mechanical science. For reflecting upon the balance, which, by its construction, moves upon a fulcrum, he perceived that the two weights exerted the same pressure on the fulcrum as if they had both rested upon it. Arrived at this result, he considered the sum of these two weights as combined with a third, and the sum of these as combined with a fourth; and discovered, that, in every such combination, the fulcrum must support their united weight; and, therefore, that

"There is, in every combination of bodies, and in every single "body, which may be conceived as made up of a number of "lesser bodies, a centre of pressure, or of gravity."

This valuable discovery of a centre of pressure, Archimedes applied to particular cases, and pointed out the method of finding the centre of gravity of plane surfaces, whether bounded by a parallelogram, a triangle, a trapezium, or a parabola.

LUCAS VALERIUS, contemporary with SIMON STEVINUS, extended the doctrine of the centre of gravity to solid bodies, for which GALILEO honours him with the distinguishing appellation of

NOVUS NOSTRÆ ÆTATIS ARCHIMEDES.

The science of mechanics assumed a new form in the hands of GALILEO, who, in 1572, wrote a small treatise on STATICS, which he reduced to this principle;—that

"It requires an equal power to raise two different bodies to "altitudes in the inverse ratio of their weights; or, that the same "power is requisite to raise 10lbs. to the height of 100 feet, and "20lbs. to the height of 50 feet."

GALILEO did not pursue this fertile principle to its different consequences. It was, however, applied to the determination of the equilibrium of machines by DES CARTES, who, it is said, had not the candour to acknowledge his obligations to the Tuscan philosopher. But with the progress of modern discovery, as we have observed, the doctrine which Archimedes applied to plane surfaces, was extended to solid bodies; and the modern analysis has conferred upon this doctrine all the improvement perhaps of which it is susceptible.

Thus we see how the discovery advanced, that, in every body, whether plane or solid, there is a certain point, which we call *the centre of gravity*, and when this point is supported, the body will remain at rest.

The weight of the body may therefore be understood to be united and collected in the centre of gravity; and hence the centre of gravity of a system of bodies is, in like manner, the centre of the weights composing the system.

Every thing upon this earth which has weight, bulk, or material form, tends to the common centre of gravity, the centre of the

earth, and seems to be acted upon by a force which urges it in the direction of a straight line perpendicular to a plane coincident with the visible horizon. This force, equal in all the particles of matter, and constant in each, we designate by the term *gravity*, without attempting to explain its nature and essence; or we may denominate it *pressure* or *weight*. Weight, however, depends upon the mass, but gravity has no dependence at all on it; for, under the exhausted receiver of the *air-pump* (where the resistance of the air is removed), the heaviest metals and the lightest feathers fall in the same time, through the same space. The reader will fully understand that we are not treating of motion, though we allude to falling bodies.

We may consider all those lines which bodies trace in their descent to the earth, as *parallel* one to another; they are frequently called lines of direction. All bodies composed of moleculæ, which are of the same size and substance, and similarly posited throughout, are considered as *homogeneous*. Such are the bodies which we shall consider in this treatise, in which we have deviated, as in the parallelogram of forces, very considerably from our predecessors and contemporaries. In the case of the trapezium, especially, we have, in a manner, shaped out a new path for ourselves, as none of the authors we have perused appear to have considered the general mode of determining the centre of gravity of that figure. They merely state its construction in detail, and confine the numerical process to a particular case, as if the general case was of no importance. We, however, take a different view of the subject, and consider the general case as of paramount utility; and the second example which we have given for illustration, will confirm this position.

This example is a practical question, the diagram being that of a floor, which fell in the new prison, Westminster, in the month of February, 1833. In the present instance we have branched the solution into two parts: the first diagram gives the floor that gave way; and the outline of the second diagram is merely the trapezium in the first, detached and drawn to a larger scale, that being the portion of the floor which was actually supported by the transverse girder; and consequently that which it was necessary to determine, for the purpose of ascertaining the strength of the beam, and settling the dispute occasioned by the accident.

We have not, however, carried the solution further than the determination of the centre of gravity, because this is all that our present subject requires; but the solution is completed further on, when treating of another subject to which it belongs.

Our mode of handling the doctrine of the centre of gravity has precluded all reference to contemporary authorities.

Except for the general principles of the science, we are not aware that we stand indebted to them, and we have carefully avoided borrowing from our predecessors. In this department of our work, as in the Parallelogram of Forces, we produce an original composition, theoretically arranged, and systematically illustrated by practical examples, which, in their solution, are generally freed from any symbols unemployed in equations of the second degree. Our labours are rendered bulky by the solution of many beautiful but practical examples, which have been worked out at full length; and our reason for doing so was this:—Having brought our notation within the limits of an equation of the second degree, we were unwilling, in figures purely rectilinear, to introduce but plane trigonometry and mensuration. We had hoped, but found it impracticable, to dispense with fluxions, or equations derived from fluxional analysis, in those problems in which the doctrine of the circle and the cone are concerned. It would tire, but not profit our readers, to detail our endeavours to divest our subject of the formidable appearance which analytical formulas alone throw over the writings of many distinguished authors who have treated of mechanics. In so far as we have succeeded, the task has been ours, the pleasure will be that of Practical Men; and we have only in conclusion here to remark, that those writers who have written before us upon Mechanics, without introducing fluxions, have studiously avoided all those cases which could not by them be solved without the appearance of this elegant branch of mathematical investigation.

THE CENTRE OF GRAVITY.

CONTENTS.

OF THE CENTRE OF GRAVITY.

Definition and Characteristics of the Centre of Gravity.

1. *The centre of gravity* of any body, or system of bodies, is that point in it which, if it be supported or fixed, the body, or system of bodies to which it belongs, will remain in equilibrio in every possible position about that centre. Or,

2. *The centre of gravity* is that point in a body, or system of bodies, into which, if the whole matter of the body or system were collected, it would produce the same effect as the body or system produces in its original state. But the most remarkable and comprehensive characteristic of

3. *The centre of gravity* is, that the mass of any body, drawn into the distance of its centre of gravity from any plane, is equal to the aggregate of the products of all the constituent particles of that body, multiplied by their respective distances from the same plane.

4. In treating of the composition of parallel forces in the foregoing treatise, we there stated, that the place of the resultant of any system of parallel forces, occurs at that point where a certain force ought to be applied, in order to maintain the system in a state of equilibrium; which point, agreeably to the custom of mechanical writers, we designated "*the centre of the parallel forces;*" hence, and from the first of the above characteristics,

The centre of gravity, and the centre of the parallel forces, are the same.

5. Scholium. When a line, or a plane, can be so drawn as to divide a surface or a solid into two equal and similar parts, or when it can be drawn in such a manner, that the elements of the surface or solid, taken two and two, shall be similarly disposed with respect to such line or plane, the body thus divided is said to be

Symmetrical with regard to the axis, or plane to which it is referred.

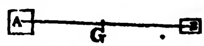
Consequently, in all such bodies, it is evident that the sum of the efforts of the several constituent elements with respect to such axis or plane will be nothing; for, if we take two particles, similarly disposed, but on different sides of the axis or plane to which they are referred, their efforts will be equal, but opposed to each other, and therefore their sum is nothing; and the same may be shewn of any other two particles similarly situated; but, by the hypothesis, all the particles or constituent elements are similarly situated; hence, the resultant of the system will be in the axis or plane by which the surface or solid is supposed to be divided, and of course its centre of gravity must be there also; for it is stated in the fourth characteristic, that the centre of gravity and the centre of the parallel forces, or the place of the resultant, are the same.

Section First

OF THE CENTRE OF GRAVITY OF TWO BODIES IN THE SAME STRAIGHT LINE.

6. Proposition 1. *The common centre of gravity oj any two bodies, is in the straight line joining their respective centres of gravity, and the distance of either body from that centre is reciprocally as the quantity of matter in it.*

Let the points A and B be the centres of gravity of the bodies A and B; and suppose AB to be an inflexible straight bar or lever, supported at the point G, and considered to be devoid of gravity or weight; then, if the point G be the centre of gravity of the bodies A and B, these bodies, by our definition, will be in equilibrio, or in a state of balanced rest; and by our principle for the composition of two parallel forces, we have



$$A : B :: BG : AG.^*$$

This property is analogous to that which we adduced for the composition of two parallel forces, acting in the same plane, and on the same side of the line of application; consequently the developement of the theory must in a great measure be similar; indeed, in so far as the consideration of parallel forces is concerned, the place of the resultant and that of the centre of gravity coincide. This circumstance identifies the composition of parallel forces with the determination of the centre of gravity; but we shall not avail ourselves of the aid thus placed within our reach, for it is a prominent feature in the plan of our work, that each article shall depend on principles peculiar to itself, and that no references be made beyond the pages of the immediate article, unless in cases where it is essentially necessary to produce the authority of some foreign principle.

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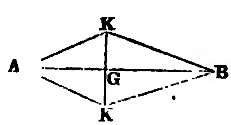
* Let A and B be the centre of gravity of the bodies A and B, and suppose AB to be an inflexible bar without weight, connecting the bodies A and B. Then, the weights of these bodies, or their tendency to descend to the earth, may be considered as two parallel forces, in which case their resultant will therefore divide AB reciprocally proportional to the weights. (Comp. of Forces, art. 167). And the direction of the resultant, must evidently pass through the centre of gravity, in order that the bodies may be in equilibrio; let it cut AB in G, then by what we have just said



$$A : B :: GB : AG.$$

Now this point (G) is the centre of gravity; for if not, let it, if possible, be K, and join AK, BK, then to have the bodies in equilibrio there shall necessarily be

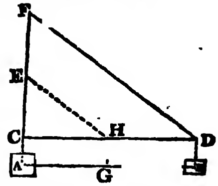
$$A \times AK = B \times BK$$
$$\therefore A : B :: BK : AK;$$
$$\text{but } A : B :: GB : AG;$$

whence BK : BG :: AK : AG, and since the angles at G are right angles; therefore, the triangles AGK and BGK are (Euc. b. vi. p. 7,) equiangular, which, since GK is common, is evidently impossible, except when $A = B$, or $AG = GK$, therefore G is the centre of gravity of the two bodies, as stated in the proposition.

This premised, we shall proceed with the solution of a few of the most important problems, as follows:—

7. PROBLEM 1. *To find the centre of gravity of two bodies, connected together by an inflexible straight line passing through their respective centres.*

Let A and B be any two bodies, connected together by the straight inflexible line AB, which is conceived to pass through the centres of gravity of the bodies A and B. At the points A and B erect the perpendiculars AC and BD, of any convenient length whatever; through C draw CD parallel to AB, and produce AC to the point F; make CE to EF, as the body B is to the body A, and join FD; through E draw EH parallel to FD, and from H let fall the perpendicular HG; then is G the point in the line AB, where the common centre of gravity of the two bodies A and B occurs, and by the proposition



$$A : B :: BG : AG.$$

For, by reason of the similar triangles FCD, ECH, it is

$$FE : EC :: HD : CH;$$

but HD is equal to BG, and CH is equal to AG by the construction; consequently,

$$FE : EC :: BG : AG;$$

now, EF is proportional to the magnitude of the body A, and EC proportional to that of B; therefore we have

$$A : B :: BG : AG;$$

hence, by equating the products of the extreme and mean terms, we have

$$A \cdot AG = B \cdot BG. \qquad (1).$$

From which we infer,

That when two bodies, connected together by a straight inflexible bar, are in equilibrio, the products of their masses, into their respective distances from the common centre of gravity, are equal.

Let a = the mass or effective energy of the body A, supposed to be concentrated in the point A;

b = the mass or effective energy of the body B, supposed to be concentrated in the point B;

d = AG, the distance of the body A from the centre of gravity G,

δ = BG, the distance of the body B from that centre,

and D = AB, the central distance, or the whole distance between the centres of the bodies A and B;

then, according to the foregoing inference, we have

$$ad = b\delta,$$

but $\delta = D - d$; consequently, we obtain

$$ad = bD - bd,$$

which, by transposition and division, gives

$$d = \frac{bD}{(a+b)}. \qquad (a)$$

And in a similar manner we obtain

$$\bar{c}=\frac{a\text{D}}{(a+b)}. \qquad (b)$$

Consequently the place of the centre of gravity is known in terms of the masses, and the distance between their respective centres.

The formulæ (a) and (b), give the following practical rules for the distance of either body from their common centre of gravity.

8. Rule.—*Multiply either body by the whole distance between their centres, and divide the product by the sum of the bodies; then the quotient will be the distance from the centre of gravity of that body opposite to the one by which the whole distance is multiplied.*

Example 1. There are two bodies connected together by a straight inflexible bar of 24 feet in length; at what distance from each body must the bar be suspended, that the whole system may remain at rest, supposing the one body to weigh 4 cwt, and the other 7?

Here, we have given, $a=4$ cwt.; $b=7$ cwt., and $\text{D}=24$ feet; then, to find the distance of the centre of gravity from the body a, we have

$$\frac{7\times 24}{4+7}=15\frac{3}{11} \text{ feet,}$$

and consequently, its distance from b, is

$$\frac{4\times 24}{4+7}=8\frac{8}{11} \text{ feet.}$$

9. Example 2. If the quantity of matter in the moon be to that of the earth, as 1 to 39, and the distance of their centres 240000 miles, where is their common centre of gravity?

$$\left.\begin{array}{l}\text{Here } \dfrac{240000\times 39}{1+39}=234000 \text{ miles from the moon's} \\ \text{and } \dfrac{240000\times 1}{1+39}=6000 \text{ miles from the earth's}\end{array}\right\}\text{centre;}$$

for $234000+6000=240000$ the distance of their centres as given in the question.

Example 3. Two masses, the one of 56 and the other of 63 stone weight, are to be suspended from a straight inflexible lever, at the distance of 9 feet and a half from each other; at what point in its length must the lever be supported, that the whole may remain at rest?

Here, we have given, $a=56$; $b=63$, and $\text{D}=9\frac{1}{2}$ feet; then the distance of the centre of gravity from the mass a, is

$$\frac{63\times 9\frac{1}{2}}{56+63}=5\frac{1}{34} \text{ feet;}$$

and, consequently, the distance of the centre of gravity from the mass b, is

$$\frac{56\times 9\frac{1}{2}}{56+63}=4\frac{16}{34} \text{ feet.}$$

9. PROPOSITION. ***When the weight of the connecting bar is taken into account, and becomes an element of the united masses.***

These two examples have been produced on the supposition that the bar or lever with which the bodies are connected, is perfectly inflexible, and void of gravity or weight; but this is obviously erroneous, for no material substance that is capable of supporting a heavy body, can in itself be supposed to possess the property of perfect levity; yet our investigation has proceeded on this hypothesis, and the resulting theorems have undergone no modification to allow for the weight of the connecting bar; but this element can be very easily taken into the account, for we have already shown (in equation 1), that the mass of one body multiplied into its distance from the common centre of gravity, is equal to the mass of the other body multiplied into its distance from the common centre of gravity; and exactly the same law must obtain with respect to the portions of the connecting bar, that are situated on opposite sides of the centre of support.

Now, we have already stated in our scholium, that the centre of gravity of a prism or cylinder, is in its middle point; hence, since both portions of a uniform connecting bar must either be prismatic or cylindrical, it follows that the centre of gravity of each arm or portion, as referred to the common centre, must be at the middle of its length, and from our definition, the whole weight of the arm must be referred to that point; consequently

if m = the mass or weight of one unit in length of the bar;
then is $\frac{1}{2}md^2$ = the effective energy of one portion,
and $\frac{1}{2}m\delta^2$ = the effective energy of the other;

and these, together with the effective energies of the respective bodies referred to opposite sides of the centre of gravity, must still be in equilibrio; therefore, we have

$$ad+\tfrac{1}{2}md^2=b\delta+\tfrac{1}{2}m\delta^2;$$

but we have shown above, that $\delta=\mathrm{D}-d$; therefore, by substitution and reduction, we obtain

$$(a+b+m\mathrm{D})d=(b+\tfrac{1}{2}m\mathrm{D})\,\mathrm{D};$$

which, by division, gives

$$d=\frac{(2b+m\mathrm{D})\,\mathrm{D}}{2(a+b+md)}. \qquad (c)$$

And in a manner exactly similar to the above, we obtain

$$\delta=\frac{(2a+m\mathrm{D})\,\mathrm{D}}{2(a+b+md)}. \qquad (d)$$

Hence, the place of the centre of gravity has been determined in terms of the central distance, the masses of the bodies and the connecting bar. The formulæ (c) and (d) afford the following practical rule for the distance of either body from the common centre of gravity, the weight of the connecting bar being taken into consideration.

10. Rule.—*To twice the weight of either body, add the whole weight of the lever or connecting bar, and multiply the sum by the central distance; then divide the product by twice the mass compounded of the bodies, and the bar, and the quotient, will be the distance of the centre of gravity from that body, opposite to the one whose double is employed in the first step of the process.*

Example 1. There are two bodies connected together by a straight inflexible bar of 24 feet in length, and weighing one cwt.; at what distance from each body must the bar be suspended, that the whole system may remain at rest, supposing the one body to weigh 4 cwt. and the other 7?

Here we have given, $a=4$ cwt.; $b=7$ cwt.; $\text{D}=24$ feet; and $m=\frac{1}{24}$ cwt.; then, to find the distance of the centre of gravity from the body a, we have

$$2b=2\times 7=14$$
$$m\text{D}=\tfrac{1}{24}\times 24=\underline{1}$$
$$2b+m\text{D}=15$$
$$\text{and } (2b+m\text{D})\,\text{D}=15\times 24=360$$
$$2(a+b+m\text{D})=2(4+7+1)=24$$

then, we get $\frac{360}{24}=15$ feet, from a,

and consequently its distance from b is,

$$24-15=9 \text{ feet.}$$

This case may also be resolved in a manner purely algebraical, as follows:—

Let $x=$ the distance of one body from the common centre of gravity; then will $24-x=$ the distance of the other body; consequently, we have

$$4x=168-7x;$$

this equation expresses the effective energies of the bodies alone, but in the example it is proposed to consider also the effect of the connecting bar; now, since the whole weight of the bar is one cwt., and its length 24 feet, it follows that the weight of one foot is $\frac{1}{24}$ part of a hundred weight; consequently, the weight of one portion of the bar is $\frac{x}{24}$, and the weight of the other is $1-\frac{x}{24}$; hence the effective energies of these portions are $\frac{x}{24}\times\frac{x}{2}=\frac{x^2}{48}$, and $(1-\frac{x}{24})\times\frac{1}{2}(1-\frac{x}{24})=12-x+\frac{x^2}{48}$; wherefore the above equation becomes

$$4x+\frac{x^2}{48}=180-8x+\frac{x^2}{48},$$

or by expunging $\frac{x^2}{48}$ from both sides, and transposing $8x$, we obtain

$$12x=180,$$

and, lastly, dividing both sides of this equation by 12, we get

$$x=15, \text{ the same as before;}$$

and $24-x=24-15=9$ feet, for the distance of b.

Example 2. The weight of a straight uniform bar of cast iron, connecting the centres of gravity of two bodies, is 252 lbs. and its length is 42 feet; at what point of its length must it be supported, that the whole system may remain in a state of quiescence, supposing the bodies to weigh respectively 13440 and 17920 lbs.?

Here we have given, $a=13440$ lbs., $b=17920$ lbs., $m=6$ lbs., and $\text{D}=42$ feet; then, to find the distance of the body a from the common centre of gravity, we have

$$2\,b=2\times 17920=35840$$
$$m\text{D}=6\times \quad 42=\underline{\quad 252}$$
$$2\,b+m\text{D}=36092$$

and $(2\,b+m\text{D})\,\text{D}=36092\times 42=1515864$

$$2\,(a+b+m\text{D})=2\,(13440+17920+252)=63224;$$

hence, we have $\dfrac{1515864}{63224}=23\tfrac{1102}{1129}$ feet from a,

and, consequently, its distance from b is

$$42-23\tfrac{1102}{1129}=18\tfrac{27}{1129} \text{ feet.}$$

The algebraic process is as follows:

Let x = the distance of the centre of gravity from the body a, then will $42-x$ = its distance from the body b; consequently, we get

$$13440\,x+3\,x^2=757932-18172\,x+3\,x^2,$$

therefore, by expunging the common term $3\,x^2$, and transposing $18172\,x$, we get

$$31612\,x=757932,$$

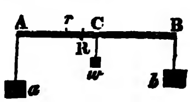
then, dividing both sides of this equation by 31612, we obtain

$$x=23\tfrac{1102}{1129} \text{ feet,}$$

and $42-x=42-23\tfrac{1102}{1129}=18\tfrac{27}{1129}$ feet, the same as before.

11. This theorem, however, may assume another form of investigation, and present a rule for practical operations somewhat simpler than the foregoing.

Thus, let AB be an inflexible bar of uniform thickness, at the extremities of which are suspended the unequal weights a and b; it is required to find the point R, the centre of gravity of these weights, and of the bar AB.



Since AB is of uniform thickness, its centre of gravity is evidently in its middle point C; therefore, conceiving the entire mass of AB to be collected at C, and represented by the weight w; our problem is identical to this:

To find the resultant of three parallel forces, a, w, b, acting perpendicularly to the straight line AB, (which in this view of the case is void of gravity).

Denoting the distance AB by D we get,

1st. $(a+b) : b :: D : Ar = \frac{Db}{(a+b)}$ which determines the resultant (or centre of gravity) of a and b.

2ndly. Suppose the weights a and b to be applied at r, we shall then have

$$rC : CR :: (a+w+b) : (a+b)$$

$$\text{therefore } CR = rC \cdot \frac{a+b}{a+w+b}.$$

$$\text{But } rC = AC - Ar = \tfrac{1}{2}D - \frac{Db}{a+b} = \frac{D(a-b)}{2(a+b)};$$

$$\therefore CR = \frac{D(a-b)}{2(a+b)} \cdot \frac{(a+b)}{(a+w+b)} = \frac{D(a-b)}{2(a+w+b)} = \frac{\frac{1}{2}D(a-b)}{(a+w+b)}.$$

And this equation affords the following easy practical rule for finding the centre of gravity, when the weight of the connecting bar is taken along with that of the bodies suspended upon it:

Rule. *Multiply half the length of the bar by the difference of the weights, and divide the product by the sum of the weights, and the weight of the bar, the quotient is the distance of the centre of gravity from the middle of the bar, and it lies on that side of the centre on which the greater weight is suspended.*

12. The preceding investigations have been conducted on the supposition, that the bar or lever which connects the centres of gravity of the two bodies, maintains a horizontal position; but it can easily be shewn that this condition is not necessary, for, according to our definition, a body, or system of bodies, is at rest when supported at the centre of gravity, whatever may be the position of the body or system; and we have shewn in the composition of parallel forces, that whether the directions of the forces be at right angles, or oblique to the lines of application, the place of the resultant will be the same, and the centre of gravity of any system of bodies, corresponds to the place of the resultant or the centre of parallel forces; hence it appears, that in determining the place of the centre of gravity, it is a matter of no importance in what position we consider the body or system of bodies to be placed, for the place of the centre will notwithstanding remain the same.

Section Second.

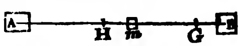
OF THE CENTRE OF GRAVITY OF THREE BODIES IN THE SAME STRAIGHT LINE.

13. Our next object will be, to determine the centre of gravity of three bodies, subsisting in the same straight line, and having their centres connected by inflexible levers or bars, considered in the

first instance without reference to gravity or weight. And the method by which this is accomplished, is obviously a simple extension of that by which we determined the centre of gravity of two bodies, under the conditions specified in the first problem; and the proposition may be enunciated as follows.

14. PROPOSITION 2. *The common centre of gravity of three bodies subsisting in the same straight line, occurs in the line connecting the centre of one body with the common centre of the other two; and it divides the distance between these centres into two parts, that are to each other, reciprocally, as the mass of one body, is to the mass compounded of the other two.*

Let us consider the points A, m, and B, as the centres of gravity of the three bodies A, m, and B; and suppose the straight line AB, passing through m, to be an inflexible bar or lever, devoid of gravity or weight; then, if the point G, be the common centre of gravity of the bodies m and B, we have, by the first proposition,



$$m : B :: BG : mG;$$

hence, the distances AG and BG are known; but by our definition, the bodies m and B may be supposed to be concentrated in the point G, which is the common centre of gravity of them both; then, by the first proposition, the point H, which is the centre of gravity of the body A and the compound mass $m+B$ acting at G, is found in the following manner:

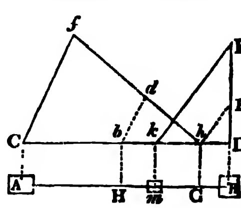
$$A : m+B :: GH : AH.$$

This analogy involves the principle enunciated in the second proposition, and the solution of the following problem will unfold its utility.

15. PROBLEM 2. *To find the common centre of gravity of three bodies, connected together by an inflexible straight line passing through their respective centres:*

Let us consider A, m, and B as any three bodies, connected together by the straight line AB, which is supposed to be inflexible, and conceived to pass through the centres of gravity of the bodies A, m, and B.

At the points A, m, and B, erect the perpendiculars AC, mk, and BD of any convenient length whatever; through the point C draw the straight line CD parallel to AB, and produce BD to F, making DE to EF as m is to B; join Fk, and through E draw Eh. parallel to Fk, and let fall the perpendicular hG; then is G the common centre of gravity of the two bodies m and B, or the point into which, if they were both collected, they would produce the same effect as they would do in their present positions, as is manifest by the definition.



From the point h draw hf in any direction at pleasure, and make hd to df as the body A is to the mass compounded of the two bodies m and B; join fc, and through d draw db parallel to fc, and let fall the perpendicular bH; then is the point H the common centre of gravity of the three bodies A, m, and B; consequently, by the proposition, we have

$$A : m+B :: GH : AH.$$

For by reason of the similar triangles fch, and dbh, it is

$$hd : df :: hb : bc,$$

but by the construction, hb is equal to GH, and bc equal to AH; consequently

$$hd : df :: GH : AH;$$

now, hd is proportional to the magnitude or mass of the body A, and df is proportional to the compound mass $m+B$; therefore, we have

$$A : m+B :: GH : AH;$$

and by equating the products of the extreme and mean terms, we get

$$A \cdot AH = (m+B) \cdot GH. \qquad (2)$$

From which we infer

That when three bodies, connected together by a straight inflexible bar, are in equilibrio, the product of one body, into its distance from the common centre of gravity of the system, is equal to the product which arises, when the sum of the other two bodies, is multiplied by the distance between their common centre, and that to which the whole system is referred.

Let a = the mass or effective energy of the body A, concentrated in the point A,

b = the mass or effective energy of the body B, concentrated in the point B,

m = the mass or effective energy of the body m, concentrated in the point m;

d = the distance between the bodies a and b,

δ = the distance between the bodies a and m,

and x = AH, the distance between the body a, and the common centre of gravity of the system, situated in H.

Then, if the common centre of gravity H, falls between the bodies a and m, we shall have $m\text{H} = \delta - x$, and $\text{HB} = d - x$; but if the common centre of gravity falls between the bodies b and m, we shall have $m\text{H} = x - \delta$, and $\text{BH} = d - x$; and in either case, we obtain, according to our inference,

$$(a+b+m)\,x = bd + m\delta,$$

which, by division, gives

$$x = \frac{bd + m\delta}{(a+b+m)}. \qquad (e)$$

This equation determines the distance of the common centre of gravity from the first body a, and its distance from each of the other bodies b and m, can easily be found, from their values as given above, in reference to the position of the body m, in respect of the common centre H.

The practical rule afforded by equation (e), is as follows.

16. RULE 3.—*Multiply each of the bodies b and m, by the respective distances from the body a; then, divide the sum of the products by the aggregate of the three masses, for the distance of the centre of gravity from the first body a, to which the distances of the other bodies b and m are referred.*

EXAMPLE 1. There are three bodies, a, m and b, whose weights are respectively 15, 20 and 25 tons, connected together by an inflexible bar or lever, at the distances of 12 and 16 feet from each other; at what point in the connecting bar does the common centre of gravity of the three bodies occur, and what is its distance from each body, supposing the bar to be perfectly straight, and uninfluenced by its own weight, or the effects of gravity.

NUMERICAL CALCULATION.

Here we have given, $a=15$ tons; $m=20$ tons; $b=25$ tons; $\delta=12$ feet, and $d=12+16=28$ feet; therefore, we have

$$bd=25\times 28=700,$$
$$m\delta=20\times 12=240,$$
$$\text{then } bd+m\delta=\overline{940};$$
$$\text{but } (a+b+m)=15+25+20=60;$$
$$\text{consequently, } \tfrac{940}{60}=15\tfrac{2}{3} \text{ feet},$$

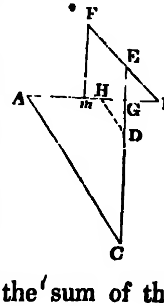
the distance of the centre of gravity from a; but since this distance is greater than the given distance between a and m, we have, for the three distances, as follows:

$$x=15\tfrac{2}{3} \text{ feet, distance from } a,$$
$$x-\delta=\ \ 3\tfrac{2}{3} \text{ feet, distance from } m,$$
$$d-x=12\tfrac{1}{3} \text{ feet, distance from } b.$$

And these results may be verified by means of the following construction.

GEOMETRICAL CONSTRUCTION.

Draw the straight line AB, and from a scale of equal parts, make Am equal to 12, and mB equal to 16 feet; through the point B, draw the straight line BF in any direction whatever with respect to AB; make BE equal to 20, and EF equal to 25, the numbers which respectively express the magnitudes of the bodies m and b, acting on the straight line AB, at the points m and B; join Fm, and through the point E, draw EG parallel to Fm, which produce to C, and make GD equal to 15, the number which expresses the magnitude of the body a acting at the point A, and make DC equal to 45, the number which expresses the sum of the



bodies m and b, acting at the point G; join CA, and through the point D, draw DH parallel to the lines CA; then is H the place of the centre of gravity of the three forces a, m and b, acting at the points A, m and B in the straight line AB; and AH, mH and BH, are the respective distances, which being measured on a scale of equal parts, will indicate, as follows, viz.

AH $=15\frac{2}{3}$ feet, or the distance of a from H,

mH $=\ 3\frac{2}{3}$ feet, or the distance of m from H,

and BH $=12\frac{1}{3}$ feet, or the distance of b from H, the same as above.

It may perhaps be of use to remark in this place, that in constructions of this kind, it is not necessary to take the numbers which express the magnitudes of the bodies from the same scale as those which express their relative distances; for since they are magnitudes dissimilar to one another, they cannot be compared; consequently, the ratio of the numbers is all that is wanted: it is, however, necessary that all magnitudes of the same kind be taken from the same scale.

The above remark became necessary, in consequence of the numbers that express the weights of the bodies in the foregoing construction, not being taken from the same scale as those that express the distances: this was purposely done, in order to preserve the diagram of a proper size, but the circumstance can have no effect whatever upon the truth of the result.

EXAMPLE 2. Three bodies are connected together by a straight inflexible bar, or beam of iron passing through their centres of gravity; the first weighs 25 tons, the second 12 tons, and the third 18 tons; and, moreover, the distance of the second from the first is 24 feet, while the distance of the third from the second is only 9 feet; at what distance from each body must the bar be supported, that the whole may remain at rest, the weight of the bar itself not being considered?

Let these weights be represented by a, m, b, and we have given, $a=25$ tons; $m=12$ tons; $b=18$ tons; and for the points of distance, $\delta=24$ feet, and $d=24+9=33$ feet; therefore, we get, by actual calculation,

$$bd=18\times33=594,$$
$$m\delta=12\times24=288,$$
$$\text{then } bd+m\delta=882;$$
$$\text{but } (a+b+m)=25+18+12=55$$

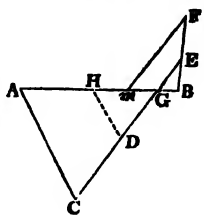
consequently, $\frac{882}{55}=16\frac{2}{55}$ feet, the distance of the common centre of gravity from the first body a; but since this distance is less than the given distance between the bodies a and m, we have for the three distances as follows, viz.

$$x=16\tfrac{2}{55} \text{ feet, distance from } a,$$
$$\delta-x=\ 7\tfrac{53}{55} \text{ feet, distance from } m,$$
$$d-x=16\tfrac{53}{55} \text{ feet, distance from } b.$$

From the results of this example, it appears, that the common centre of gravity of the system is very nearly in the middle of the connecting bar, or very nearly at equal distances from the extreme bodies a and b; and that this is actually the case, will become manifest from the following

GRAPHICAL OR GEOMETRICAL CONSTRUCTION.

Draw the straight line AB, and from a scale of equal parts, make Am equal to 24, and mB equal to 9 feet; through the point B, draw the straight line BF, making any angle whatever with AB; and make BE, EF, equal respectively to the numbers 12 and 18, being the numbers which express the weights of the bodies m and b; join Fm, and through the point E, draw EG parallel to mF, which produce to C, and make GD equal to 25, and DC equal to 30, the numbers which express the weight of the body a and the aggregate weights of the bodies m and b; join CA, and through the point D, draw DH parallel to CA; then is H the place of the common centre of gravity of the three forces a, m and b; and AH, mH and BH, are its respective distances from the bodies, which being applied to a scale of equal parts, will be found to measure respectively as below, viz.



$$\text{AH} = 16\tfrac{2}{55} \text{ feet,}$$
$$m\text{H} = 7\tfrac{53}{55} \text{ feet,}$$
$$\text{and BH} = 16\tfrac{53}{55} \text{ feet, the same as before.}$$

The difference between the distances BH and AH is only $\frac{51}{55}$ parts of a foot; let us therefore inquire what change must take place in the position of the body m, so that the point H shall bisect the line of communication between the bodies A and B.

In the present instance, the whole distance is 33 feet, and its half is 16·5 feet; consequently, substituting this number for x in equation (e), we have

$$\frac{bd+m\delta}{(a+b+m)} = 16\cdot 5,$$

in which equation, δ only is unknown; hence, by multiplication and transposition, we get

$$m\delta = 16\cdot 5\,(a+b+m) - bd,$$

therefore, by division, it becomes

$$\delta = \frac{16\cdot 5\,(a+b+m) - bd}{m};$$

but we have shewn above, that the sum of the weights, or $(a+b+m) = 55$, and $bd = 594$; consequently, by taking $m = 12$, its original value, we obtain

$$\delta = \frac{907\cdot 5 - 594}{12} = 26\tfrac{1}{8} \text{ feet.}$$

Therefore, if the place of the body m be removed $2\frac{1}{8}$ feet further from the place of the extreme body a, towards that of the other extreme body b, all other things remaining; the common centre of gravity of the system, shall then bisect the line of communication which passes through the respective centres.

17. *The same thing however, may be determined generally in the following manner:*

Since the letter d in our notation, denotes the length of the connecting line, or the whole distance between the centre of a and b the extreme bodies of the system; it follows from the conditions of the question, that

$$x=\tfrac{1}{2}d,$$

where x denotes the distance between the centre of the extreme body a, and the common centre of gravity of the system; therefore, from the equation marked (e), by substitution, we obtain

$$d=\frac{2\,(bd+m\delta)}{(a+b+m)},$$

in which expression δ is the only unknown quantity; by multiplication, we get

$$2\,(bd+m\delta)=d\,(a+b+m),$$

and finally, by transposition and division, we obtain

$$\delta=\frac{d}{2m}(a+m-b). \qquad (f)$$

18. From this equation it is manifest, that if b be greater than a and m together, δ is negative, and the body m lies on the other side of a; but this is contrary to the supposition, or the conditions of the question, which distinctly imply, that the body m must exist somewhere in the line between the extreme bodies a and b, which cannot be the case if δ is negative; for then the body m exists at a point in the line produced beyond a, and instead of being the middle body, changes its character, and assumes the position of one of the extreme bodies of the system, connected to the other bodies by a line which has no existence; hence we infer, that in order that the question may be consistent, it is a necessary condition, that the sum of the bodies a and m together, shall be greater than the body b.

And, moreover, in order that the question shall be consistent, it is also a necessary condition, that the sum of the bodies b and m together, shall be greater than the body a; for, if otherwise, then δ is greater than d, and this also, is contrary to our premises; for, as we have already stated, the body m must exist somewhere in the line between the bodies a and b, which cannot be the case if δ is greater than d; because d in our notation, has been taken to denote the distance between the centres of a and b, the extreme bodies of the system.

It does not follow however, as a necessary condition, that the sum of the extreme bodies a and b, shall be greater than the middle body m; for, whatever may be the proportion, or relation between the sum of the extreme bodies and the middle one, it can always be so situated as to produce an equilibrium, provided that its magnitude is such, as being added to either extreme body, the sum shall exceed the other extreme; under this limitation, there are three cases, in which we can compare the magnitude of the middle body with the sum of the two extremes, and in which the premises of the question shall always be fulfilled.

If b is equal to the sum of a and m together, then δ vanishes, and the system becomes that of two equal bodies, sustained in a state of rest at a point equally distant from both; that is, at the middle of their connecting line.

If the extreme bodies a and b are equal, whatever may be the magnitude of the other body, then δ is equal to $\frac{1}{2}d$, and consequently, the common centre of gravity of the system coincides with that of the body m; for they both occur at the middle point of the connecting line.

19. We shall now endeavour to give a numerical exemplification of these inferences, as the process of reasoning by which they are directly obtained from the equation, may not at first sight be sufficiently explicit.

The several inferences which we propose to exemplify numerically, when brought into one view, are as below, viz.

1. When b is greater than $a+m$, δ is negative.
2. When a is greater than $b+m$, δ is greater than d.
3. When m is less than $a+b$, but such, that $a+m$ is greater than b, and also $b+m$ greater than a, δ is less than d.
4. When m is equal to $a+b$, δ is less than d.
5. When m is greater than $a+b$, δ is less than d.
6. When a is equal to $b+m$, δ is equal d.
7. When b is equal to $a+m$, δ is nothing.
8. When a and b are equal, and m of any magnitude, δ is equal [to $\frac{1}{2}d$.

In the examples by which we propose to illustrate these inferences, it may be of use to consider the distance d as constant, or the same in all, as by so doing, we shall preclude the necessity of repeating the condition of distance for each example, and the conclusions will nevertheless, be equally convincing; we shall therefore assume 16 feet as the constant distance, and apply it instead of d in the following examples:—

Example 1. The weights of three bodies, acting on a straight inflexible bar or lever, are equal respectively to 12, 5 and 18 tons; in what manner are the bodies situated on the bar, the centre of gravity being at the middle of d?

Here we have given, $a=12$; $m=5$; $b=18$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{10}(12+5-18)=-1\cdot 6 \text{ feet.}$$

This result exemplifies the first inference, and indicates by its negative affection, that the body m, which ought to be situated on the line between the bodies a and b, occurs at the distance of $1\cdot 6$ feet on the other side of the body a.

EXAMPLE 2. The weights of three bodies, acting on a straight inflexible bar or lever, are respectively equal to 18, 5 and 12 tons; what is the situation of the bodies with respect to each other, supposing the common centre of the system to be at the middle of d?

Here we have given, $a=18$; $m=5$; $b=12$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{10}(18+5-12)=17\cdot 6 \text{ feet.}$$

This result exemplifies the second inference, and since the calculated value of δ exceeds the given value of d, it indicates that the body m, which ought to be situated between the bodies a and b, occurs at the distance of $1\cdot 6$ feet on the other side of b; but, because the body m, although it is situated beyond b, lies in the same direction with respect to a, its distance beyond b must be regarded as a positive quantity, notwithstanding that it falls without the limits prescribed for the operation of the system.

EXAMPLE 3. The weights of three bodies, acting on a straight inflexible bar or lever, are equal respectively to 12, 6 and 14 tons; what is the position of the bodies with respect to one another, the common centre of gravity being at the middle point of the distance between the two extremes?

Here we have given, $a=12$; $m=6$; $b=14$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{12}(12+6-14)=5\tfrac{1}{3} \text{ feet.}$$

This result exemplifies the third inference, and by its magnitude and positive character, it indicates a situation between the bodies a and b, being at the distance of $5\frac{1}{3}$ feet from a towards b, and at the distance of $10\frac{2}{3}$ feet from b towards a; consequently, the body m is situated at the distance of $2\frac{2}{3}$ feet from the common centre of gravity of the system.

EXAMPLE 4. The weights of three bodies, acting on an inflexible bar or lever, are equal respectively to 8, 15 and 7 tons; what is the situation of the bodies with respect to one another, in case of an equilibrium, the connecting bar being supported at the middle of its length?

Here we have given, $a=8$; $m=15$; $b=7$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{30}(8+15-7)=8\tfrac{8}{15}\text{ feet.}$$

This result exemplifies the fourth inference, and the same remarks apply to it as those which we made use of for the third inference preceding; the body m being situated at the distance of $8\frac{8}{15}$ feet from a, and $7\frac{7}{15}$ feet from b, being $\frac{8}{15}$ of a foot from the common centre of gravity.

Example 5. The weights of three bodies, acting on an inflexible bar, or lever in equilibrio, are respectively equal to 8, 20 and 11 tons; how are the bodies situated with respect to each other, the lever being supported at the middle of its length?

Here we have given, $a=8$; $m=20$; $b=11$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{40}(8+20-11)=6\tfrac{4}{5}\text{ feet.}$$

This result exemplifies the fifth inference, and indicates that the body m is situated between the extremes a and b, at the distance of $6\frac{4}{5}$ feet from the former, and $9\frac{1}{5}$ feet from the latter; being $1\frac{1}{5}$ feet distant from the centre of the system.

If we compare the results of the third, fourth, and fifth examples with one another, it will appear, that when the magnitude of the middle body m is less than, equal to, or greater than the sum of the extreme bodies a and b, its situation in the system is nearest to the lighter body; this happens as a necessary consequence, of the centre of the system being limited to the middle point of the connecting line.

Example 6. The weights of three bodies, acting on an inflexible bar or lever, are equal respectively to 12, 5 and 7 tons; how are the bodies situated with respect to each other in the case of an equilibrium, the lever being supported at the middle of its length?

Here we have given, $a=12$; $m=5$; $b=7$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{10}(12+5-7)=16\text{ feet.}$$

This result exemplifies the sixth inference; it indicates that the bodies m and b coincide and operate as one body, which reduces the system to two masses of equality, kept in equilibrio by a power equal to their sum, applied at the middle of the connecting line or distance between the centres of the extreme masses.

Example 7. The weights of three bodies, acting on an inflexible bar or lever, are equal respectively to 9, 5 and 14 tons; how are the bodies situated in the case of an equilibrium, the line of communication being supported at its middle point?

Here we have given, $a=9$; $m=5$; $b=14$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{10}(9+5-14)=0 \text{ feet.}$$

This result exemplifies the seventh inference; it indicates that the bodies a and m coalesce in the point A, and form one body of an equality with b, thus reducing the system to two bodies in equilibrio about the middle of their connecting line, the same as in the preceding example.

EXAMPLE 8. The weights of three bodies, acting on an inflexible bar or lever, are equal respectively to 8, 15, and 8 tons; how are the bodies disposed with respect to each other when in equilibrio, the lever or bar on which they act being supported at its middle point?

Here we have given, $a=8$; $m=15$; $b=8$, and $d=16$ feet; let these numbers be substituted in equation (f), and it becomes

$$\delta=\frac{16}{30}(8+15-8)=8 \text{ feet.}$$

This result exemplifies the eighth inference; we learn from it, that the bodies a and b being equal, they have no effect whatever upon the system, and therefore the equilibrium would be equally complete, if no other body but m existed; for the distance of m from a is the very same as its distance from b, and consequently its situation coincides with that of the common centre of gravity, or that point at which the bar is supported.

Thus have we completed the numerical exemplification of the several inferences derived from equation (f). It now remains to select from among them, those that can be made available in a practical point of view; for it is only by furnishing the Practical Man with valuable deductions, and shewing him in what manner those deductions are applied to mechanical constructions, that our labours become appreciable, and important to the community, for whom they have been written.

Cases of Utility Consistent with the theorem.

20. The only useful cases then, consistent with the conditions of the question, are the following, viz.

1. When m is less than $a+b$, but such, that $a+m$ is greater than b, and $b+m$ greater than a.
2. When m is equal to $a+b$.
3. When m is greater than $a+b$.

All the remaining cases of the preceding class, are either inconsistent or need no calculation; but to shew this, it was necessary to enter into the proofs, as we have done by numerical calculations.

These three cases, then, being all that are useful to the practical mechanic, we shall proceed to show the utility of our theorem, by applying it to the solution of two or three useful examples.

EXAMPLE 1. At the extremities of an iron shaft 22 feet long, are fixed two wheels of the weights of 2, and $2\frac{1}{2}$ cwt., and somewhere between these, is fixed another wheel of $1\frac{1}{2}$ cwt.; at what distance from each of the extreme wheels must the intermediate one be fixed, in order that the whole weight may come upon the middle of the shaft, at which point it is supported by a transverse bearer?

Here, we have given, $a=2$; $m=1\cdot5$; $b=2\cdot5$, and $d=22$ feet, supposed to be the distance between the centres of the extreme wheels; then, since the shaft is supported on its gudgeons at the extremities, and on the journal at the transverse bearer, we may consider it as having no effect upon the system of wheels as regards the place of the centre of gravity; therefore, by substituting the above numbers in equation (f), we get

$\frac{22}{6}(2+1\cdot5-2\cdot5)=7\frac{1}{3}$ feet distant from the lighter wheel,
$14\frac{2}{3}$ feet distant from the heavier wheel,
and $3\frac{2}{3}$ feet from the middle of the shaft.

EXAMPLE 2. The weights of three bodies, suspended on a straight inflexible metal bar, are equal respectively to 224, 460, and 236lbs.; at what distance from the extreme weights must the intermediate one be suspended, that an equilibrium may obtain when the bar is supported at its middle point, supposing the whole length of the bar, or distance between the extreme weights to be 38 feet?

Here, we have given, $a=224$ lbs.; $m=460$ lbs.; $b=236$ lbs., and $d=38$ feet; therefore, by substitution, equation (f) becomes

$\frac{38}{920}(224+460-236)=18\frac{58}{115}$ feet distant from the lighter body,
$19\frac{57}{115}$ feet distant from the heavier body,
and $\frac{57}{115}$ of a foot distant from the middle of [the bar.

EXAMPLE 3. The weights of three bodies, acting on a straight inflexible bar or lever 40 feet in length, are respectively equal to 20, 60, and 30 tons; at what distance from each of the extreme bodies must the intermediate one be placed, so that the whole may be in equilibrio, when the connecting bar is suspended from the middle of its length?

Here, we have given $a=20$ tons; $m=60$ tons; $b=30$ tons, and $d=40$ feet; therefore, by substitution, equation (f), becomes

$\frac{40}{120}(29+60-30)=16\frac{2}{3}$ feet distant from the lesser extreme weight;
$23\frac{1}{3}$ feet from the greater extreme,
and $3\frac{1}{3}$ feet distant from the middle of the bar.

When the distance is known, or otherwise limited by situation.

21. The preceding enquiry has had respect only to the determination of δ, the distance between the place of the middle body m,

and that of a, the first extreme; but in practice it may frequently happen, that this distance is known, or at least limited by the circumstances of situation, and therefore, in order that the conditions of the question may obtain, it is requisite to ascertain the value of d, the distance between the extreme bodies a and b. This is, perhaps, the more important case of the two, and consequently it is entitled to a separate discussion. Our next object then, will be to resolve the following problem.

22. Problem 3. *Given the magnitudes or weights of three bodies a, m and b, acting in the same straight line, and the distance between the middle body m, and the first extreme a; to find d, the distance between the extreme bodies a and b such, that the common centre of gravity, or the centre of the system shall occur at the middle of that distance.*

In the equation maked (e), x expresses generally the distance between the common centre of gravity, and the first extreme body a, and we have shown, that $d-x$, is the distance between that centre and the other extreme body b; let x' denote that distance; then, we have

$$x'=d-\frac{bd+m\delta}{a+b+m}=\frac{d\,(a+m)-m\delta}{a+b+m},$$

but according to the conditions of the problem, $x'=x$; that is, the common centre of gravity is at the middle of the straight line connecting the centre of the extreme bodies a and b; therefore, we have

$$\frac{d\,(a+m)-m\delta}{a+b+m}=\frac{bd+m\delta}{a+b+m},$$

or by cancelling the common denominator, we obtain

$$d=\frac{2m\delta}{a+m-b}. \qquad (g)$$

This equation, it is evident might have been deduced, or inferred at once from equation (f), but we have thought it preferable, for the sake of system, to deduce it in the above manner, as it then has a more direct allusion to the principle exhibited in equation (e), and consequently has a tendency to keep the mind of the reader more steadily directed to the property enunciated in the proposition.

Practical examples illustrating the last Problem.

23. The following examples will show the utility of the formula, and exemplify the manner of its application to the resolution of practical mechanical questions.

Example 1. Suppose that three bodies, a, m and b, whose weights are respectively, 20, 36 and 48 tons, act together on a straight inflexible bar or lever, considered without weight; what must be the length of the lever to admit the system to be at rest when supported at its middle point, the distance between the first extreme and intermediate body being 18 feet?

Here we have given, $a=20$, $m=36$, $b=48$, and $\delta=18$; let these *numbers be substituted for the literals in equation (g), and it* becomes

$$d=\frac{2\times36\times18}{20+36-48}=162 \text{ feet};$$

consequently, the common centre of gravity is 81 feet distant from each of the extreme bodies a and b, and 63 feet distant from the intermediate body m.

Verification of the foregoing result.

The truth of this result may be verified in the following manner, viz.; find the common centre of gravity of the extreme body b, and the intermediate body m, on the supposition of the distance d being determined or known; thus,

$$\frac{48\times\overline{63+81}}{36+48}=82\tfrac{2}{7} \text{ feet, distant from the place}$$

of the body m, but by the preceding process, the centre of the system is 63 feet distant from the same body; consequently, the distance between the centre of the two bodies m and b, and the centre of the three bodies a, m, and b, is $82\frac{2}{7}-63=19\frac{2}{7}$ feet; then, by equation (2), we get

$$81\,a \quad =19\tfrac{2}{7}\,(b+m), \text{ that is}$$
$$81\times20=84\times19\tfrac{2}{7}=1620,$$

therefore, the conclusion at which we have arrived is rigorously correct.

The same thing may be determined analytically in the following manner, viz.

Let $2\,x=$ the whole length of the connecting line, or distance between the extreme bodies a and b.

$x=$ the distance of each from the common centre of gravity of the system.

and $x-18=$ the distance of the body m from the common centre;

then by the principle implied in equation (2), we get

$$20x+(x-18)\times36=48x$$
$$36x-648=28x$$
$$8\,x=648$$
$$x=\ 81$$

or, $2\,x=81\times2=162$ feet, the same as before.

Example 2. The shaft of a mill has to sustain three wheels, of the weights of 2, 7 and 3 cwt.; what must be the length of the shaft from centre to centre of the extreme wheels, in order that a transverse girder placed at the middle of its length, shall remove the pressure from the gudgeons, and sustain the system at rest, the distance between the first extreme and intermediate wheels being 12 feet?

Here we have given, $a=2$; $m=7$; $b=3$, and $\delta=12$; let these numbers be substituted in equation (g), and we have

$$d=\frac{2\times7\times12}{2+7-3}=28 \text{ feet, the distance between centre and}$$

centre of the extreme wheels a and b; consequently, the place of the common centre of gravity is 14 feet from each extreme, and 2 feet from the place of the intermediate wheel.

Verification of the foregoing result.

To verify the result, compute the place of the common centre of the two wheels m and b, by the first problem, thus,

$$\frac{16\times3}{7+3}=4\cdot8 \text{ feet distant from } m; \text{ consequently,}$$

the distance between the centre of the system and that of the two bodies m and b, is $4\cdot8-2=2\cdot8$ feet; then, by equation (2), we have

$$14\,a=2\cdot8\,(m+b), \text{ that is}$$
$$14\times2=2\cdot8\times10=28.$$

The analytical operation is as below, thus,

Let $2\,x$ = the whole length of the connecting line or shaft,
x = the distance of each extreme wheel from the centre of the shaft, and
$x-12$ = the distance of the intermediate wheel; then, by the principle of equation (2), we have

$$2\,x+(x-12)\times7=3x,$$
$$7\,x-84=x$$
$$6x=84$$
$$x=14,$$

or $2\,x=14\times2=28$ feet, the same as before.

Analytical Deductions from the foregoing Equation.

24. Referring to equation (g), if we attentively examine the relation that subsists among the quantities of which it is composed, it will occur to us, that:

1. If b be greater than $a+m$, the value of d is negative, and the shaft must lie in a different direction with respect to the wheel a, from what we have supposed in the question; or, in other words, a becomes the intermediate wheel, and b and m the extremes, which is contrary to the supposition; consequently, in order that the conditions may be fulfilled, it is necessary that $a+m$ shall be greater than b.

2. If b be equal to $a+m$, the value of d is infinitely great; that is, the value of d must be such, that the value of δ in respect of it shall vanish; but because δ is supposed to be a real positive quantity, its value as compared with that of d cannot vanish, unless d be con-

sidered as greater than any assignable quantity whatever; that is, infinitely great: but the same thing is obvious from the equation itself; for if b be equal to $a+m$, the denominator vanishes, and by the operations of algebra, we know that if any quantity is divided by *zero*, the quotient is to be considered as greater than any real assignable magnitude whatever.

3. If b be less than $a+m$, then the value of d is positive; but in order that b may always occupy a prominent place in the system, it is necessary that $2m$ shall be greater than $(a+m)-b$; otherwise, the value of δ will exceed that of d, and m will assume the place of an extreme body, and b will become the intermediate one, which is contrary to the supposition.

Hence, we infer, that the only limitation necessary in the present instance, is, that $(a+m)$ shall be greater than b, and $2m$ greater than $(a+m)-b$.

If $(a+m)-b=2m$, then $d=\delta$, and the bodies m and b coalesce, reducing the system to that of two bodies only, which brings us back to the solution of the first problem, alrèady fully exemplified.

PROBLEM 4. *To find the centre of gravity of three bodies acting in the same straight line, when the weight and the connecting bar becomes an element of the general mass.*

25. In what we have hitherto performed with regard to the common centre of gravity of three bodies, acting in the same straight line, the connecting bar or lever has been considered as devoid of gravity or weight, but in order that the subject may be rendered as complete as possible, we shall, in what follows, take into account the weight of the bar itself.

For this purpose let the foregoing notation remain; and let w denote the weight of one unit in length of the bar; then if the bar be considered uniform in shape and density throughout the whole of its length, the centre of gravity of each segment made by the centre of the system, will occur at the middle of its length, and the weight of the segments will be wx, and $w\,(d-x)$ respectively, making the effective energies, $\frac{1}{2}wx^2$ and $\frac{1}{2}w\,(d-x)^2$: consequently, in case of an equilibrium, we have

$$ax+m\,(x-\delta)+\tfrac{1}{2}wx^2=b(d-x)+\tfrac{1}{2}w\,(d-x)^2:$$
$$\text{or } ax+\tfrac{1}{2}wx^2=b\,(d-x)+m\,(\delta-x)+\tfrac{1}{2}w\,(d-x)^2:$$

but in either case, after the requisite reductions, we find generally, that

$$x=\frac{(2b+wd)\,d+2m\delta}{2\,(a+b+m+wd)}. \qquad (h)$$

This equation, as might be expected, is more complicated than any of the former, nevertheless it is not difficult to reduce, as will become manifest from the following examples.

Example 1. A cast iron shaft, 4 inches square and 36 feet between gudgeon and gudgeon, is required to sustain three wheels of the weights of 4, 7 and 6 cwt., placed at the distances of 14 and 22 feet from each other; at what point of the shaft must an upright support be placed to remove the whole pressure from the gudgeons, and balance the shaft with its appendages?

In this example there are given, $a=4$; $m=7$; $b=6$; $d=36$, and $\delta=14$; let these numbers be substituted for the literals in equation (h), and it becomes

$$x=\frac{(2\times 6+36w)36+2\times 7\times 14}{2(4+6+7+36w)}$$

Now, since the material of which the shaft is constructed is cast iron, the weight of one foot in length, or the value of w is $4\times 4\times 3\cdot 2=51\cdot 2$lbs.*; consequently, by employing $51\cdot 2$ instead of w in the foregoing value of x, we shall obtain

$$x=\frac{(\overline{2\times 6+36\times 51\cdot 2})\times 36+2\times 7\times 14}{2(4+6+7+\overline{36\times 51\cdot 2})}=18 \text{ feet, very nearly;}$$

therefore, the place where the support ought to be fixed, is 18 feet from each of the extreme wheels, and 4 feet from the intermediate one; but if the weight of the shaft had not been taken into the account, equation (e) would have given

$$x=18\tfrac{8}{17} \text{ feet;}$$

hence, the effect that the weight of the shaft produces on the place of the centre, is $\frac{8}{17}$ of a foot, or very nearly half a foot, a quantity which may safely be disregarded in large constructions.

Example 2. Three bodies, whose weights are respectively, 24, 40, and 53 tons, are supported on a straight uniform beam of cast iron, at the distances of 16 and 29 feet from each other; at what point ought the beam to be supported, so that the whole system may remain in equilibrio, the depth of the beam being 18 inches, and its breadth 2 inches?

Here, we have given, $a=24$; $m=40$; $b=53$; $d=16+29=45$, and $\delta=16$, and since the depth of the beam is 18 inches, while its breadth is 2 inches, the weight of one foot, or the value of w, is $18\times 2\times 3\cdot 2=115\cdot 2$lbs., let these numbers be substituted for the literals in equation (h), and it becomes

$$x=\frac{(\overline{2\times 53+115\cdot 2\times 45})45+2\times 40\times 16}{2(24+53+40+\overline{115\cdot 2\times 45})}=22\cdot 574 \text{ feet,}$$

hence, the common centre of gravity of the bodies and the beam is, in the beam, 22·574 feet from one extreme, 22·426 from the other extreme, and 6·574 feet from the intermediate body.

* The weight of a bar of cast iron, one inch square, and 12 inches long, is 3·2 pounds.

If the place of the centre be calculated by equation (e), disregarding the weight of the beam, it will be

$$x=25 \cdot 854 \text{ feet};$$

consequently, the effect of the weight of the beam in this example, is very considerable, being not less than $3 \cdot 28$ feet.

An equation to express, generally, the difference between the value of x, as calculated by the equations (e) and (h), would be very complicated, and at the same time, it would lead to nothing of very great practical utility. We shall not, therefore, attempt to give a formula for that purpose in this place, but proceed.

PROBLEM 5. *To determine the relative positions of the bodies with respect to each other, when the common centre of gravity of the system, is situated in the middle of the connecting line.*

26. Since d is taken to denote the whole distance between the centres of the extreme bodies, and x to denote the distance between the first extreme and the centre of the system, it follows, that when this centre falls in the middle of the connecting line, we have

$$x=\tfrac{1}{2}d;$$

therefore, if $\frac{1}{2}d$ be substituted for x in equation (h), it becomes

$$d=\frac{(2b+wd)\,d+2m\delta}{(a+b+m+wd)},$$

which, by reduction, gives

$$d=\frac{2m\delta}{(a+m-b)},$$

$$\text{and } \delta=\frac{d}{2m}(a+m-b);$$

equations perfectly identical with (g) and (f), already established; from which we infer, that when the system is in equilibrio, at the middle of the connecting line, the weight of such line or bar, has no tendency to disturb the equilibrium, and consequently, in that case, it need not be considered in the calculation.

SECTION THIRD.

OF THE CENTRE OF GRAVITY OF FOUR OR MORE BODIES SITUATED IN THE SAME RIGHT LINE.

27. In a similar manner to that which we have exhibited for determining the centre of gravity of three bodies, may the centre of gravity of four or more bodies be found, supposing them to be

all situated in the same right line; indeed, the law of continuation is so obvious, that it is almost unnecessary to dwell longer on the subject; we shall, therefore, pursue it only one step farther, by deriving the equation for four bodies, leaving its full developement, and the extension to a greater number, for exercise to the reader.

Let a, m, n and b, represent the four bodies taken in order, from the first a, to the last b; and let δ denote the distance from a to m; δ' the distance from a to n, and d the distance from a to b; x denoting the distance from a to the place of the common centre of the system.

Then, if we suppose that the bodies a and m subsist on one side of the common centre, while the bodies n and b subsist on the other; we shall have x; $(x-\delta)$; $(\delta'-x)$, and $(d-x)$, for the respective distances of the bodies from the centre of gravity; consequently, by the principle indicated in equations (1) and (2), we have

$$ax+m(x-\delta)=n(\delta'-x)+b(d-x);$$

which, by transposition and division, gives

$$x=\left(\frac{m\delta+n\delta'+bd}{a+m+n+b}\right) \qquad (i)$$

The following is the practical rule furnished by this equation.

Rule. *Multiply the magnitude or density of each body, by its respective distance from the beginning of the system, and divide the sum of the products by the sum of the bodies for the distance of the centre of gravity sought.*

The following examples will suffice for the illustration of equation (i).

Example 1. Four bodies connected together by a straight inflexible bar or lever, have their weights equal respectively to 18, 26, 12, and 30 cwt.; at what point in the length of the bar is the common centre of gravity, the distance between the bodies being as below,

distance from a to m, 17 feet,
———— a — n, 23——,
———— a — b, 40——?

Here, we have given, $a=18$; $m=26$; $n=12$; $b=30$; $\delta=17$; $\delta'=23$; and $d=40$; let these values be substituted in equation (i), and it becomes

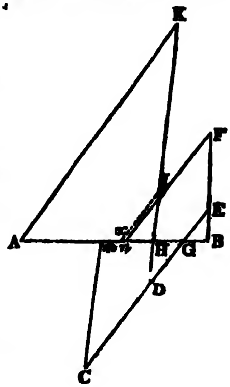
$$x=\frac{\overline{26\times17}+\overline{12\times23}+\overline{30\times40}}{18+26+12+30}=22\cdot3 \text{ feet from } a;$$

$5\cdot3$ feet from m; $\cdot7$ from n, and $17\cdot7$ from b.

Geometrical Construction.

It will however, render the subject somewhat more complete, to exhibit the method of continuation graphically; for which purpose,

Let AB be a straight line passing through the centres of the bodies a, m, n and b. Make AB equal to 40 feet, taken from a scale of equal parts of any convenient magnitude whatever; on the straight line AB, set off Am and An equal respectively to 17 and 23 feet, taken from the same scale as AB; then are the points A, m, n, and B, the positions of the four bodies, whose weights are given, and whose common centre of gravity is required to be determined.



Through the point B, draw the straight line BF in any direction at pleasure; make BE proportional to the weight of the body n, and EF proportional to the weight b; join Fn, and through the point E, draw EG parallel to Fn; then is the point G the common centre of gravity of the bodies b and n.

Produce EG to C, making GD proportional to the weight of the body m, and DC equal to BF, or proportional to the sum of the bodies b and n; join Cm, and through the point D, draw DH parallel to Cm; then is the point H, the common centre of gravity of the three bodies m, n, and b.

Produce DH to the point K, making HI proportional to the weight of the body a, and IK equal to GC, or proportional to the sum of the bodies m, n, and b; join KA, and through the point I, draw Ix parallel to KA, then is the point x the centre sought.

Demonstration. Since the preceding construction involves the combinations exhibited in equation (i), it is obvious that the method of finding the place of the common centre of gravity, may be traced with great facility from the diagram, in the following manner.

In the similar triangles BEG and BFn, it is

$$\text{BF} : \text{B}n :: \text{EF} : n\text{G} : \text{that is,}$$

$$(n+b) : (d-\delta') :: b \cdot n\text{G} = \frac{b\,(d-\delta')}{(n+b)};$$

then in the similar triangles HDG and mCG, it is

$$\text{GC} : \text{G}m :: \text{DC} : m\text{H}, \text{ that is,}$$

$$(m+n+b) : \frac{b\,(d-\delta)+n\,(\delta'-\delta)}{(n+b)} :: (n+b) : m\text{H} = \frac{b\,(d-\delta)+n\,(\delta'-\delta)}{(m+n+b)}.$$

Again, in the similar triangles HIx and HKA, it is

HK : HA :: KI : Ax; that is

$$(a+m+n+b):\left(\frac{bd+m\delta+n\delta'}{m+n+b}\right):(m+n+b):\text{A}x;$$

which, by equating the product of the extremes and means, gives

$$\text{A}x=\left(\frac{bd+m\delta+n\delta}{a+m+n+b}\right), \text{ the same as equation } (i).$$

The reader will be the better enabled to trace the terms of the three foregoing proportions, by observing the following comparisons, viz.

$$\text{BF}=n+b; \qquad \text{B}n=d-\delta'; \qquad \text{EF}=b$$

$$\text{GC}=m+n+b; \qquad \text{G}m=\frac{b(d-\delta)+n(\delta'-\delta)}{n+b}; \qquad \text{DC}=n+b;$$

$$\text{HK}=a+m+n+b; \qquad \text{HA}=\frac{bd+m\delta+n\delta'}{n+m+b}; \qquad \text{KI}=m+n+b.$$

The formation of these terms from the parts of the figure, is too obvious to require illustration.

EXAMPLE 2. There are four cast iron wheels, whose weights are respectively 4, 5, 6 and 7 cwts., and their distances from each other taken in order, are 8, 10 and 12 feet; what is the distance of each wheel from the common centre of gravity, supposing the shaft on which the wheels are suspended, to have no influence on its position?

Here, we have given, $a=4$; $m=5$; $n=6$; $b=7$; $\delta=8$; $\delta'=18$ and $d=30$; let these numbers be substituted in equation (i), and it becomes

$$x=\frac{\overline{5\times 8}+\overline{6\times 18}+\overline{7\times 30}}{4+5+6+7}=16\tfrac{3}{11} \text{ feet from the body } a,$$

$8\frac{3}{11}$ from m, $1\frac{8}{11}$ from n, and $13\frac{8}{11}$ from b.

The numerical operation deduced from the preceding diagram, consists of three elegant proportions, which are these:

$$6+7:30-18::7:n\text{G}=\tfrac{84}{13},$$
$$5+6+7:\tfrac{84}{13}+(18-8)::6+7:m\text{H}=\tfrac{214}{18},$$
$$4+5+6+7:\tfrac{214}{18}+8::5+6+7:\text{A}x=\tfrac{358}{22};$$

consequently, by division we get

$$\text{A}x=16\tfrac{3}{11} \text{ feet, the same as before.}$$

Thus have we deduced from principles of great simplicity, the theory of the common centre of gravity for several bodies situated in the same right line; we shall in the next section proceed to determine its position, when the bodies are placed under any other circumstances whatever.

Section Fourth.

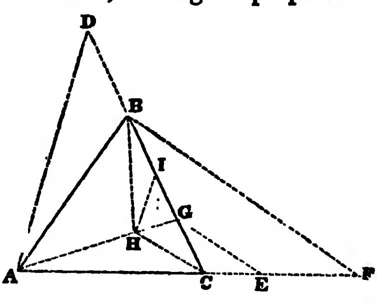
OF THREE BODIES ANY HOW POSITED IN SPACE, AND CONNECTED TWO AND TWO.

28. Of three bodies any how posited in space, and connected two and two by straight lines, which meeting in the centres of the three bodies, become the sides of a plane triangle, in whose surface the common centre of gravity of the system must occur.

If the distance between the centre of gravity of one body, and the common centre of the other two were known, the resolution of this case, would differ but very little from that of three bodies existing in the same straight line; but since that distance depends on the position of the common centre of the two bodies taken in connection, the complete developement involves the consideration of other principles, as will become manifest from the following problem.

Problem. *Let the points* A, B *and* C, *be the positions of the three bodies, whose magnitudes or weights are represented by a, b and c, and whose common centre of gravity is required to be found.*

Draw AB, BC and AC, and produce AC to F, making CE proportional to the magnitude or weight of the body *b*, and EF proportional to that of the body *c*; draw FB, and through the point E, draw the straight line EG parallel to FB, meeting BC in G; then is G the common centre of gravity of the two bodies *b* and *c*.



Draw the straight line AG, and produce CB to the point D, making GI proportional to the magnitude or weight of the body *a*, and ID equal to CF, or proportional to the sum of the bodies *b* and *c*; join DA, and through the point I, draw IH parallel to AD, meeting AG in the point H; then shall H be the common centre of gravity of the three bodies *a*, *b* and *c*, whose situations in space are indicated by the positions of the points A, B, and C. Join BH and CH; then are AH, BH and CH, the distances of the common centre from each of the bodies.

By examining the above construction, it is obvious, that the magnitudes or weights of the two bodies *b* and *c*, may bear such a relation to one another, as to cause their common centre of gravity G, to fall at any point in the line BC; consequently, the magnitude and position of the line AG, in which the common centre of the system occurs, must remain unknown until the position of the point G has been ascertained.

Let $d=$ AB, the distance between the centre of the two bodies a and b,
$\delta=$ BC, ——————————————————— b—— c,
$\delta'=$ AC, ——————————————————— a—— c,
$x=$ BG, ———————————— centre of the body b and the common centre of gravity of the two bodies b and c,
$x'=$ AH, the distance between the centre of the body a and the common centre of gravity of the three bodies a, b and c,

and $\varphi=$ the angle ABC, which can always be found, when the three sides of the triangle, d, δ and δ', or the respective distances between the centres of the bodies are known.

Then, by Plane Trigonometry, we obtain

$$\text{AG}=\sqrt{d^2+x^2\mp 2\,dx\cos.\varphi},$$

but by equation (a), we have

$$x=\frac{c\delta}{b+c}$$

and consequently, by involution, its square is

$$x^2=\frac{c^2\,\delta^2}{(b+c)^2}$$

and moreover, by Plane Trigonometry, we get

$$\cos.\varphi=\frac{d^2+(\delta+\delta')(\delta-\delta')}{2d\delta}.$$

Let these values of x, x^2 and cos. φ be substituted instead of them, in the above expression for the value of AG, and it becomes

$$\text{AG}=\sqrt{d^2+\frac{c^2\,\delta^2}{(b+c)^2}\mp\frac{c\,\{d^2+(\delta+\delta')\,(\delta-\delta')\}}{b+c}}.$$

By reducing the fractions to a common denominator, and collecting the terms, the above equation will then become transformed into the following, viz.

When φ is acute, or less than a right angle, we have

$$\text{AG}=\frac{1}{b+c}\sqrt{(b+c)(bd^2+c\delta'^2)-bc\delta^2} \qquad (k)$$

When φ is obtuse, or greater than a right angle, we have

$$\text{AG}=\frac{1}{b+c}\sqrt{(b+c)\{bd^2+c(\delta'^2-\delta^2)\}+c^2\delta^2} \qquad (l)$$

29. Although these equations involve no Trigonometrical value of the angle φ, yet it is easy to perceive, that their form is in a great measure dependent on its magnitude, and consequently, its relation to a right angle or 90 degrees, must always be correctly ascertained before the calculation is begun, in order that the proper formulæ may be employed; for it is evident, that the equations (k) and (l), although they are both adapted to the determination of the line subtending the angle φ, are by no means similar, and therefore, if they should be used indiscriminately in any case, we may probably arrive at an erroneous result. We are, however, in the possession of means to guard against this contingency, and since these means are very easily applied, we shall, before proceeding to the application of the foregoing theory, specify them in detail.

1. If the square of δ', the side subtending the angle φ, is less than the sum of the squares of d and δ, the sides containing that angle; then is φ less than a right angle, in which case, equation (k) must be applied.

2. But if the square of δ', the side subtending the angle φ, is greater than the sum of the squares of d and δ, the sides containing it; then is φ greater than a right angle, and equation (l) has place.

These particulars being properly attended to, will guard the operator against the probability of error, arising from the employment of the wrong formula; and in this view of the matter the reader will consider them equivalent to a *rule*

The following examples will suffice to exemplify what we have said on this point.

They comprise appropriate methods of practically applying the formulæ of equations k and l, first, when two of the bodies form the denominator of the fraction which precedes the radical sign.

Example 1. The weights of three bodies situated in space, but not in the same right line, are 4, 6 and 8 tons respectively, while their distances asunder are 12, 14 and 16 yards; what is the length of the line in which the common centre of gravity of the system occurs, or that which joins the body a to the common centre of the bodies b and c?

Numerical Calculation.

In this example, there are given, $a=4$; $b=6$; $c=8$; $d=12$, $\delta=14$, and $\delta'=16$; then, since δ'^2 is less than the squares of d and δ taken conjointly, we know, from what is stated above, that φ, the angle which is subtended by δ' is less than a right angle, or 90 degrees; consequently, the question as it now stands, must be resolved by the first of the preceding reduced equations, or that marked (k), in the following manner.

$$bd^2=6\times 12\times 12=\ \ 864$$
$$c\delta'^2=8\times 16\times 16=2048$$
$$bd^2+c\delta'^2=\overline{2912};$$

therefore, $(b+c)\,(bd^2+c\delta'^2)=\ 14\times 2912\ =\ 40768$, the first member under the radical sign;

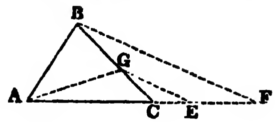
and again, $bc\delta^2=-6\times 8\times 14^2=-9408$, the second member under the radical sign;

therefore the sum is $=\ 31360$, and the square root is, $177\cdot 0875$; wherefore, by division, we have

$$\text{AG}=\frac{177\cdot 0875}{6+8}=12\cdot 65 \text{ yards, very nearly.}$$

The following geometrical construction will verify the result obtained from the equation marked k, and traced in the numerical operation agreeably to article 29, and its first process of application.

Geometrical Construction.



Make AB, BC and AC, respectively equal to 12, 14 and 16 yards, taken from a scale of equal parts of any convenient magnitude at pleasure; then shall the angular points A, B and C represent the positions of the three bodies *a*, *b* and *c*, whose weights or magnitudes are respectively measured by the numbers 4, 6 and 8.

Produce AC to the point F, making the lines CE and EF, respectively proportional to the numbers 6 and 8, which measure the weights of the bodies *b* and *c*; join FB, and through the point E, draw EG parallel to FB, meeting BC in the point G; then is G the common centre of gravity of the bodies *b* and *c*; join GA, which is the line required to be found, and which passes through the common centre of the bodies *a*, *b* and *c*; if AG be taken in the compasses, and applied to a scale of equal parts, of the same dimensions as that from which the lines AB, BC and AC were measured, it will indicate 12·65 yards, the same as before.

Example 2. The masses of three bodies are in proportion to each other, as the numbers 37, 18 and 49, while the distances between their centres taken two and two, are as the numbers 56, 43 and 72; what is the length of the line connecting the common centre of gravity of the greatest and least bodies with the centre of the intermediate one?

Numercal Calculation.

In this example we have given, not the absolute magnitudes of the bodies, nor the absolute distances, but merely the ratio which they bear to one another; yet this circumstance can make no difference whatever in the method of solution, for the resulting line will bear exactly the same relation to its absolute magnitude, which the given lines bear to their absolute magnitudes; we may therefore, proceed to resolve the question, precisely as if the numbers which express the ratios, were those which express the magnitudes.

Here then, we have given, from the data in question, $a=37$; $b=18$; $c=49$; $d=56$; $\delta=43$, and $\delta'=72$.

Then, because the square of δ', the side subtending the angle φ, exceeds the sum of the squares of d and δ, the sides containing it, we know that φ, the angle subtended by δ' is greater than a right angle, and consequently, the question must be resolved by equation (l), in the following manner, where the separate members under the radical sign, are traced in conformity with the means which we pointed out in article 29, and which were limited in their application by the remark numbered 2, page 39.

$$bd^2 = 18 \times 56 \times 56 = 56448$$
$$c(\delta'^2 - \delta^2) = 49(5184 - 1849) = \underline{163415}$$
$$(b+c)\{bd^2 + c(\delta'^2 - \delta^2)\} = 67 \times 219863 = 14730821, \text{ first member,}$$
$$c^2\delta^2 = (49 \times 43)^2 = 2107^2 = \underline{4439449}, \text{ second member,}$$
$$\text{sum of the members} = 19170270,$$

and the square root of 19170270 is 4378·38;

wherefore, by division, we have,

$$\text{AG} = \frac{4378 \cdot 38}{18+49} = 65 \cdot 35, \text{ very nearly;}$$

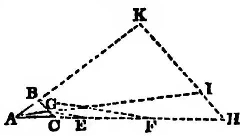
then, suppose that the absolute magnitude of δ', the side opposite to the obtuse angle φ, is 432 instead of 72; that is, six times greater, then we have

$$72 : 432 :: 65 \cdot 35 : 392 \cdot 2$$

consequently, the length of the line which joins the centre of the body a, with the common centre of gravity of the bodies b and c, has thus been determined, and the following construction will verify the result.

Geometrical Construction.

Construct the triangle ABC such, that its sides AB, BC and AC shall be respectively equal to the numbers 56, 43 and 72, taken from a scale of equal parts of any magnitude at pleasure; then shall the angular points B and C, denote the relative positions of the bodies b and c, while the point A denotes the absolute position of the body a.



Produce the side AC indefinitely towards H, making CE proportional to 18, and EF proportional to 49, the numbers which express the ratio of the masses b and c; join FB, and through the point E draw EG parallel to BF, meeting BC in the point G; join GA, which will fix the position of the line required.

Make AH equal to six times the length of AC, and through the point H draw the line HK parallel to BC; produce AB and AG, to meet HK in the points K and I; then shall the points K and H denote the absolute position of the bodies b and c, and I the position of their centre of gravity.

Then, if the line AI be taken in the compasses, and applied to the same scale of equal parts, on which the lines AB, BC and AC were measured, it will be found to indicate 392·2, the same as was found by the calculation.

The object of this example is to show, that although the numbers concerned in the question are large and unmanageable, either by calculation or construction, they may be reduced in any pro-

portion whatever to accommodate them to practice, and yet the result be strictly accurate, for the line AG, which obtains from the first part of the construction, being applied to the scale, will indicate 65·35, which being taken six times, will give the whole length of the line AI.

It is scarcely necessary to remark, that any other proportion may be adopted instead of that of one to six, and the result be equally rigorous, but the absolute magnitude of the original numbers will always suggest the most convenient relative reduction. The original numbers in the present example are, for the masses 222, 108 and 294; and for the distances, they are 336, 258 and 432 respectively, which numbers would have greatly increased the labour of calculation, and to have taken them from a scale of such dimensions, as to be susceptible of accurate measurement, would have required a diagram inconveniently large; and to avoid this inconvenience, the preceding method of solution has been adopted, and may be very advantageously resorted to in all similar cases.

Secondly. When the sum of the three bodies enter the denominator of the fraction which precedes the radical sign.

30. Having thus determined the magnitude of the line AG, in which the common centre of gravity of the system occurs; we have next to suppose the bodies b and c, to be brought together as a single body in the point of their common centre of gravity o; then, by equation (a), we obtain,

For the acute value of φ,

$$\text{AH}=\frac{1}{a+b+c}\sqrt{(b+c)\,(bd^2+c\delta'^2)-bc\delta^2}\,. \qquad (m)$$

For the obtuse value of φ,

$$\text{AH}=\frac{1}{a+b+c}\sqrt{(b+c)\{bd^2+c\,(\delta'^2-\delta^2)\}+c^2\delta^2}\,. \quad (n)$$

Where the point H is the common centre of the system. (See the general diagram).

If we compare the equations (m) and (n) just derived, with those marked (k) and (l) preceding; it will be found that they differ in nothing but the introduction of the third body a, in the denominator of the fraction which precedes the radical sign; consequently, the operation will differ in nothing from that displayed in the two preceding examples, except in using the sum of the three bodies a, b and c for a divisor, instead of the two bodies b and c.

One example will suffice for the exemplification of each of the foregoing equations, according to the acute and obtuse values of the angle φ.

Example 1. The weights of three bodies, not situated in the same straight line, are equal respectively to 4, 6 and 8 tons, and the distances between their centres, are 12, 14 and 16 yards; at what distance from each body is the common centre of gravity, supposing the straight lines by which they are connected, to have no influence whatever on the equilibrium?

Numerical Calculation.

In this example, there are given, $a=4$; $b=6$; $c=8$; $d=12$; $\delta=14$, and $\delta'=16$.

Then because δ'^2 is less than $d^2+\delta^2$, the angle φ which is subtended by the side δ', is less than a right angle or 90 degrees; consequently, the question must be resolved by equation (m), and the operation is as below, viz.

$$bd^2=6\times 12\times 12=\ 864,$$
$$c\delta'^2=8\times 16\times 16=2048,$$
$$bd^2+c\delta'^2=\overline{2912};$$

therefore, $(b+c)\,(bd^2+c\delta'^2)=14\times 2912=40768$, the first member under the radical sign,

and $bc\delta^2=-6\times 8\times 14\times 14=-\ 9408$, the second member under the radical sign,

sum of the members $=31360$,

its square root is $177\cdot 0875$;

wherefore, by division, we obtain

$$\text{AH}=\frac{177\cdot 0875}{4+6+8}=9\cdot 84 \text{ yards, very nearly.}$$

The value of AG is found by equation (k), but having already determined the radical or surd part of that equation, it is not necessary to repeat the whole process to discover the value of AG; for as we have already observed, the only difference between the expressions for the values of AH and AG is, that the multiplier for the one is $\frac{1}{a+b+c}$, and for the other, it is $\frac{1}{b+c}$; therefore, if the radical member as determined for AH, be multiplied by $\frac{1}{c+b}$, we have the value of AG; that is

$$177\cdot 0875\times \tfrac{1}{14}=12\cdot 65, \text{ nearly.}$$

We have shown, that by equation (a),

$$\text{BG}=\frac{c\delta}{b+c},$$

which in the present example, is

$$\text{BG}=\frac{8\times 14}{6+8}=8;$$

consequently, $\text{CG}=14-8=6$.

Then, in each of the triangles BAG and CAG, there are given the three sides, to find the angles GAB and GAC, which by Plane Trigonometry, are determined in the following manner, to wit,

$$\cos.\ \text{GAB}=\frac{12^2+(12\cdot 65)^2-8^2}{2\times 12\times 12\cdot 65}=\cdot 79058,$$

and similarly

$$\cos.\ \text{GAC}=\frac{16^2+(12\cdot 65)^2-6^2}{2\times 16\times 12\cdot 65}=\cdot 94878.$$

Then in the triangles BAH and CAH, we have given the sides BA, AH and CA, AH, with the natural cosines of the contained angles; to find the sides BH and CH subtending those angles.

Consequently, by Plane Trigonometry, we have

$$\text{BH}=\sqrt{12^2+9\cdot 84^2-2\times 12\times 9\cdot 84\times\cdot 79058}=7{\cdot}35 \text{ yards nearly.}$$

$$\text{CH}=\sqrt{16^2+9\cdot 84^2-2\times 16\times 9\cdot 84\times\cdot 93878}=7{\cdot}56 \text{ yards nearly.}$$

Therefore, the distances of the three bodies from the common centre of gravity, are respectively as follows, viz.

Distance of the common centre from a, is 9·84 yards,
—————————————— b, — 7·35 ——
—————————————— c, — 7·56 ——

31. But instead of finding the values of the lines BH and CH, by the method exemplified above, which may be justly objected to as a tedious operation, they can be found directly from the data, without ascertaining beforehand the values of the parts BG, GC, AG and AH, which must always be known previously, if the foregoing mode of operation is to be applied.

The two following symmetrical equations, investigated after the manner of equation (n), are applicable to this purpose. viz.

$$\text{BH}=\frac{1}{a+b+c}\sqrt{(a+c)\{ad^2+c(\delta^2-\delta'^2)\}+c^2\delta'^2}. \qquad (o)$$

$$\text{CH}=\frac{1}{a+b+c}\sqrt{(a+b)\{a\delta'^2+b(\delta^2-d^2)\}+b^2d^2}. \qquad (p)$$

We shall calculate the value of BH and CH by these equations, as the coincidence of the results will manifestly be the means of establishing the truth of both methods.

And first, for the value of BH, we have

$$ad^2=4\times 12^2=576,$$

$$c(\delta^2-\delta'^2)=8(14^2-16^2)=-480,$$

$(a+c)\{ad^2+c(\delta^2-\delta'^2)\}=12\times 96=1152$, the first member under the radical sign,
$c^2\delta'^2=8^2\times 16^2=16384$, the second member under the rad. sign,
therefore, the sum of the memb. $=17536$, its square root is 132·42, which being divided by 18 the sum of the bodies, gives

$$\text{BH} = \frac{132 \cdot 42}{18} = 7 \cdot 35 \text{ yards.}$$

Again for the value of CH, we have

$a\delta'^2 = 4 \times 16^2 = 1024,$

$b(\delta - d^2) = 6(14^2 - 12^2) = \ 312,$

$(a+b)\{a\delta'^2 + b(\delta^2 - d^2)\} = 10 \times \overline{1336} = 13360$, the first memb. under the rad. sign,

$b^2 d^2 = 6^2 \times 12^2 = \ 5184$, the secd. mem. under the rad. sign,

therefore the sum of the members $= \overline{18544}$, its square root is 136·1, which being divided by 18, the sum of the bodies, gives

$$\text{CH} = \frac{136 \cdot 1}{18} = 7 \cdot 56 \text{ yards.}$$

Here then, the results are the same as found before, but it is easy to perceive that the labour is considerably less by this latter method, and since it affords the advantage of computing each distance independently of the others, and directly from the given quantities, besides familiarizing the reader with the use and application of the formulæ, we earnestly recommend its general adoption.

EXAMPLE 2. The masses or weights of three bodies not situated in the same straight line, are equal respectively to 28, 12 and 44 tons, and the respective distances are 56, 42 and 84 feet; at what distance from each body is the common centre of gravity, supposing the lines connecting the bodies to have no influence on the equilibrium?

NUMERICAL CALCULATION.

Here, we have given, $a=28$; $b=12$; $c=44$; $d=56$; $\delta=42$, and $\delta'=84$.

Then, because the square of δ', the side subtending the angle φ, is greater than the sum of the squares of d and δ, the sides containing it, φ is greater than a right angle or 90 degrees; therefore, in order to find the distance AH, we must employ equation (n) as follows:

$bd^2 = 12 \times 56 = \ 37632,$

$bd^2 = 12 \times 56 \times 56 = \ 37632,$ [lum.

$b \times c)\{bd^2 + c(\delta'^2 - \delta^2)\} = 56 \times \overline{270480} = 15146880$, first term under the vincu-

$c^2\delta^2 = 44^2 = 42^2 = 3415104$ second term under the vincul.

sum of the terms $= \overline{18561984}$, its square root is 4308·36, wherefore, by division, we have

$$\text{AH} = \frac{4308 \cdot 36}{28 + 12 + 44} = 51 \cdot 29 \text{ feet distant from } a.$$

Then to find the value of BH, equation (o) gives the following operation, viz.

$ad^2 = 28 \times 56 \times = \ 87808,$

$c(\delta^2 - \delta'^2) = 44(42^2 - 84^2) = 232848,$ [member

$(a + c)\{ad^2 + c(\delta^2 - \delta'^2) = 72 \times - \overline{145040} = -10442880$, the first

$c^2\delta'^2 = 44^2 \times 84^2 = 3696^2 = \ 13660416$, the sec. member; therefore, the sum of the members $= \overline{3217536}$, and its

square root is 1793·75; which being divided by 84, the sum of the masses, gives

$$\text{BH}=\frac{1793\cdot75}{84}=21\cdot35 \text{ feet, very nearly, distant from } b.$$

Again, to find the value of CH, equation (p) gives the following operation, viz.

$$a\delta'^2 = \quad 28\times84\times84=197568,$$
$$b(\delta^2-d^2)=12(42^2-56^2)=-\ \underline{15264},$$
$$(a+b)\{a\delta'^2+b(\delta^2-d^2)\}=40\times\overline{182304}=7292160, \text{ the first member,}$$
$$b^2d^2=12^2\times56^2=672\times672 \ =\ \underline{451584}, \text{ the second mem.}$$

therefore, the sum of the members $=\overline{7743744}$, and its square root is, 2782·76; which being divided by 84, the sum of the masses, gives

$$\text{CH}=\frac{2782\cdot76}{84}=33\cdot13 \text{ feet nearly, distant from } c.$$

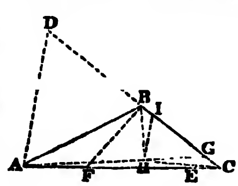
Consequently, the distances of the three bodies, from the common centre of gravity, are respectively as below, viz.

Distance of the common centre from a, is 51·29 feet,
———————————————— b,—21·
———————————————— c,—33·1[illegible]

However simple and elegant the foregoing operations may appear, yet it is a fact not to be denied, that in questions of this sort, the numerical process in point of facility falls far short of the graphic construction; but then, what is lost in this respect is amply compensated by the superior accuracy of the results obtained by calculation; nevertheless, if the construction be performed with good instruments and great care, the results obtained in this way will be found sufficiently accurate for every practical purpose; but when the results are sought to be incorporated with ulterior mathematical operations, they must be determined by calculation.

Here follows the construction of the above example.

Construct the triangle ABC, whose sides AB, BC and AC shall be respectively equal to 56, 42 and 84 feet; in AC, take CE and EF proportional to the numbers 12 and 44 respectively, and join FB; through the point E, draw EG parallel to FB, meeting the side BC in the point G; then is G the common centre of gravity of the two bodies b and c.



Join AG, and produce CB to D, making GI proportional to the number 28, and ID equal to CF, or proportional to 12+44=56; join DA, and through the point I draw IH parallel to AD, meeting AG in the point H; then is H the common centre of gravity of the three bodies a, b and c.

Draw BH and CH; then shall AH, BH and CH be the distances sought; which being taken in the compasses and applied to a scale

of equal parts will indicate the numbers 51·29; 21·35, and 33·13 respectively.

We have varied the mode of procedure in this instance, purposely to show, that we are under no restraint with respect to the process of construction, for, by whatever steps the operation is conducted, the result must be the same, because in whatever position the system of bodies may be presented, the place of the system, or the common centre of gravity of the bodies composing it, is a fixed and assignable point.

Thirdly. When the three lines, by which the centres of the bodies are connected, are equal among themselves, and the sum of the bodies becomes the denominator of the fraction which precedes the radical sign, as in the equations of equilibrium, marked (m) and (n).

32. If the three lines d, δ and δ' by which the centres of the bodies are connected, are equal among themselves; then, the distance of each body from the common centre of gravity of the system, is expressed by the following class of symmetrical equations, viz.

$$\left.\begin{aligned} &1.\ \text{AH}=\frac{d}{a+b+c}\sqrt{(b^2+bc+c^2)} \\ &2.\ \text{BH}=\frac{d}{a+b+c}\sqrt{(a^2+ac+c^2)} \\ &3.\ \text{CH}=\frac{d}{a+b+c}\sqrt{(a^2+ab+b^2)} \end{aligned}\right\} \qquad (q)$$

The following example will suffice to exemplify the use of these equations.

Example. The weights of three bodies, whose centres are at the angular points of an equilateral triangle, are equal respectively to 10, 20 and 30 lbs; whereabouts in the surface of the triangle is the common centre of gravity, supposing its side to be 36 inches in length?

Numerical Calculations.

Here, we have given, $a=10$; $b=20$; $c=30$, and $d=36$; then, to find the length of the line AH, by No. 1 class (q), it is

$b^2=20\times20=400$, first term under the radical sign,
$bc=20\times30=600$, second ———— ————,
$c^2=30\times30=900$, third ————————————; therefore,
$b^2+bc+c^2=1900$, its square root is 43·5889; which being multiplied by 36, the side of the triangle, and divided by 60, the sum of the bodies, gives

$$\text{AH}=\frac{43\cdot5889\times36}{60}=26\cdot15 \text{ inches distance from } a.$$

Then, to find BH, or the distance from b, No. 2, class (q) gives

$a^2 = 10 \times 10 = 100$, first term under the radical sign,
$ac = 10 \times 30 = 300$, second——————————,
$c^2 = 30 \times 30 = \underline{900}$, third——————————;

therefore, $a^2 + ac + c^2 = 1300$, its square root is 36·0555, which being multiplied by 36 the side of the triangle, and divided by 60 the sum of the bodies, gives $BH = \frac{36 \cdot 0555 \times 36}{60} = 21 \cdot 63$ inches, from b.

Again, to find CH, or the distance from c, No. 3, class (q), gives

$a^2 = 10 \times 10 = 100$, first term under the radical sign,
$ab = 10 \times 20 = 200$, second ——————————,
$b^2 = 20 \times 20 = \underline{400}$, third——————————;

therefore, $a^2 + ab + b^2 = 700$, its square root is 26·4556; which being multiplied by 36 the side of the triangle, and divided by 60 the sum of the bodies, gives

$$CH = \frac{26 \cdot 4556 \times 36}{60} = 15 \cdot 87 \text{ inches, distance from } c.$$

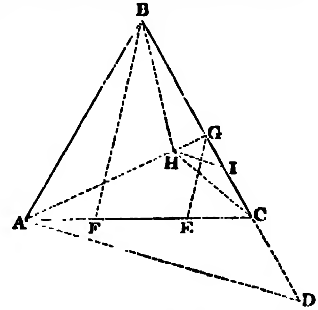
Therefore, the distances of the three bodies from the common centre of gravity, are respectively, as follows, viz.

Distance of the common centre from a, is 26·15 inches,
——————————————— b,—21·63———,
——————————————— c,—15·87———.

Geometrical Construction verifying the results obtained Numerically.

The following construction will serve to verify the results which the above operations have produced, and if the process be conducted with attention, the distances may be found from an accurate scale to the greatest nicety.

Describe the equilateral triangle ABC, such, that its sides AB, BC and AC, shall be respectively equal to 36 inches; in the side AC, take CE and EF proportional to the numbers 20 and 30, being the measure of the bodies which act at the points B and C; join FB, and through the point E, draw EG parallel to BF, meeting BC in G; then shall the point G be the common centre of the two bodies b and c.



Join AG, and produce BC to D in such a manner, that GI shall be equal or proportional to the number 10, and ID equal to CF, or proportional to 20+30=50; join AD, and through the point I, draw IH parallel to AD, meeting AG in the point H; then is H the centre of the system, or the common centre of gravity of the three bodies, a, b and c.

Draw BH and CH; then shall AH, BH and CH be the distances required, which being applied to a scale of equal parts, will be found to measure 26·15; 21·63 and 15·87 inches respectively, the same as before.

If the magnitudes of the bodies, as well as their mutual distances are equal; then, the distance of each body from the place of the system or common centre of gravity, will be expressed as follows, viz.

$$\text{AH} = \text{BH} = \text{CH} = \tfrac{d}{3}\sqrt{3} = \cdot 57735d. \qquad (r)$$

In this equation, the bodies which we have supposed to become equal to one another do not enter; this is in consequence of the supposition of equality, for since the bodies oppose each other mutually with equal intensities and at equal distances, it is manifest that their effects become neutralized with respect to the centre of gravity, and consequently, whatever may be the magnitudes of the bodies, provided they retain their equality, it is evident, that they cannot have any influence on the equilibrium or the position of the common centre; we may therefore, entirely disregard the effects of the bodies in our enquiry, and confine our attention to the consideration of the connecting lines only, and this brings us to a branch of the subject peculiarly adapted to all those departments of mechanical operations which are concerned in building, carpentry, and masonry.

SECTION FIFTH.

OF THE CENTRE OF GRAVITY OF THE PERIMETER OF RIGHT LINED FIGURES, WITH THE METHOD OF DETERMINING THE SIDES OF TRIANGULAR FIGURES WHEN THE DISTANCES OF THE ANGULAR POINTS FROM THE CENTRE OF GRAVITY OF THE FIGURES ARE KNOWN.

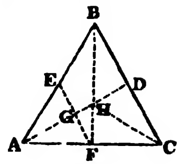
PROBLEM. *To determine the centre of gravity of the perimeter of a right lined figure; and first of a triangle.*

33. The figure now under consideration is an equilateral triangle, which, on account of the number and equality of its parts, is the least complicated of all straight lined superficies, and consequently, from this circumstance, as well as from its dependence on the foregoing principles, it claims the first of our attention.

By our scholium (art. 5) the centre of gravity of a straight line, is at the middle of its length; therefore,

Let ABC, be an equilateral triangle, the centre of gravity of whose perimeter is required to be found.

Bisect the sides AB, BC and AC, in the points E, D and F; then are these points the centres of gravity of the straight lines AB, BC and AC, considered individually, and because the lines AB and AC are supposed to be placed wholly in the points E and F; join EF, and the common centre of gravity of the lines AB and AC, considered conjointly, must occur in EF, the line connecting their individual centres.



Bisect EF in G; then shall the point G be the common centre of gravity of the two lines AB and AC, in which they are supposed to be wholly placed.

But the point D is the centre of gravity of the line BC; therefore, join DG and the common centre of gravity of the two lines AB, AC placed in G, and the line BC placed in D, occurs in DG, ~~and divides~~ it into two parts, which are to each other reciprocally as the line BC taken singly, to the two lines AB and AC taken conjointly; that is, as one to two

Join BF to cut DG in H; then shall H be the centre of gravity of the three lines AB, BC and AC; join BH and CH, and produce DG to A; then are the distances AH, BH and CH, equal among themselves, and the value of each is indicated in equation (r); where d denotes the distance between any two of the points A, B and C, or which is the same thing, the side of the equilateral triangle ABC.

Because the triangle ABC is equilateral, and F the middle point of AC, the line BF is perpendicular to AC; and for the same reason, AD is perpendicular to BC; consequently, the triangles BFC and BDH are similar, and we have

$$BF : BC :: BD : BH,$$

but $BF = \sqrt{BC^2 - CF^2}$, and $BD = \frac{1}{2} BC$; that is, $BF = \frac{d}{2}\sqrt{3}$, and $BD = \frac{d}{2}$;

therefore, the above analogy becomes

$$\frac{d}{2}\sqrt{3} : d :: \frac{d}{2} : BH = \frac{d}{3}\sqrt{3} = \cdot 57735d,$$

which is the identical value of BH indicated in equation (r);

but $BH \div BF$; that is, $\frac{d}{3}\sqrt{3} \div \frac{d}{2}\sqrt{3} = \frac{2}{3}$;

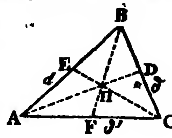
hence we infer,

That the centre of gravity of the perimeter of an equilateral triangle, is distant from each of the angular points by two thirds of the line drawn from any of the angles to the middle of the opposite side.

34. This is obviously the case with respect to the equilateral triangle; let us therefore inquire if the principle holds generally; for this purpose we must recur to the general equations (n), (o) and (p), in which, if we suppose the bodies a, b and c, to become equal to one another, the distance of each from the common centre of gravity, will be respectively expressed by the following class of symmetrical formulas; that is,

$$\left.\begin{array}{ll} 1. & AH=\frac{1}{3}\sqrt{2\,(d^2+\delta'^2)-\delta^2} \\ 2. & BH=\frac{1}{3}\sqrt{2\,(d^2+\delta^2)-\delta'^2} \\ 3. & CH=\frac{1}{3}\sqrt{2\,(\delta^2+\delta'^2)-d^2} \end{array}\right\} \quad (s)$$

Now, the writers on Geometry have demonstrated, that in any plane triangle ABC, whose sides are respectively denoted by the letters d, δ and δ'; if straight lines AD, BF and CE, be drawn from the angular points A, B and C, to the middle of the opposite sides BC, AC and AB; then, we have



$$\left.\begin{array}{ll} 1. & AD=\frac{1}{2}\sqrt{2\,(d^2+\delta'^2)-\delta^2} \\ 2. & BF=\frac{1}{2}\sqrt{2\,(d^2+\delta^2)-\delta'^2} \\ 3. & CE=\frac{1}{2}\sqrt{2\,(\delta^2+\delta'^2)-d^2} \end{array}\right\} \quad (t)$$

Let numbers 1, 2 and 3 in class (s), be respectively divided by the corresponding numbers in class (t), and the constant quotient $\frac{2}{3}$, implies, that AH, BH and CH are respectively equal to two thirds of AB, BF and CE; hence, we infer generally,

That the centre of gravity of the perimeter of any right lined triangle, is distant from each of the angular points by two thirds of the straight lines, drawn from the respective angles to the middle of the opposite sides.

But the centre of gravity of the perimeter of a triangle, must obviously* be the centre of gravity of the triangular surface, or the area bounded by the lines whose common centre of gravity has just been determined; consequently,

* For suppose the area of $\triangle$ ABC (see the preceding figure) to be composed of an infinite number of straight lines parallel to AC; then BF drawn from the vertex B of the triangle to the middle of the base AC, divides each of these straight lines into two equal parts, therefore the centres of gravity of all these lines are in the straight line BF, and the $\triangle$ will therefore balance itself upon the straight line BF. If therefore the straight line BF be supported, the $\triangle$ ABC will be kept in equilibrio.

In like manner, it may be proved, that if the straight line AD be supported, the $\triangle$ ABC will remain in equilibrio.

Therefore H, the intersection of AD and BF, is the point which, when supported, the $\triangle$ ABC would be in equilibrio; and therefore H is the centre of gravity of the triangular surface, and of the perimeter of the figure. Q. E. D.

If two lines be drawn from any two angles of a plane triangle to the middle of the opposite sides, the centre of gravity of the triangle, is at the point of intersection;
and the distances of that point from each of the angles, are expressed generally by the equations in class (s).

We shall now propose an example or two, to show the manner in which the formulæ are to be applied.

EXAMPLE 1. If the sides of a plane triangle are respectively, 24, 32 and 40 inches; how far is its centre of gravity distant from each of the angular points?

NUMERICAL CALCULATION.

In this example, there are given $d=24$; $\delta=32$, and $\delta'=40$ inches.

Then, to find the distance AH, by No. 1, class (s), we have the following process,

$d^2=24\times24=\ 576$
$\delta'^2=40\times40=1600$
$2(d^2+\delta'^2)=2\times\overline{2176}=4352$, first member under the radical sign,
$\delta^2=\ 32\times32=1024$, second ———————————,
diff. of the members $=\overline{3328}$, its square root is, $57\cdot69$ nearly;
therefore, we have

$$\text{AH}=\frac{57\cdot69}{3}=19\cdot23 \text{ inches.}$$

To find the distance BH, No. 2, class (s), gives the following process,

$d^2=24\times24=\ 576,$
$\delta^2=32\times32=1024,$
$2(d^2+\delta^2)=2\times\overline{1600}=3200$, first member under the radical sign,
$\delta'^2=40\times40=1600$, second ———————————,
diff. of the members $=\overline{1600}$, its square root is 40;
consequently, we have

$$\text{BH}=\tfrac{40}{3}=13\tfrac{1}{3} \text{ inches.}$$

To find the distance CH, No. 3, class (s), gives the following process,

$\delta^2=32\times32=1024$
$\delta'^2=40\times40=1600$
$2(\delta^2+\delta'^2=2\times\overline{2624}=5248$, first member under the radical sign,
$d^2=24\times24=\ 576$, second ———————————
difference of the mem. $=\overline{4672}$, its square root is, $68\cdot64$, consequently, we have

$$\text{CH} = \frac{68 \cdot 64}{3} = 22 \cdot 88 \text{ inches,}$$

$$\text{BH} = \frac{40}{3} = 13 \cdot 33,$$

$$\text{AH} = \frac{57 \cdot 69}{3} = 19 \cdot 23.$$

EXAMPLE 2. A thin triangular plate of cast iron, whose sides are respectively 20, 28 and 39 feet, is to be supported on the point of an upright spindle; at what distance from the angular points of the plate must the spindle be applied, that the plate may remain steadily at rest in a horizontal position?

NUMERICAL CALCULATION.

In this example we have given, $d=20$; $\delta=28$, and $\delta'=39$; consequently,

To find AH, No. 1, class (s), gives the following operation:

$d^2 = 20 \times 20 = 400$,

$\delta'^2 = 39 \times 39 = 1521$,

$2(d^2+\delta'^2) = 2 \times \overline{1921} = 3842$, first member under the radical sign,

$\delta^2 = 28 \times 28 = 784$, second ———————,

difference of the mem. $= \overline{3058}$, its square root is $55 \cdot 29$; consequently,

$$\text{AH} = \frac{55 \cdot 29}{3} = 18 \cdot 43 \text{ feet, nearly.}$$

To find the distance BH, No. 2, class (s), gives this process,

$d^2 = 20 \times 20 = 400$,

$\delta^2 = 28 \times 28 = 784$,

$2(d^2+\delta^2) = 2 \times \overline{1184} = 2368$, first member under the radical sign,

$\delta'^2 = 39 \times 39 = 1521$, second ———————,

diff. of the members $= \overline{847}$, its square root is $29 \cdot 103$; consequently,

$$\text{BH} = \frac{29 \cdot 103}{3} = 9 \cdot 701 \text{ feet, nearly.}$$

To find the distance CH, No. 3, class (s), gives the following process:

$\delta^2 = 28 \times 28 = 784$

$\delta'^2 = 39 \times 39 = 1521$

$2(\delta^2+\delta'^2) = 2 \times \overline{2305} = 4610$, first member under the radical sign,

$d^2 = 20 \times 20 = 400$, second. ———————

difference of the memb. $= \overline{4210}$, its square root is $64 \cdot 88$; consequently, we have

$$\text{CH} = \frac{64 \cdot 88}{3} = 21 \cdot 63 \text{ feet nearly,}$$

$$\text{BH} = \frac{29 \cdot 103}{3} = 9 \cdot 701 \text{———},$$

$$\text{AH} = \frac{55 \cdot 29}{3} = 18 \cdot 43 \text{———}.$$

for the respective distances from each angle of the plate where the spindle must be placed.

Transformation of the formulæ in class (s), to determine the sides of the triangle.

35. The formulæ in class (*s*), since they involve both the sides of the triangle, and the distances of the centre of gravity from the angular points, can be so transformed, as to determine the sides of the triangle, in a manner nearly similar to that which we have just employed in determining the position of the centre of gravity; and since this problem may be useful on very many occasions, we shall here lay down the method of its resolution. For which purpose, let each side of the three equations in class (*s*) be multiplied by the number 3 and afterwards squared, and we shall have

$$9\text{AH}^2=2d^2+2\delta'^2-\delta^2$$
$$9\text{BH}^2=2d^2+2\delta^2-\delta'^2$$
$$9\text{CH}^2=2\delta^2+2\delta'^2-d^2,$$

and these again by reduction, become

$$\left.\begin{array}{l} 1.\ d=\sqrt{2(\text{AH}^2+\text{BH}^2)-\text{CH}^2}, \\ 2.\ \delta=\sqrt{2(\text{BH}^2+\text{CH}^2)-\text{AH}^2}, \\ 3.\ \delta'=\sqrt{2(\text{AH}^2+\text{CH}^2)-\text{BH}^2}, \end{array}\right\} \qquad (u)$$

Which three equations are not only symmetrical with respect to one another, but are, with the exception of the fractional coefficient analogous in form to those in class (*s*), and, consequently, the method of applying them will also be similar, as will appear from the resolution of the following examples.

Example 1. The distances of three angular points of a plane triangle, from the centre of gravity of the figure, are respectively 12, 17 and 25 feet; what are the sides of the triangle?

Numerical Calculation.

Here, we have given AH $=12$; BH $=17$, and CH $=25$ feet; consequently, to find the value of d, No. 1 class (u) gives the following process

$\text{AH}^2=12\times12=144,$

$\text{BH}^2=17\times17=289,$

$2(\text{AH}^2+\text{BH}^2)=2\times433=866$, first member under the rad. sign,

$\text{CH}^2=25\times25=625$, second ———————,

difference of the members $=241$, hence we get

$$d=\sqrt{241}=15\cdot52 \text{ feet.}$$

To find the value of δ, No. 2, class (u) gives this process,

$\text{BH}^2=17\times17=289,$

$\text{CH}^2=25\times25=625,$

$2(\text{BH}^2+\text{CH}^2)=2\times914=1828$, first member under the rad. sign,

$\text{AH}^2=12\times12=\ 144$, second ———————,

difference of the members $=1684$; therefore, we have

$$\delta=\sqrt{1684}=41\cdot03 \text{ feet.}$$

To find the value of δ', No. 3, class (u) gives

$AH^2 = 12 \times 12 = 144,$

$CH^2 = 25 \times 25 = 625,$

or $2(AH^2 + CH^2) = 2 \times \overline{769} = 1538$, first member under the radical sign,

and $BH^2 = 17 \times 17 = 289$, second ——————————,

difference of the memb. $= \overline{1249}$, hence we get,

$$\delta' = \sqrt{1249} = 35 \cdot 34 \text{ feet.}$$

EXAMPLE 2. There is a mahogany board in the form of a triangle, such, that the straight lines drawn from the centre of gravity to the angular points, are respectively 36, 40 and 48 inches; what are the sides of the board?

NUMERICAL CALCULATION.

Here, we have given, $AH = 36$; $BH = 40$, and $CH = 48$ inches; consequently, to find the value of d, No. 1, class (u), gives

$AH^2 = 36 \times 36 = 1296,$

$BH^2 = 40 \times 40 = 1600,$

or $2(AH^2 + BH^2) = 2 \times \overline{2896} = 5792$, first member under the rad. sign,

and $CH^2 = 48 \times 48 = 2304$, second ——————————,

difference of the memb. $= \overline{3488}$, hence

$$d = \sqrt{3488} = 59 \cdot 06 \text{ inches.}$$

To find the value of δ, No. 2, class (u), gives

$BH^2 = 40 \times 40 = 1600,$

$CH^2 = 48 \times 48 = 2304,$

or $2(BH^2 + CH^2) = \overline{2 \times 3904} = 7808$, first member under the rad. sign,

and $AH^2 = 36 \times 36 = 1296$, second ——————————

difference of the memb. $= \overline{6512}$; hence we get

$$\delta = \sqrt{6512} = 80.69 \text{ inches.}$$

To find the value of δ', No. 3, class (u) gives

$AH^2 = 36 \times 36 = 1296,$

$CH^2 = 48 \times 48 = 2304,$

or $2(AH^2 + CH^2) = 2 \times \overline{3600} = 7200$, first member under the rad. sign,

and $BH^2 = 40 \times 40 = 1600$, second ——————————,

difference of the memb. $= \overline{5600}$, hence we have

$$\delta' = \sqrt{5600} = 74 \cdot 83 \text{ inches.}$$

If we take the squares of the formulæ in class (u), and collect the members on both sides of these squared equations, we shall get

$$d^2 + \delta^2 + \delta'^2 = 3(AH^2 + BH^2 + CH^2) \qquad (v)$$

From which we infer,

That the sum of the squares of the three sides of any plane triangle, is equal to three times the sum of the squares of the distances of its centre of gravity from the angular points.

This is a very curious inference; it does not however, appear to be of any great practical utility; because we have already got independent expressions for each of the distances of the centre of gravity from the angular points in terms of the sides, and also independent expressions for each of the sides in terms of the angular distances; it would, therefore, be a matter of mere curiosity, and would lead to nothing useful, to advance any thing in the shape of examples for the illustration of equation (v); we shall, therefore, drop the consideration of the plane triangle, and proceed with the next order of rectilineal figures which we meet with in buildings generally, namely such as are four sided.

Section Sixth.

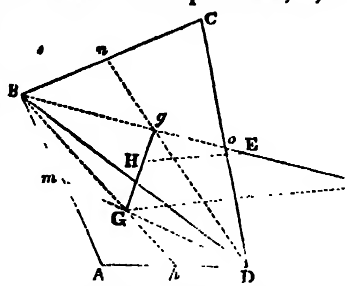
OF THE CENTRE OF GRAVITY OF TRAPEZIUMS.

Problem. *To determine the centre of gravity of a Trapezium or four sided figure.*

36. The developement of this problem depends on the same, or similar principles to those of the plane triangle just established.

Let ABCD be a trapezium, of which the sides AB, BC, CD and AD, together with the diagonal BD are given, and let it be required to determine the centre of gravity of the figure ABCD.

Bisect the four sides AB, BC, CD and AD in the points m, n, o and p, and draw Dm, Bp and Bo, Dn meeting each other two and two in the points G and g; then are G and g, the centres of gravity of the plane triangles ABD and BCD, which constitute the trapezium, or four sided figure ABCD.



Join Gg; then, if we conceive the triangles ABD and BCD, to be collected in the points G and g; it is evident that their common centre of gravity, which is the same as the centre of gravity of the trapezium, must lie in the line Gg, and divide it into two parts

Which are to each other reciprocally as the areas of the triangles ABD *and* BCD.

Produce go to F, and make gE and EF respectively proportional to the measures of the triangles ABD and BCD; join FG, and through the point E draw EH parallel to FG, and meeting Gg in the point H: then is H the centre of gravity of the trapezium ABCD.

We shall not attempt to give an independent formulæ for the calculation of this case, it would be too complicated to admit of an

easy reduction, and besides, it would increase the number of our symbols to a greater extent than our plan allows. Nevertheless, since a numerical operation must be resorted to when great accuracy is required, it is necessary that we point out the method by which the place of the point H is to be determined; and, for this purpose, we shall resolve at full length, the two following examples, where the data are selected in such a manner, that all the parts of the figure, necessary for the display of a complete solution, are separately calculated; and in order to have a check on the operation and prove our results, we shall also give a graphical solution of both examples.

EXAMPLE 1. The four sides of a trapezium are respectively 18, 26, 38 and 46 feet, and its diagonal connecting the remote extremes of the two shortest sides is 42 feet; at what point in the surface of the figure ABCD, is its centre of gravity?

NUMERICAL CALCULATION.

In this example, we have given, AD = 18; AB = 26; BC = 38; CD = 46, and BD = 42 feet; then, since the trapezium may be divided into the two triangles ABD, BCD; to find, first, the area of the triangle ABD it is,

$$\begin{aligned}
AD &= 18,\\
AB &= 26,\\
BD &= 42,\\
AD+AB+BD &= 86\\
\tfrac{1}{2}\{AD+AB+BD\} &= 43, \ldots\ \text{log. } 1\cdot633468,\\
43-18 &= 25, \ldots\ \text{log. } 1\cdot397940,\\
43-26 &= 17, \ldots\ \text{log. } 1\cdot230449,\\
43-42 &= 1, \ldots\ \text{log. } 0\cdot000000,\\
&\ 4\cdot261857, \text{ sum of the logarithms,}
\end{aligned}$$

nat. number 135·185 .... log. 2·130928, ½ sum of the logs.;

hence, the area of the triangle ABD, is 135·185 square feet.

Secondly. To find the area of the triangle BCD, it is

$$\begin{aligned}
BC &= 38,\\
BD &= 42,\\
CD &= 46,\\
BC+BD+CD &= 126,\\
\tfrac{1}{2}\{BC+BD+CD\} &= 63, \ldots\ \text{log. } 1\cdot799341,\\
63-38 &= 25, \ldots\ \text{log. } 1\cdot397940,\\
63-42 &= 21, \ldots\ \text{log. } 1\cdot322219,\\
63-46 &= 17, \ldots\ \text{log. } 1\cdot230449.\\
&\ 5\cdot749949, \text{ sum of the logarithms,}
\end{aligned}$$

nat. number 749·85 .... log. 2·874974, ½ sum of the logs.;

hence the area of the triangle BCD, is 749·85 square feet.

Then, for the distances BG and Bg, by equation (s), we have

$$\text{BG}=\tfrac{1}{3}\sqrt{2(\text{AB}^2+\text{BD}^2)-\text{AD}^2},$$

$$\text{and B}g=\tfrac{1}{3}\sqrt{2(\text{BC}^2+\text{BD}^2)-\text{CD}^2},$$

which formulæ afford the following operations for the distances of the centres of gravity G and g, from the point B, viz.

For the distance BG, it is

$\text{AB}^2=26\times26=\ \ 676$
$\text{BD}^2=42\times42=1764$
$2\{\text{AB}^2+\text{BD}^2\}=\ 2\times\overline{2440}=4880$, first member under the radical sign,
$\text{AD}^2=18\times\ 18\ =\ \ 324$, second ——————————,
difference of the members $=\overline{4556}$, its square root is, 67.49;

consequently, we get $\text{BG}=\frac{67\cdot49}{3}=22.5$ feet very nearly.

For the distance Bg, it is

$\text{BC}^2=38\times38=1444$
$\text{BD}^2=42\times42=1764$
$2\{\text{BC}^2+\text{BD}^2\}=\ 2\times\overline{3208}=6416$, first member under the radical sign,
$\text{CD}^2=46\times\ 46\ =2116$, second ——————————,
difference of the members $=\overline{4300}$, its square root is, 65.574;

consequently, we get $\text{B}g=\frac{65\cdot574}{3}=21.858$ feet nearly.

But $\text{BG}=\frac{2}{3}\,\text{B}p$, and $\text{B}g=\frac{2}{3}\,\text{B}o$, as we have already shown by the general inference under class (t); therefore, we have

$$\text{B}p=22.5+\frac{22\cdot5}{2}=33\cdot75,$$

$$\text{and B}o=21\cdot858+\frac{21\cdot858}{2}=32.787.$$

Hence then, in the triangles pBD and oBD, we have given the three sides pB, BD and pD, to find the angle DBp; and the three sides oB, BD and oD, to find the angle DBo.

Therefore, by Plane Trigonometry, we have

$$\text{vers. DB}p=\frac{\text{D}p^2-(\text{BD}\backsim\text{B}p)^2}{2\text{BD}.\text{B}p},$$

$$\text{and vers. DB}o=\frac{\text{D}o^2-(\text{BD}\backsim\text{B}o)^2}{2\text{BD}.\text{B}o},$$

which formulæ afford the following operations for the versed sines of the angles sought, viz.

To find the angle DBp, it is

$$
\begin{array}{llll}
\text{D}p=\tfrac{1}{2}\,\text{AD}= 9 \ldots 9 & & & \\
\text{BD}=42\,.\ \text{B}p=33.75 & & & \\
\overline{51} \quad \overline{42.75} & & & \\
33.75 \quad 42 & & & \\
\overline{17.25} \quad \overline{75} & \ldots\ldots & \text{log.} & 9.875061, \\
 17.25 & \ldots\ldots & \text{log.} & 1.236789, \\
 33.75 & \text{.ar. co.} & \text{log.} & 8.471726, \\
42\times 2=84 & \text{.ar. co.} & \text{log.} & 8\cdot075721, \\
\text{nat. vers. DB}p=5^\circ\ 28' \ldots\ 0\cdot00456 & \ldots & \text{log.} & \overline{7.659297}, \\
\end{array}
$$

To find the angle DBo, we have

$$
\begin{array}{llll}
\text{D}o=\tfrac{1}{2}\,\text{DC}=23 \ldots 23 & & & \\
\text{BD}=42\ .\ \text{B}o=32.787 & & & \\
\overline{65} \quad \overline{55.787} & & & \\
32.787 \quad 42 & & & \\
\overline{32\cdot213} \quad \overline{13.787} & \ldots\ldots & \text{log.} & 1.139469, \\
 32.213 & \ldots\ldots & \text{log.} & 1\cdot508031, \\
 32.787 & \text{. ar. co.} & \text{log.} & 8.484298, \\
42\times 2=84 & \text{. ar. co.} & \text{log.} & 8.075721, \\
\text{nat. vers. DB}o=34^\circ 2' \ldots\ 0.16126 & \ldots & \text{log.} & \overline{9.207519}, \\
\end{array}
$$

Here then, we have the angle $o\text{B}p=\text{DB}p+\text{DB}o=5^\circ\ 28'+34^\circ\ 2'=39^\circ\ 30'$; then, in the triangle GBg, we have given BG$=22.5$; B$g=21.858$, and the angle GB$g=39^\circ\ 30'$; to find the side Gg, opposite to the given angle GBg.

For which purpose, find φ an angle such, that

$$\tan.\ \varphi=\frac{2\sin.\ \frac{1}{2}\,\text{GB}g\sqrt{\text{BG}\cdot\text{B}g}}{\text{BG}\backsim\text{B}g};$$

then, for the distance Gg, between the centres of gravity of the constituent triangles ABD and DBC, we have

$$\text{G}g=(\text{BG}\backsim\text{B}g)\ \sec.\ \varphi.$$

Now, the first of these equations for the value of tan. φ, gives the following process:

$$
\begin{array}{lll}
\text{BG}=\ 22\cdot\ \ 5\ \ldots & \text{log. } 1\cdot352183, & \\
\text{B}g=\ 21\cdot858\ \ldots & \text{log. } 1\cdot339610, & \text{[logs.} \\
\text{BG. B}g=491\cdot805 & \text{log. } \overline{2\cdot691793}, & \text{sum of the} \\
\sqrt{\text{BG. B}g.}=\ 22\cdot476\ . & \text{log. } 1\cdot345896, & \text{half sum of} \\
 2\cdot\ \ \ldots & \text{log. } 0\cdot301030, & \text{[the logs.} \\
\tfrac{1}{2}\,\text{GB}g=\ 19^\circ\ 45' & \text{log. sin. } 9\cdot528810, & \\
\text{BG}-\text{B}g=22\cdot5-21\cdot858=0\cdot642 & \text{ar. co. log. } 0\cdot192465, & \\
\text{hence we have } \varphi=87^\circ\ 32'\ 50 & \text{log. tan. } \overline{11\cdot368201}, & \\
\end{array}
$$

Then, to find the value of the distance Gg, the second equation above gives

$$\text{BG}-\text{B}g=22\cdot5-21\cdot858=0.642 \text{ log. } 9\cdot807535$$
$$\varphi=87^\circ\ 32'\ 50'' \ \ldots\ldots \text{ log. sec. } 11\cdot368581$$

hence we have $\text{G}g=15$ feet . . . . log. $\overline{1\cdot176116}$

But we have shown in the construction of the preceding diagram, that the centre of gravity of the trapezium lies in the line Gg, and that Gg is divided by it into two parts, which are to each other reciprocally, as the areas of the triangles ABD, and DBC. Now, the areas of these triangles are respectively 135·185 and 749·85 square feet; or, they are to each other, as the numbers 1 and 5·546 very nearly; consequently, we have

$$1+5\cdot546 : 15 :: 5\cdot546 : 12\cdot7=\text{GH};$$

hence, the distance of the point H from G is 12·7 feet, and, consequently, its distance from g is 2·3 feet; but the positions of the points G and g are known; wherefore the position of the point H is known also.

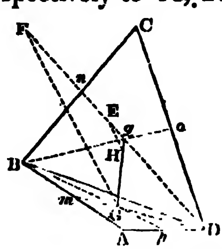
37. The foregoing operation is very laborious, but no part of it can be dispensed with, when it is found necessary to establish the position of the point H by calculation; indeed, if it were required to fix its situation with respect to the angular points of the figure, the labour would be very much increased, as several more operations would in that case be necessary, in order to determine the four distances, which fix the position of the centre of gravity.

It is probably owing to the vast labour attending the numerical calculation of this problem generally, that writers on the subject have omitted to exhibit it, otherwise than in particular cases; such as, when two sides of the figure are parallel to one another; but our object being more especially, to point out the methods of calculation generally, for the advantage of a numerous class of mechanicians, it is only in accordance with our plan, that we have dwelt so largely on the preceding solution.

Geometrical construction, verifying the preceding calculation.

The following graphical construction will serve to verify the truth of the conclusion to which we have come respecting the position of the point H, and other parts of the figure, which we have made the subject of calculation.

On the three lines AD, AB and BD, equal respectively to 18, 26 and 42 feet, taken from a scale of equal parts of any convenient dimensions, construct the triangle ABD, which will obviously limit the position of the figure. Then, on BD as a base, with the sides BC and CD, equal respectively, to 38 and 46 feet, construct the triangle BCD; then shall ABCD be the trapezium whose centre of gravity is required to be found.



Bisect the four sides AB, BC, CD and AD in the points m, n, o and p, and draw the lines Dm, Bp, Bo and Dn, meeting

each other two and two in the points G and g; then are G and g the centres of gravity of the constituent triangles ABD and DBC.

Join Gg, and conceive the triangles ABD and DBC to be placed wholly in the points G and g; then does their common centre of gravity, or that of the trapezium, exist in the line Gg, dividing it into two parts, which are to each other reciprocally as the areas of the triangles ABD and DBC. Now, the areas of these triangles, as determined by calculation, are to each other as 1 to 5·546 very nearly; therefore,

Produce gn to the point F, making gE and EF respectively proportional to the numbers 1 and 5·546; join FG, and through the point E draw EH parallel to FG, meeting the line Gg in the point H; then shall the point H be the centre of gravity of the trapezium ABCD.

Then if the lines BG, Bg, Gg and GH, be respectively taken in the compasses, and applied to the same scale of equal parts from which the figure was constructed, they will be found to measure respectively 22·5; 21·858; 15, and 12·7 feet, the very same as we found them to be by calculation; and moreover, if the measures of the angles DBp and DBo be taken in the compasses, and applied to a scale of chords, they will indicate respectively 5° 28′ and 34° 2′, thereby confirming the truth of every part of our calculation.

This method of proof by geometrical construction is very satisfactory to practical men; and if the operation is skilfully performed with delicate instruments, the coincidence of the results by both methods will often be very remarkable.

EXAMPLE 2. There is a certain building, the plan of which is in the form of a rectangular parallelogram 41½ feet long, and 24½ feet wide; from one corner of which is separated a triangular space, cut off by a diagonal wall originating at one angle of the building, and terminating in the opposite side, at the distance of 17 feet from its remote extremity, thereby forming a figure compounded of a rightangled triangle and a rectangular parallelogram. At the junction of the diagonal wall with the side of the building, is placed a transverse girder, stretching directly from side to side, parallel to the ends, for the purpose of supporting the binding joists on which the floor is to rest.

Now, the question is, what part of the floor is actually sustained by the transverse girder, supposing it to press equally on the walls all around, and whereabouts is the centre of gravity of that portion which the girder sustains, its form being that of a trapezium whose diagonal is the length of the girder, and whose sides are such as the theory of gravity assigns to them?

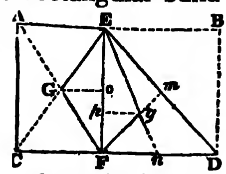
The resolution of this question, it is manifest, branches itself into two parts.

1. *What portion of the floor is sustained by the transverse girder?*
2. *Whereabouts is the centre of gravity of that portion?*

Now, in order to render the solution as simple and explicit as possible, we shall consider these parts separately; and to avoid confusion in the diagrams, we shall also give a separate construction for each; and first,

To find what portion of the floor is supported by the transverse girder.

Let the figure ABDC represent the plan of the rectangular building, whose length AB or CD is $41\frac{1}{2}$ feet, and breadth AC or BD $24\frac{1}{2}$ feet; the part EBD being separated by the diagonal wall DE, commencing at the angle D, and terminating at the point E, 17 feet distant from A, the extremity of the side wall AB.



Through the point E, draw EF parallel to AC; then shall EF represent the transverse beam or girder; by which, together with the surrounding walls, the whole floor AEDC has to be sustained.

Now, it is clear, that by reason of the transverse girder EF, stretching from side to side of the building, and resting on the walls, the floor AEDC is divided into two straining portions, one of which is rectangular as AEFC, and the other triangular as EDF, and the portion of the floor, which is actually and wholly sustained by the girder, must obviously be bounded by the straight lines connecting its extremities with the centres of gravity of the straining portions AEFC and EDF.

Draw the diagonals AF and CE, meeting each other in the point G; then by our scholium, the point G is the centre of gravity of the rectangle AEFC. Bisect the lines ED and FD in the points m and n, and draw the lines Fm and En, cutting each other in the point g; then, as we have already shown, the point g is the centre of gravity of the triangle EDF; consequ^ntly, the trapezium GEgF is the portion of the floor which is sustained by the girder EF.

Then, because the diagonals of a rectangular parallelogram are equal to one another, and mutually bisected in the point G, GE is equal to GF; that is,

$$\text{GE}=\text{GF}=\tfrac{1}{2}\sqrt{\text{EF}^2+\text{CF}^2};$$

but EF is the breadth of the building equal to $24\frac{1}{2}$ feet, and $\text{CF}=17$ feet by construction; therefore, we have

$$\text{GE}=\text{GF}=\tfrac{1}{2}\sqrt{24\cdot5^2+17^2}=14\cdot91 \text{ feet.}$$

Then, by equation (s), we have

$$\text{E}g=\tfrac{1}{3}\sqrt{2\,(\text{EF}^2+\text{ED}^2)-\text{FD}^2},$$

$$\text{and F}g=\tfrac{1}{3}\sqrt{2\,(\text{EF}^2+\text{FD}^2)-\text{ED}^2}.$$

Which two equations, by reason of the right angle EFD, and the equality of the lines EF, FD, in the present instance, become simply

$$Eg = \frac{EF}{3}\sqrt{5}$$

$$\text{and } Fg = \frac{EF}{3}\sqrt{2};$$

from which we get

$$Eg = \frac{24.5 \times 2\cdot236}{\cdot} = 18.26 \text{ feet, and}$$

$$Fg = \frac{24.5 \times 1.4142}{3} = 11.55 \text{ feet, very nearly;}$$

consequently, the four sides of the trapezium GEgF, taken in order from the point G, are equal respectively to 14.91; 18.26; 11.55, and 14.91 feet, the diagonal being $24\frac{1}{2}$ feet.

Let fall the perpendiculars Go and gp; then it is evident, that Go is equal to one half of CF, and gp equal to one third of FD; therefore, the area of the trapezium GEgF, or that portion of the floor which presses on the transverse girder, is expressed by the product

$$\tfrac{1}{2}EF \times (\tfrac{1}{2}CF + \tfrac{1}{3}FD);$$

but $\frac{1}{2}$ EF $=12\frac{1}{4}$; $\frac{1}{2}$ CF $=8\frac{1}{2}$, and $\frac{1}{3}$ FD $=8\frac{1}{6}$ feet; consequently, the area of the trapezium GEgF $=12\frac{1}{4}(8\frac{1}{2}+8\frac{1}{6})=12\frac{1}{4}\times 16\frac{2}{3}=204\frac{1}{6}$ square feet.

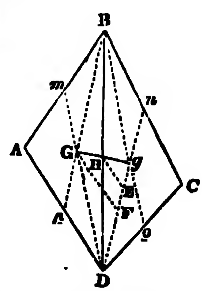
This satisfies the first demand of the question, and reduces the second to the determination of the centre of gravity of a trapezium, whose sides are respectively 14.91; 18.26; 11.55, and 14.91, and whose diagonal, which joins the remote extremities of the two contiguous and equal sides, is $24\frac{1}{2}$ feet.

Hence then,

To find the centre of gravity of that portion of the floor supported by the transverse girder.

Let ABCD be a trapezium, whose sides AB, BC, CD and AD, are equal respectively, to 14·91; 18·26; 11·55, and 14·91 feet, and whose diagonal BD is equal to $24\frac{1}{2}$ feet.

Bisect the four sides AB, BC, CD and AD in the points m, n, o and p, and draw the lines Dm, Bp, Bo and Dn, cutting each other two and two in the points G and g; then are the points G and g the centres of gravity of the constituent triangles ABD and CBD.



Join the points G and g by the straight line Gg, and conceive the triangles ABD and CBD to be collected in the points G and g; then it is obvious, that their common centre of gravity exists in the line Gg, which connects their individual centres; but the common centre of gravity of the triangles ABD and CBD, composing the trapezium ABCD, must be the same as the centre of gravity of the trapezium

itself; the centre of gravity of the trapezium ABCD, therefore lies in the line Gg, and divides it into two parts, which are to each other reciprocally as the areas of the triangles ABD and CBD, of which the trapezium is constituted.

Now, the area of the triangle ABD, as calculated from the foregoing diagram, is

$$12\tfrac{1}{4} \times 8\tfrac{1}{2} = 104\tfrac{1}{8} \text{ square feet;}$$

and the area of the triangle CBD, calculated in the same manner, is

$$12\tfrac{1}{4} \times 8\tfrac{1}{6} = 100\tfrac{1}{24} \text{ square feet; therefore}$$

In the straight line gD, take gE and EF, respectively proportional to the numbers $104\frac{1}{8}$ and $100\frac{1}{24}$; join FG, and through the point E draw EH, parallel to FG, meeting Gg in H; then shall H be the centre of gravity of the trapezium ABCD.

Then, if the lines BG, Bg, Gg, GH and gH, be taken in the compasses, and applied to the same scale of equal parts from which the figure was constructed, they will indicate 12·57; 13·88; 5·75; 2·82, and 2·93 feet respectively; and if the measures of the angles DBp and DBo, be taken in the compasses and applied to a scale of chords, they will indicate 13° and 11° 28′.

The areas of the triangles ABD and CBD, having been already calculated, it is needless to repeat the operation, or to determine the areas by an independent process arising from the present data; we, therefore, proceed to determine the distances of the centres of gravity of the constituent triangles, ABD and CBD from the angular point B; for which purpose equation (s) gives

$$BG = \tfrac{1}{3}\sqrt{2(AB^2 + BD^2) - AD^2},$$

$$Bg = \tfrac{1}{3}\sqrt{2(BC^2 + BD^2) - CD^2};$$

and these formulæ afford the following operations, viz.

For the distance BG, it is

$AB^2 = 14·91 \times 14·91 = 222·3081,$

$BD^2 = 24.5 \times 24.5 = 600.25,$

$2(AB^2 + BD^2) = 2 \times 822·5581 = 1645·1162,$ first term under the rad. sign,

$AD^2 = 14·91 \times 14·91 = 222·3081,$ second ———,

difference of the members $= 1422·8081$ its square root is, 37·72;

consequently, we get $BG = \dfrac{37·72}{3} = 12·57$ feet nearly.

For the distance Bg, it is

$BC^2 = 18·26 \times 18·26 = 333·4276$

$BD^2 = 24·5 \times 24·5 = 600·25$

$2(BC^2 + BD^2) = 2 \times 933·6776 = 1867·3552$ first term und. the rad. sign.

$CD^2 = 11·55 \times 11.55 = 133·4025,$ second ——,

difference of the members $= 1733·9527$, its sq. root is 41·64; consequently, we get $Bg = \dfrac{41·64}{3} = 13·88$ feet.

But $BG=\frac{2}{3}Bp$, and $Bg=\frac{2}{3}Bo$, as we have already shown by the general inference under class (t); therefore, we have

$$Bp=12\cdot57+\frac{12\cdot57}{2}=18\cdot855,$$

$$\text{and } Bo=13\cdot88+\frac{13\cdot88}{2}=20\cdot82;$$

Consequently, in the triangles pBD and oBD, we have given pB, BD and pD, to find the angle DBp; and the three sides oB, BD and oD, to find the angle DBo.

Therefore, by plane Trigonometry, we have

$$\text{vers. } DBp=\frac{Dp^2-(BD\backsim Bp)^2}{2\,BD\,.\,Bp}$$

$$\text{and vers. } DBo=\frac{Do^2-(BD\backsim Bo)^2}{2\,BD\,.\,Bo}.$$

which formulæ afford the following operations, for the versed sines of the angles sought, viz.

To find the angle DBp, it is

$Dp=\frac{1}{2}AD=$ 7.455 . . .	7.455			
$BD=$24.5	. $Bp=$18.855			
31.955	26.31			
18.855	24.5			
13.1	1.81	. . . . . .	log.	0.257679,
	13.1	. . . . . .	log.	1.117271,
	18.855	. . ar. co.	log.	8.724574,
	24.5 × 2=49	. ar. co.	log.	8.309804,
nat. vers. $DBp=13°$	0.02566	. . .	log.	8.409328,

To find the angle DBo, we have

$Do=\frac{1}{2}DC=$ 5.775 . . .	5.775			
$BD=$24.5	. $Bo=$20.82			
30.275	26.595			
20.82	24.5			
9.455	2.095	. . . . .	log.	0.321184,
	9.455	. . . .	log.	0.975662,
	20.275	. . ar. co.	log.	8.693039,
	24.5 × 2=49	. . ar. co.	log.	8.309804,
nat. vers. $DBo=11°\,28'$.	0.01994	. . . . . .	log.	8.299689,

But the angle $oBp=DBp+DBo=13°+11°28'=24°\,28'$; then in the triangle GBg, we have given $BG=12.57$; $Bg=13.88$, and the angle $GBg=24°\,28'$; to find Gg, the side opposite to the given angle.

In order to which, let us find an angle φ, such that

$$\tan. \varphi = \frac{2 \sin. \frac{1}{2} \text{GB}g \sqrt{\text{BG} . \text{B}g}}{\text{BG} \backsim \text{B}g};$$

then, for the distance Gg, between the centres of gravity of the constituent triangles ABD and DBC, we have

$$\text{G}g = (\text{BG} \backsim \text{B}g) \sec. \varphi.$$

Now, the expression for tan. φ gives the following operation, viz.

$$\begin{array}{lll}
\text{BG} = 12.57 \ldots\ldots & \log. & 1.099335, \\
\text{B}g = 13.88 \ldots\ldots & \log. & 1.142389, \\
\text{BG} \cdot \text{B}g = 174.471 \ldots & \log. & \overline{2.241724}, \text{ sum of the logs.} \\
\sqrt{\text{BG} . \text{B}g} = 13.2 \quad \ldots & \log. & 1.120862, \ \tfrac{1}{2} \text{ sum of the logs.} \\
\qquad \ldots & \log. & 0.301030, \\
\tfrac{1}{2} \text{GB}g = 12^\circ\, 14' \ldots\ldots & \log. \sin. & 9.326117, \\
\text{B}g - \text{BG} = 13.88 - 12.57 = 1.31 & \text{ar.co.log.} & 9.882729, \\
\text{hence we have } \varphi = 76^\circ\, 49'\, 42'' & \log. \tan. & \overline{10.630738}.
\end{array}$$

Now, the expression for the distance Gg gives

$$\begin{array}{lll}
\text{B}g - \text{BG} = 13.88 - 12.57 = 1.31 & \log. & 0.117271, \\
\varphi = 76^\circ\, 49'\, 42'' \ldots\ldots & \log. \sec. & 0.642313, \\
\text{hence we have the dist. G}g = 5.75 \text{ feet} \ldots\ldots & \log. & \overline{0.759584}.
\end{array}$$

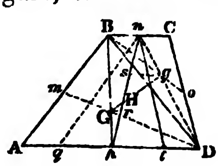
And by the principle of construction, we get

$$100\tfrac{1}{24} + 104\tfrac{1}{8} : 5\tfrac{3}{4} :: 104\tfrac{1}{6} : 2.93 \text{ feet} = g\text{H},$$
$$\text{and } 5.75 - 2.93 = 2.82 \text{ feet} = \text{GH}.$$

38. Such is the method of calculating generally the place of the centre of gravity of a quadrilateral figure, whose sides and one of its diagonals are given; the process exhibited in its full extent is very prolix and laborious; but there are particular cases of the problem, in which the operation may be greatly curtailed, and some of these we shall in the next place exemplify.

First. When two of the sides of the trapezium are parallel to one another, as the side walls of a building.

Let ABCD be a trapezium or quadrilateral figure, of which the sides AB, BC, CD and AD, together with the diagonal BD are given, and suppose the sides AD and BC to be parallel to one another; it is required to determine the place of the centre of gravity of the figure ABCD.



Bisect the sides AB, BC, BC and AD in the points m, n, o and p; join Dm, Bp, Bo and Dn, meeting each other two and two in the points G and g; then are G and g, the centres

of gravity of the constituent triangles ABD and DBC. Join Gg, the common centre of gravity of the triangles ABD, DBC, or the centre of gravity of the trapezium ABCD, is situated in the line Gg; join also np, the middle points of the parallel sides AD and BC; then because np bisects AD and BC, it will bisect all lines parallel to them, and consequently, by our scholium, the areas ABnp and CDnp are symmetrical with respect to the line np; but they are also equal between themselves; therefore, it is evident, that if the line np be supported, the whole figure will be supported; hence, the centre of gravity of the figure ABCD is in the line np; but we have already shown, that it is in the line Gg, therefore it must be at the point of intersection; that is, at the point H.

Through the points G and g, draw Gr and gs parallel to AD or BC, and through the point n, draw nq and nt parallel to the sides AB and DC.

Then, because H is the common centre of gravity of the triangles ABD and DBC, we get

$$GH : gH :: DBC : ABD,$$

and since the triangles ABD and DBC are of the same altitude, lying between the same parallels, their areas are to one another as their bases; therefore, we have

$$GH : gH :: BC : AD$$

and by the similar triangles GrH, gsH, we have,

$$GH : gH :: rH : sH, \text{ that is}$$
$$rH : sH :: BC : AD,$$

or by composition of ratios, we have

$$rH : rH + sH :: BC : BC + AD$$

and again, by composition, we obtain

$$2rH + sH : rH + sH :: 2BC + AD : BC + AD; \qquad (1)$$

and moreover

$$rH + sH : sH :: BC + AD : AD$$
$$\text{or } rH + sH : rH + 2sH :: BC + AD : BC + 2AD. \qquad (2)$$

Then, by expunging the common terms in the analogies (1) and (2), or by comparison *ex æquo*, we have

$$2rH + sH : 2sH + rH :: 2BC + AD : 2AD + BC,$$

but $2rH + sH = pH$, and $2sH + rH = nH$*; therefore, it is finally,

$$pH : nH :: 2BC + AD \quad 2AD + BC;$$

or by equating the products of the mean and extreme terms, we get

$$pH(2AD + BC) = nH(2BC + AD) \qquad (w)$$

From which we infer, that

* This is obvious, for pr, rs and sn are each of them equal to one third of np.

If the parallel sides of a trapezium be bisected, and if the line, which joins the points of bisection, be divided in the ratio of 2BC+AD *to* 2AD+BC, *the point* H, *where the division occurs, will be the centre of gravity of the figure* ABCD.

Because nq, is by construction parallel to AB, and by hypothesis, BC is parallel to AD, nq is equal to AB and Aq to Bn; and for the same reason, nt is equal to CD and tD to nC; consequently, Aq+tD =BC, and qt=AD−BC; but because Ap=Dp and Aq=Dt, we have $pq=pt$: that is, qt is bisected by the line np.

Let d=AB or nq, one side of the trapezium ABCD, or of the triangle qnt,
δ=CD or nt, another side opposite to the former,
and δ'=AD∽BC=qt, the third side of the triangle, or the difference between the two parallel sides of the trapezium, AD and BC.

Then, according to No. 2. class (t), we get

$$np=\tfrac{1}{2}\sqrt{2(d^2+\delta^2)-\delta'^2},$$

and by the above inference, or by the principle indicated in equation (w), we obtain

$$\left.\begin{aligned} 1.\quad n\text{H} &= \frac{(2\text{AD}+\text{BC})\sqrt{2(d^2+\delta^2)-\delta'^2}}{6(\text{AD}+\text{BC})} \\ 2.\quad p\text{H} &= \frac{(2\text{BC}+\text{AD})\sqrt{2(d^2+\delta^2)-\delta'^2}}{6(\text{AD}+\text{BC})} \end{aligned}\right\} \qquad (x)$$

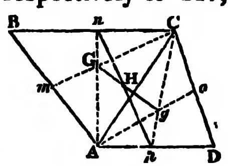
These expressions determine the segments into which the line np is divided by the point H, which marks the position of the centre of gravity of the trapezium ABCD, whose two sides AD and BC, are parallel to one another.

We shall exemplify the use of these equations, by the resolution of the following numerical examples, and that we may have a check upon the operation and verify the results, it will be necessary to employ a graphic construction as well as a numerical calculation; we shall take these in the order now written.

EXAMPLE 1. In the trapezium ABCD, whose sides AD and BC are parallel to one another, there are given, the sides AB, BC, CD and AD, respectively equal to 317, 348, 278 and 254 feet, and the diagonal AC equal to 308 feet; at what point in the surface of the figure is its centre of gravity?

The geometrical construction proceeds thus.

With the three sides AB, BC and AC, equal respectively to 317, 348 and 308 feet, taken from a scale of equal parts of any dimensions at pleasure, construct the triangle ABC, and on AC as a base, with the two sides CD and AD, equal respectively to 278 and 254 feet, construct the triangle ACD; then shall ABCD be the trapezium whose centre of gravity is required to be found.



Bisect the sides AB, BC, CD and AD, in the points m, n, o and p; join cm, An, Ao and cp, meeting each other two and two in the points G and g; then shall G and g be the centre of gravity of the constituent triangles ABC and ACD. Then, if we conceive the triangles ABC and ACD to be collected into the points G and g, they may be considered as exerting the whole of their energies in these points only; join Gg, then shall the common centre of gravity of the triangles ABC and ACD, or the centre of gravity of the trapezium ABCD exist in the line Gg; join also np cutting Gg in H; then shall H be the centre of gravity of the trapezium ABCD; and if the lines nH and pH be taken in the compasses and applied to the same scale as was employed in constructing the figure, they will indicate respectively 139·54 and 154·87 feet.

The numerical operation indicated in equation (x), is as follows; viz.

To find nH, No. 1 class (x), gives

$d^2 = 317 \times 317 = 100489$

$\delta^2 = 278 \times 278 = \underline{\ 77284}$ [sign.

$2(d^2+\delta^2) = 2 \times 177773 = 355546$ first member under the rad.

$\delta'^2 = (348-254)^2 = 94 \times 94 = \underline{\ 8836}$, second ————

difference of the members $= 346710$, its square root is, 588·82;

$$\frac{2\text{AD}+\text{BC}}{6(\text{AD}+\text{BC})} = \frac{2\times 254+348}{6(254+348)} = 0\cdot 237$$ very nearly; the co-efficient;

then $n\text{H} = 588\cdot 82 \times 0\cdot 237 = 139\cdot 54$ feet.

But we have already stated, that the whole line np is expressed, thus:

$$n\text{P} = \tfrac{1}{2}\sqrt{2(d^2+\delta^2)-\delta'^2};\ \text{that is,}$$

$$np = \frac{588\cdot 82}{2} = 294\cdot 41 \text{ feet};$$

consequently, by substraction we have

$$p\text{H} = 294\cdot 41 - 139\cdot 54 = 154\cdot 87 \text{ feet.}$$

Now, the position of the line np is known, therefore, the position of the point H is known.

But the distance pH may also be found by the second number of equation (x), in the following manner, viz.

$d^2 = 317 \times 317 = 100489$

$\delta^2 = 278 \times 278 = \underline{\ 77284}$

$2(d^2+\delta^2) = 2 \times 177773 = 355546$, first mem. under the radical,

$\delta'^2 = (348-254)^2 = 94 \times 94 = \ 8836$, second ————————,

difference of the members $= 346710$, its square root is, 588·82,

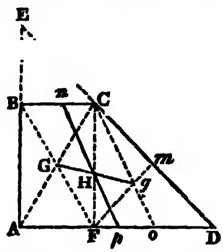
$$\frac{2\,\text{BC}+\text{AD}}{6(\text{BC}+\text{AD})} = \frac{2\times 348+254}{6(348+254)} = 0\cdot 263,$$ the value of the coefficient:

then $p\text{H} = 588\cdot 82 \times 0.263 = 154.87$ feet very nearly, the same as before.

Example 2. The ground plan of a building is in the form of an isosceles right angled triangle, whose side is $41\frac{1}{2}$ feet; at the distance of 17 feet from one of the acute angles, a wall is built, cutting off a triangular portion similar to the complete plan, and the remainder is to be covered with a fire proof floor of brick work, supported by the walls all round, and by a column placed directly under its centre of gravity; it is required to determine the position of the column.

Geometrical Construction.

Let AED be the plan of the triangular building, of which the sides AE and AD, are perpendicular to each other, and respectively equal to $41\frac{1}{2}$ feet; and let BC be the wall running parallel to AD, at the distance of 17 feet from the angular point E, and cutting off the triangular portion EBC, exactly similar to the complete plan EAD; then is the trapezium ABCD, the portion whose centre of gravity is required to be found.



Through the point C, draw CF parallel to AE forming the rectangular parallelogram ABCF; draw the diagonals AC, BF cutting each other in the point G; then shall G be the place of the centre of gravity of the rectangle ABCF.

Bisect CD and FD in the points m and o; join Fm and Co cutting each other in the point g; then is g the centre of gravity of the triangular space CFD; join Gg, then, the common centre of gravity of the rectangle ABCF, and the triangle CFD must fall somewhere in the line Gg; that is, the centre of gravity of the trapezium is situated in the line Gg. Bisect AD and BC in the points p and n; join pn; then, the areas of the figures ABnp and DCnp, are equal to one another, and consequently, the centre of gravity of the trapezium ABCD lies in the line pn, but we have shown that it lies also in the line Gg, it must therefore be in the point H, where these lines intersect each other; then, if nH and pH be taken in the compasses and applied to the scale employed in constructing the figure, they will indicate 15·61 and 11.77 feet respectively.

The numerical operation indicated in equation (x), *is as follows, viz.*

To find nH, No. 1 equation (x), gives

$d^2 = 24{\cdot}5 \times 24.5 \quad = 600.25$
$\delta^2 = 24.5 \times 24.5 \times 2 = 1200.5$ [radical sign.
$2(d^2 + \delta^2) = 2 \times 1800.75 = 3601{\cdot}5$, first term under the
$\delta'^2 = (41.5 - 17)^2 = 24.5 \times 24.5 = 600.25$, second——,
difference of the terms $= 3001.25$, its square root is 54.78,

$$\frac{2\,\text{AD}+\text{BC}}{6(\text{AD}+\text{BC})}=\frac{2\times 41.5+17}{6\;(41.5+17)}=0.285 \text{ nearly, the coefficient,}$$

then $nH=54.78\times 0.285=15.61$ feet very nearly.

To find pH, No. 2, equation (x), gives

$d^2=24.5\times 24.5 \quad = 600.25$

$\delta^2=24.5\times 24.5\times 2=1200.5$ [radical sign,

$2\;(d+\delta^2)=2\times \overline{1800.75}=3601.5$, first term under the

$\delta'^2=(41.5-17)^2=24.5\times 24.5=\underline{600.25}$, second ———

difference of the terms $=\overline{3001.25}$, its square root is 54.78, the same as above

$$\frac{2\,\text{BC}+\text{AD}}{6(\text{BC}+\text{AD})}=\frac{2\times 17+41.5}{6\;(17+41.5)}=0.215, \text{ nearly, the coefficient.}$$

then $pH=54.78\times 0.215=11.77$ feet.

Now, the position of the line np is known; consequently, the position of the point H is known; and an upright column placed under the floor at the point H, will produce the most advantageous effect in sustaining the superincumbent load.

*Secondly, When the rectilineal figure takes the form of a trapezoid.**

39. Recurring to the diagram immediately preceding equation (x); if we suppose the sides AB and CD to be equal to each other, then the figure ABCD becomes a trapezoid, and the value of the line np, in which the centre of gravity is situated, is thus expressed, viz.

$$np=\tfrac{1}{2}\sqrt{4d^2-\delta'^2},$$

and equation (x), becomes,

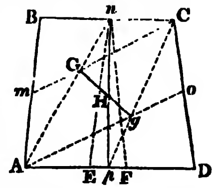
$$\left.\begin{aligned} &1.\;\; nH=\frac{(2\text{AD}+\text{BC})\sqrt{4\,d^2-\delta'^2}}{6\;(\text{AD}+\text{BC})}, \\ &2.\;\; pH=\frac{(2\text{BC}+\text{AD})\sqrt{4\,d^2-\delta'^2}}{6\;(\text{BC}+\text{AD})}. \end{aligned}\right\}\quad (y)$$

And these formulæ will find their application in the following numerical examples, where, as in the two preceding cases, we shall resolve the questions both graphically and numerically, that the one may prove a check upon the other.

EXAMPLE 1. The two parallel sides of a trapezoid are respectively 29 and 37 feet, and the two equal sides each 32 feet; at what point in the surface of the figure does the centre of gravity occur?

* We limit the term trapezoid to that form of quadrilateral figures, which has two of its sides parallel, and the other two equal to one another.

Geometrical Construction.



Construct the trapezoid thus; make AD equal to the longer of the two given parallel sides, and at the points A and D, set off AE and DF, each equal to one half of BC the shorter parallel side; bisect EF perpendicularly in the point p, and from the points E and F, with the distances En and Fn equal to the other given sides, describe arcs, cutting the perpendicular line pn in the point n; through n draw BC parallel to AD, and make nB and nC equal to EA and FD; join AB and DC, then shall the figure ABCD, be the trapezoid whose centre of gravity is required to be found.

To find the centre of gravity: bisect the sides AB, BC, CD and AD, in the points m, n, o and p; join An, Cm and Ao, Cp, meeting each other two and two in the points G and g; join Gg cutting the perpendicular np in the point H; then is H the position of the centre of gravity of the figure ABCD.

If nH and pH be taken in the compasses, and applied to a scale of equal parts, of the same magnitude as that employed in constructing the figure; they will indicate 16·52 and 15·23 feet respectively.

The numerical operation implied in equation (y) is as follows: viz.

To find nH, No. 1, equation (y), gives

$4d^2=4\times 32\times 32=4096$, first member under the radical sign,

$\delta'^2=(37-29)^2=\quad 64$, second ———

difference of the members $=\overline{4032}$, its square root is, 63·49;

$$\frac{2\text{AD}+\text{BC}}{6(\text{AD}+\text{BC})}=\frac{2\times 37+29}{6(37+29)}=0\cdot 26 \text{ nearly, the value of the coefficient,}$$

then $n\text{H}=63\cdot 49\times 0\cdot 26=16\cdot 52$ feet very nearly.

To find pH, No. 2, equation (y), gives

$4d^2=4\times 32\times 32=4096$, first member under the radical sign,

$\delta'^2=(37-29)^2=\quad 64$, second ———————,

$\overline{4032}$, its square root is, 63·49;

$$\frac{2\text{BC}+\text{AD}}{6(\text{BC}+\text{AD})}=\frac{2\times 29+37}{6(29+37)}=0\cdot 24 \text{ very nearly the coefficient,}$$

then $63\cdot 49\times 0\cdot 24=15\cdot 23$ feet, for the distance pH.

We have shown above, that the value of the whole line np for the case of the trapezoid, is expressed by the equation

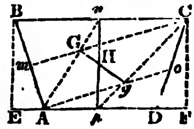
$$np=\tfrac{1}{2}\sqrt{4d^2-\delta'^2}\text{; therefore}$$

$$np=\frac{63\cdot 49}{2}=15\cdot 23+16\cdot 52=31\cdot 75 \text{ feet.}$$

EXAMPLE 2. A marble slab in the form of a trapezoid, has its parallel sides respectively equal to 12 and 18 feet; whereabouts in its superficies ought it to be supported, to remain in equilibrio, supposing each of its equal sides to measure $9\frac{1}{2}$ feet?

GRAPHICAL CONSTRUCTION.

Construct the trapezoid ABCD in the following manner; make AD equal to the shorter of the two given parallel sides; bisect AD in the point p, and produce pA pD to the points E and F, making pE and pF each equal to one half of BC, the longer of the two parallel sides. At the points E and F erect the perpendiculars EB and FC, and at the points A and D, with the distance AB or DC the other sides of the figure, describe arcs crossing ED and FC in the points B and C; join BC, so shall ABCD be the trapezoid whose centre of gravity is required to be found.



To find the centre of gravity; bisect the sides AB, BC, CD and AD in the points m, n, o and p; join An, Cm and Ao, Cp cutting each other two and two in the points G and g; join also Gg and np, cutting each other in the point H; then shall H be the place of the centre of gravity of the trapezoid ABCD.

If nH and pH be taken in the compasses and applied to the same scale as that from which the figure was constructed, they will be found to measure 4.2 and 4.81 respectively.

The numerical operation derived from equation (y), is as follows.

To find nH, No. 1, equation (y), gives

$4d^2 = 4 \times 9.5 \times 9.5 = 361$, first member under the radical sign,
$\delta'^2 = (18-12)^2 = \underline{36}$, second ———————————,
difference of the terms $= 325$, its square root is, 18.027,

$$\frac{2\text{AD}+\text{BC}}{6(\text{AD}+\text{BC})} = \frac{2 \times 12+18}{6(18+12)} = 0.233, \text{ the value of the coefficient.}$$

then $n\text{H} = 18.027 \times 0.233 = 4.2$ feet nearly.

To find pH, No. 2, equation (y), gives

$4d^2 = 4 \times 9.5 \times 9.5 = 361$, first member under the radical sign,
$\delta'^2 = (18-12)^2 = \underline{36}$, second ———————————,
diff. of the members $= 325$, its square root is, 18.027,

$$\frac{2\text{BC}+\text{AD}}{6(\text{BC}+\text{AD})} = \frac{2 \times 18+12}{6(18+12)} = 0.266, \text{ the value of the coefficient.}$$

then $p\text{H} = 18.027 \times 0.266 = 4.81$ feet.

Now, we have already stated that the value of the line np, which connects the middle points of the parallel sides AD and BC, is expressed by the formula

$$np=\tfrac{1}{2}\sqrt{4d^2-\delta'^2};$$

consequently, we have

$$np=\frac{18.027}{2}=4.2+4.81=9.013 \text{ feef.}$$

COROL. If we compare the two particular cases which we have just exemplified, with the general case of the trapezium formerly treated of; it will appear, that the nearer the figure approaches to regularity, the simpler do its elements become, and the more elegant is the theory which those elements unfold. We shall therefore descend one step further in the line of induction, in order to show that our inquiries will ultimately lead us to a perfect figure, in which the centre of magnitude and the centre of gravity are the same.

40. If we suppose the parallel sides AD and BC to be equal to each other, then the figure will be a rhomboid or a rectangle, according as the other sides meet those that are parallel, obliquely, or at right angles; and in either case, since δ' the difference of the two parallel sides vanishes, the equations marked (y) become

$$n\text{H}=\tfrac{1}{2}d;$$
$$\text{and } p\text{H}=\tfrac{1}{2}d;$$

and moreover, if we consider all the sides to be equal among themselves, the figure will be either a rhombus or a square; and here also the values of nH and pH are expressed by $\tfrac{1}{2}d$; hence we infer, that

If in a rectangle, a rhomboid, a square or rhombus, any two opposite sides be bisected, the centre of gravity of the figure is situated in the middle of the line that joins the points of bisection.

But in all the figures just enumerated, the centre of magnitude is in that point; consequently,

In a rectangle, a rhomboid, a square or a rhombus, the centre of gravity and the centre of magnitude are situated in the same point.

These, with the general, and the two particular forms above alluded to, are all the varieties of the quadrilateral figure that can occur; consequently, the theory as we have now expounded it, may, as far as regards the quadrilateral only, be considered as complete; hence, it is quite unnecessary to dwell longer on this branch of the subject, as it is presumed that but little more can be advanced respecting it.

We therefore proceed to the determination of the centre of gravity of some other right lined figures, which, from their usefulness, do certainly merit a place in the theory which we are now endeavouring to establish; and first of irregular polygons.

SECTION SEVENTH.

OF THE CENTRE OF GRAVITY OF IRREGULAR POLYGONS.

PROBLEM. *To determine the centre of gravity of any irregular polygon whatever.*

41. This problem, from its similarity to that which we have now resolved, will have its solution in a great measure depending on the same or similar principles; but for the sake of variety, and for the purpose of introducing other modes of calculation, we shall not confine ourselves entirely to the methods heretofore employed, but shall avail ourselves of the aid of other methods whenever they are found to answer our purpose.

Every polygon whose sides and angles are given, may be divided into triangles or trapezia, whose areas can be found by the rules of trigonometry, and whose centres of gravity can be determined by the methods already explained.

A polygon consisting of five sides, is divided into one triangle and one trapezium, by drawing a line from any angle to any other angle not immediately adjacent; or it is divided into three triangles, by drawing two lines from any angle, to the extremities of the side directly opposed to it.

In like manner, a polygon of six sides may be divided into two trapezia, and these again into two triangles; hence, a polygon of six sides is divisible into four triangles, either by joining the remote extremities of two contiguous sides, or by lines drawn from any angle to those which are not immediately adjacent.

A polygon of seven sides is divisible into five triangles, and a polygon of eight sides into six; hence, generally, a polygon of n sides is divisible into $n-2$ triangles; that is,

A polygon of any number of sides, is divisible into as many triangles, as are denoted by two less than the number of its sides.

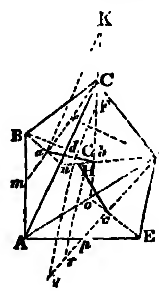
And since three operations are necessary to determine the centre of gravity of a quadrilateral or four sided figure, four operations will be required to determine the centre of gravity of a figure of five sides, five operations for a figure of six sides, and so on, always one operation less than there are sides in the polygon.

Hence it appears, that a formula to calculate generally, the centre of gravity of a polygon of any number of sides, would be excessively intricate, and after all, it would involve the several operations necessary for determining individually, the centres of gravity of all the lower orders of polygons, from the triangle upwards to the polygon of n sides.

We shall not therefore, attempt to resolve the present problem generally by calculation, nor yet to carry it to any high degree, for the law of continuity being obvious, and the principle of construction general, we are apprehensive that but little advantage would be derived by pursuing the process of induction, especially since the reader can have no difficulty in carrying it to whatever extent the subject of his immediate inquiries may demand.

42. In the first place then, let it be proposed to determine the position of the centre of gravity of the five-sided figure ABCDE, whose sides and angles are either all given or determinable from one another.

Draw the diagonals AC and AD, then is the figure ABCDE divided into the three triangles ABC, ACD and ADE, whose common centre of gravity must be the centre of gravity of the figure ABCDE.



Bisect the straight lines AB, AC, AD and AE in the points m, n, o and p, and join Cm, Bn; Dn, Co; Eo and Dp, cutting each other two and two in the points a, b and c; then shall a, b and c be the centres of gravity of the component triangles ABC, ACD and ADE respectively.

Conceive the triangles ABC and ACD to be wholly collected in the points a and b, and join ab; then shall the common centre of gravity of the triangles ABC and ACD, or the centre of gravity of the trapezium ABCD, occur in the line ab, and divide it into two parts that are to each other, reciprocally, as the areas of the triangles ABC and ACD; but the areas of the triangles ABC and ACD subsisting on the same base AC, are to one another as the perpendiculars Bd and De; consequently the common centre of gravity of the triangles ABC and ACD, divides the line ab into two parts, that are to one another reciprocally as the perpendiculars Bd and De.

Produce bc to K, and make bF and FK respectively proportional to the perpendiculars Bd and De; join Ka, and through the point F, draw FG parallel to Ka meeting ab in the point G; then shall G be the common centre of gravity of the triangles ABC and ACD.

Imagine the triangles ABC, ACD, or the trapezium ABCD, to be wholly collected in the point G, while the triangle ADE is in like manner supposed to be collected in the point c; join Gc, then shall the common centre of gravity of the trapezium ABCD and the triangle ADE, or the centre of gravity of the five sided figure ABCDE, occur in

the line GC, and divide it into two parts, that are to each other reciprocally as the areas of the trapezium ABCD, and the triangle ADE.

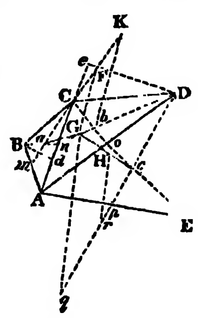
Produce *cp* to *q*, and make *cr* and *rq*, respectively proportional to the areas of the trapezium ABCD and the triangle ADE; join G*q*, and through the point *r* draw *r*H parallel to *q*G, meeting G*c* in the point H, then shall H be the common centre of gravity of the component triangles ABC, ACD and ADE, or the centre of gravity of the five sided polygon ABCDE, which is composed of these triangles.

43. This is the general method of construction, and its application to particular examples will appear in what follows, where the solutions drawn out at length, will, it is presumed, suffice to show in what manner the position of the point H is to be determined, relatively to the positions of the points G and *c*.

EXAMPLE 1. In the five sided figure ABCDE there are given the sides AB, BC, CD, DE and AE, equal respectively to 20, 30, 40, 50 and 60 feet; at what point in the superficies of the figure ABCDE does its centre of gravity occur, supposing the angles ABC and BAE to be respectively equal to 102°, 35′ and 126° 46′?

GEOMETRICAL CONSTRUCTION.

Draw the straight line AE, and from a scale of equal parts of any convenient magnitude at pleasure, take AE equal to 60 feet; at the point A, in AE, make the angle EAB equal to 126° 46′ taken from the scale of chords, and set off AB equal to 20 feet; at the point B, in AB, make the angle ABC equal to 102° 35′ taken from a scale of chords, and set off BC equal to 30 feet. Then from the points C and E, with distances CD equal to 40 and ED equal to 50 feet, describe arcs cutting each other in the point D; draw CD and ED; then shall ABCDE be the given five sided figure, whose centre of gravity is required to be found.



Draw the diagonals AC and AD, then is the figure ABCDE divided into the three triangles ABC, ACD and ADE; bisect the sides AB, AC, AD and AE in the points *m*, *n*, *o* and *p*, and join C*m*, B*n*; D*n*, C*o*; E*o* and D*p*, cutting each other two and two in the points *a*, *b* and *c*; then shall *a*, *b* and *c* be the centres of gravity of the constituent triangles ABC, ACD and ADE respectively.

Produce the diagonal AC to the point *e*, and let fall the perpendiculars B*d* and D*e*; then conceive the triangles ABC and ACD to be wholly concentrated in the points *a* and *b*, and join *ab*. Produce *ac* to the point K, and make *a*F and FK respectively proportional to the perpendiculars D*e* and B*d*; join K*b*, and through the point F draw FG parallel to K*b* meeting *ab* in the point G, then shall G be the

common centre of gravity of the triangles ABC and ACD. Join GC and produce cp to q, making cr and rq respectively proportional to the areas of the trapezium ABCD, and the triangle ADE; join qG, and through the point r draw rH parallel to qG, meeting Gc in the point H; then shall H be the place of the centre of gravity of the figure ABCDE.

If GH and cH be taken in the compasses, and applied to the same scale of equal parts from which the figure was constructed, they will indicate respectively 14·75 and 10·33 feet. Hence, the position of the point H relating to the positions of the points G and c has been determined.

The numerical solution is effected in the following manner, viz.

In the plane triangle ABC, there are given the two sides AB and BC, respectively equal to 20 and 30 feet, and the included angle ABC, equal to 102° 35′; to find the side AC and the angle BAC.

First then for the side AC.

Find φ an angle such, that

$$\tan. \varphi = \frac{2 \sin. \frac{1}{2}\text{ABC}\sqrt{\text{AB}\cdot\text{BC}}}{\text{BC}-\text{AB}},$$

here follows the operation:

$\text{AB}=20$	log. 1·301030
$\text{BC}=30$	log. 1·477121
$\text{AB}\cdot\text{BC}=20\times30=600$ . . .	log. 2·778151 sum of the logs.
$\sqrt{\text{AB}\cdot\text{BC}}=\sqrt{600}=24\cdot49$. . .	log. 1·389075 half sum of the logs.
$\frac{1}{2}\text{ABC}=\frac{1}{2}(102°\,35')=51°\,17'30''$	log. sin. 9·892284
constant .	log. 0·301030
$\text{BC}-\text{AB}=30-20=10$ . .	ar. co. log. 9,000000
hence we have $\varphi=75°\,20'\,27''$	log. tan. 10.582389;

having thus found the value of the angle φ, the value of the diagonal AC is found by the following very simple equation: viz.

$$\text{AC}=(\text{BC}-\text{AB}) \sec. \varphi;$$

now, the natural secant of φ, that is, the natural secant of 75° 20′ 27″ is 3.955; consequently, by the preceding equation, we have

$$\text{AC}=10\times3.955=39.55 \text{ feet.}$$

Then, to find the angle BAC, it is

$$39.55 : 30 :: \sin. 102°\,35' : \sin. 47°\,45'\,29'';$$

that is. $\frac{30\times\cdot97598}{39.55}=\cdot74031=$nat. sin. 47° 45′ 29″

But the angle BAE is given equal to $126° 46'$; therefore, by subtraction, we have

$$\text{the angle CAE} = 126° 46' - 47° 45' 29'' = 79° 0' 31''.$$

Join CE, then in the triangle ACE, there are given, the two sides AC and AE, respectively equal to $39 \cdot 55$ and 60 feet, and the included angle CAE equal to $79° 0' 31''$; to find the side CE and the angle ACE.

First then, for the side CE,

Find φ an angle such, that

$$\tan. \varphi = \frac{2 \sin. \frac{1}{2} \text{CAE} \sqrt{\text{AC} \cdot \text{AE}}}{\text{AE} - \text{AC}},$$

the operation is as follows,

$\text{AC} = 39.55$	log.	1.597146	
$\text{AE} = 60$	log.	1.778151	
$\text{AC} . \text{AE} = 39.55 \times 60 = 2373$.	log.	3.375297	sum of the logs.
$\sqrt{\text{AC.AE}} = \sqrt{2373} = 48 \cdot 71$ . .	log.	1.687648	$\frac{1}{2}$ sum of the logs.
$\frac{1}{2}\text{BAE} = \frac{1}{2}(79° 0' 31'') = 39° 30' 15\frac{1}{2}''$	log.sin.	9.803550	
constant . .	log.	0.301030	
$\text{AE} - \text{AG} = 60 - 39.55 = 20.45$.	ar.co.log.	8.689307	
hence we have $\varphi = 71° 44' 20''$	log. tan.	10.481535;	

Having thus found the value of the angle φ, the value of the diagonal CE, is found by the following very simple equation, viz.

$$\text{CE} = (\text{AE} - \text{AC}) \sec. \varphi;$$

Now, the natural secant of φ, that is, the natural secant of $71° 44' 20''$, is 3.19134; consequently, by the foregoing theorem, we have

$$\text{CE} = 20.45 \times 3.19134 = 65.24 \text{ feet}.$$

Then, to find the angle ACE, it is,

$$65.24 : 60 :: \sin. 79° 0' 31'' : \sin. 64° 31';$$

$$\text{that is, } \frac{60 \times .98165}{65.24} = .90281 = \text{nat. sin. } 64° 31'.$$

Again, in the triangle ECD, there are given the three sides EC, CD and DE, respectively equal to 65.24, 40 and 50 feet; to find the angle ECD; for which purpose we have the following equation, viz.

$$\text{vers. ECD} = \frac{\text{ED}^2 - (\text{CE} - \text{CD})^2}{2\text{CE} . \text{CD}},$$

which formula supplies the following numerical operation, viz.

$$
\begin{array}{llll}
\text{ED}=50 & & 50 & \\
\text{CD}=40 & \text{EC}= & 65.24, \text{ add} & \\
\overline{90} & & \overline{115.24}, & \\
65.24 & & 40\quad, \text{ subtract} & \\
\overline{24.76} & & \overline{75.24} \ldots\ldots \text{log.} & 1.876449 \\
& & 24.76 \ldots\ldots \text{log.} & 1.393751 \\
& & 65.24 \text{ ar. co. log.} & 8.185486 \\
& 40\times 2= & 80 \quad \text{ar. co. log.} & \underline{8.096910} \\
& \text{nat. vers. } 49^\circ\, 59'= & 0.35694\ldots \text{log.} & 9.552596;
\end{array}
$$

consequently, the whole angle ACD$=$ACE$+$ECD; that is, ACD$=$ $64^\circ\,31'+49^\circ\,59'=114^\circ\,30'$; therefore, in the triangle ACD, we have given the two sides AC and CD, equal respectively to 39.55 and 40 feet, and the included angle ACD equal to $114^\circ\,30'$; to find the diagonal AD.

For which purpose, find φ an angle such, that

$$\tan. \varphi=\frac{2\sin.\frac{1}{2}\text{ACD}\sqrt{\text{AC.CD}}}{\text{CD}-\text{AC}},$$

the numerical operation is as follows,

$$
\begin{array}{rll}
\text{AC}=39.55 \ldots\ldots\ldots\ldots \text{log.} & 1.597146 & \\
\text{CD}=40 \quad \ldots\ldots\ldots\ldots \text{log.} & \underline{1.602060} & \\
\text{AC.CD}=39.55\times 40=1582 \ldots \text{log.} & 3.199206 & \text{sum of the logs.} \\
\sqrt{\text{AC.CD}}=\sqrt{1582}=39.77 \ldots \text{log.} & 1.599603 & \tfrac{1}{2}\text{ sum of the logs.} \\
\tfrac{1}{2}\text{ACD}=\tfrac{1}{2}(114^\circ 30')=57^\circ 15' \text{log. sin.} & 9.926029 & \\
\text{constant} \ldots\ldots \text{log.} & 0.301030 & \\
\text{CD}-\text{AC}=40-39.55=.45 \text{ ar. co. log.} & \underline{0.346787} & \\
\text{hence we have } \varphi=89^\circ\, 36'\, 58'' \text{ log. tan.} & 12.173449; &
\end{array}
$$

having thus found the value of the angle φ, the value of the diagonal AD is found by the following very simple and concise equation, viz.

$$\text{AD}=(\text{CD}-\text{AC})\sec.\varphi.$$

Now the natural secant of φ, that is, the natural secant of $89^\circ\,36'\,58''$, is 149.005; consequently, by the foregoing theorem, we have

$$\text{AD}=.45\times 149.005=67.05 \text{ feet.}$$

Hence then, in the five sided figure ABCDE, we have given the sides AB, BC, CD, DE and AE, respectively equal to 20, 30, 40, 50 and 60 feet, and the diagonals AC and AD, respectively equal to 39.55 and 67.05 feet; from which, the areas of the triangles ABC, ACD and ADE, together with the position of their common centre of gravity, can easily be found.

From the equations marked (t), we have

$$cm=\tfrac{1}{2}\sqrt{2(\text{BC}^2+\text{AC}^2)-\text{AB}^2}.$$

Whence we obtain the following process,

$BC^2 = 30 \times 30 = \quad 900$
$AC^2 = 39.55 \times 39.55 = 1564.2025$
$2(BC^2 + AC^2) = 2 \times \overline{2464.2025} = 4928.405$, first term under rad.
$AB^2 = 20 \times 20 = 400$, second ———————,
difference of the terms, $= \overline{4528.405}$, its sq. root, is 67.293;

consequently, by halving, we get

$$cm = \frac{67.293}{2} = 33.646 \text{ feet.}$$

Again, from the equations marked (t), we have

$$co = \tfrac{1}{2}\sqrt{2(AC^2 + CD^2) - AD^2},$$

which gives the following numerical process,

$AC^2 = 39.55 \times 39.55 = 1564.2025$
$CD^2 = 40 \quad \times 40 \quad = 1600.$
$2(AC^2 + CD^2) = 2 \times \overline{3164.2025} = 6328.405$, first term under rad.
$AD^2 = 67.05 \times 67.05 = 4495.7025$, second———————
difference of the terms $= \overline{1832.7025}$, its sq. root is 42.81;

consequently, dividing by 2, we get

$$co = \frac{42.81}{2} = 21.41 \text{ feet very nearly.}$$

Now $Am = \frac{1}{2} AB$, and $Ao = \frac{1}{2} AD$; that is, $Am = \frac{20}{2} = 10$, and $Ao = \frac{67.05}{2} = 33.525$ feet; therefore, in the triangles ACm and ACo; there are given, the sides AC, Cm and Am; AC, Co and Ao; to find the angles ACm and ACo, whose sum is equal to mCo.

To find the angle ACm, we have

$$\text{vers. } ACm = \frac{Am^2 - (AC - Cm)^2}{2AC \cdot Cm},$$

from which equation we derive the following process,

$Am = 10$————————	10	
$AC = 39.55$———— Cm	$= 33.646$. add	
49.55	43.646,	
33.646	39.55 subtract	
15.904	4.096. . . . . log.	0.612360
	15.904. . . . . log.	1.201506
	33.646 . ar. co. log.	8.473067
$39.55 \times 2 = 79.1$	. ar. co. log.	8.101824
nat. vers. $12^\circ\ 42'$ . . . . $=$	0.02448 . . . log.	8.388757.

hence, we have the angle $ACm = 12^\circ 42'$

To find the angle ACo, we have

$$\text{vers. ACo} = \frac{\text{A}o^2 - (\text{AC} - \text{C}c)^2}{2\,\text{AC}\,.\,\text{C}o}$$

from which expression we derive the following process.

Ao=33.525	——— 33.526		
AC=39.55	——— Co=21.41 , add		
73.075	54.935,		
21.41	39.55 , subtract		
51.665	15.385	log.	1.187098
	51.665	log.	1.713196
	21.41 . ar. co.	log.	8.669383
	79.1 . ar. co.	log.	8.101824
nat. vers. 57° 57′=	0.46935 . .	log.	9.671501

hence we have the angle ACo=57° 57′.

Now, the angle mCo=ACm+ACo; that is,

mCo=12° 42′+57° 57′=70° 39′,

and from what we have shown, in treating of the centre of gravity of the plane triangle, we get

$$ca = \tfrac{2}{3}\,cm, \text{ and } cb = \tfrac{2}{3}\,co;$$

$$\text{that is } ca = \frac{33.646 \times 2}{3} = 22.431 \text{ feet nearly.}$$

$$\text{and } cb = \frac{21.41 \times 2}{3} = 14.27 \text{ feet nearly.}$$

Consequently, in the triangle acb, there are given, the two sides ac and bc, equal respectively to 22.431 and 14.27 feet, and the included angle acb, equal to 70° 39′; to find the side ab, the distance between the centres of gravity of the triangles ABC and AOD, together with the angle abc.

Find φ an angle such, that

$$\tan.\,\varphi = \frac{2 \sin.\,\frac{1}{2}\,acb\,\sqrt{ac\,.\,bc}}{ac - bc},$$

the numerical operation supplied by this formula is as follows, viz.

ac=22.431	log.	1.350848
bc=14.27	log.	1.154424
$ac\,.\,bc$=22.431 × 14.27=320.09	log.	2.505272 sum of logs,
$\sqrt{ac\,.\,bc}=\sqrt{320.09}$= 17.89	log.	1.252636 ½ sum of logs,
½ acb=½ (70° 39′)=35° 19′ 30″	log. sin.	9.762088
Constant	log.	0.301030
$ac-bc$=22.431 − 14.27=8.161 ar. co.	log.	9.088257
hence we have φ=68° 28′ 24″	log. tan.	10.404011,

having thus found the value of the angle φ, the value of the line ab, is found by the following very simple expression, viz.

$$ab = (ac - bc) \text{ sec. } \varphi,$$

but the natural secant of φ, that is, the natural secant of $68° 28' 24''$ is $2 \cdot 725$; consequently, by the preceding equation, we have

$$ab = 8 \cdot 161 \times 2 \cdot 725 = 22 \cdot 23 \text{ feet, very nearly;}$$

and because the straight line ab, connects the centres of gravity of the triangles ABC and ACD; if we conceive these triangles to be wholly concentrated in the points a and b, their common centre of gravity, or that of the trapezium ABCD, shall divide the line ab, into two parts such, that they are to each other, reciprocally, as the areas of the triangles ABC and ACD.

In the geometrical construction of this example, the line ab was divided into two parts, reciprocally proportional to the perpendiculars from B and D on the diagonal AC and AC produced, by which means we avoided the calculation of the areas of the triangles; but in the present instance, since the angles ABC and ACD are known, it is easier to compute the areas of the triangles, than it is to determine the perpendiculars; and, because the areas of the triangles must ultimately be known, before the centre of gravity of the whole figure can be found; it is obvious, therefore, that the best and easiest mode of procedure in this case, is to calculate directly the areas of the triangles ABC and ACD.

Now, the writers on Trigonometry have shown that in any plane triangle, when two sides and the included angle are given, the area is equal

To half the product of the two given sides, drawn into the natural sine of their included angle.

Therefore, let A represent the area of any plane triangle, a and b the two given sides, and φ the included angle; then we have generally, for the area

$$A = \tfrac{1}{2}ab \text{ sin. } \varphi \qquad (z)$$

Which formula, being adapted to the triangles now under consideration, gives

$$A = \tfrac{1}{2} \text{ AB} \cdot \text{BC sin. ABC, for the } \Delta \text{ABC,}$$

and $$A = \tfrac{1}{2} \text{ AC} \cdot \text{CD sin. ACD; for the } \Delta \text{ACD,}$$

or by adopting the numerical valu· · of the several factors in each, we have

$$A = \tfrac{1}{2} (20 \times 30) \text{ sin. } 102° 35', \text{ for the } \Delta \text{ ABC,}$$
$$A = \tfrac{1}{2} (39 \cdot 55 \times 40) \text{ sin. } 114° 30'; \text{ for the } \Delta \text{ ACD;}$$

but the natural sine of $102° 35'$, is $\cdot 97595$, and that of $114° 30'$ is $\cdot 90996$: hence, the respective areas are as beneath, viz.

The triangle $\text{ABC} = 20 \times 15 \times \cdot 97598 = 292 \cdot 794$ square feet,
The triangle $\text{ACD} = 39 \cdot 55 \times 20 \times \cdot 90996 = 719 \cdot 778$ square feet.

Then, to find aG and bG, the segments of the line ab, which joins the centres of gravity of the triangles ABC and ACD, we have by the first principle

$292 \cdot 794 + 719 \cdot 778 : 22 \cdot 23 :: 719 \cdot 778 : 15 \cdot 8$ feet,

and $292 \cdot 794 + 719.778 : 22 \cdot 23 :: 292 \cdot 794 : 6 \cdot 43$ feet; consequently, the position of the point G, with respect to the surface of the trapezium ABCD, is known; it therefore remains to determine the position of the point H, with respect to the surface of the polygon or five sided figure ABCDE, or rather, with respect to the positions of the points G and c which mark the centres of gravity of the trapezium ABCD, and the triangle ADE.

Next to find the value of the angle abc, we have

$22 \cdot 23 : 22 \cdot 431 :: \sin. 70^\circ 39' : \sin. 72^\circ 11'$;

that is, $\dfrac{22 \cdot 431 \times \cdot 94351}{22 \cdot 23} = \cdot 95204 =$ nat. sin. $72^\circ 11'$.

Recurring to the equations marked (t), we have

$$\mathrm{D}n = \tfrac{1}{2}\sqrt{2(\mathrm{AD}^2 + \mathrm{CD}^2) - \mathrm{AC}^2},$$

Which formula supplies the following process,

$\mathrm{AD}^2 = 67 \cdot 05 \times 67 \cdot 05 = 4495 \cdot 7925$,

$\mathrm{CD}^2 = 40 \times 40 = 1600 \cdot$ [radical sign.

$2(\mathrm{AD}^2 + \mathrm{CD}^2) = 2 \times \overline{6095 \cdot 7025} = 12191 \cdot 405$, the first term under the

$\mathrm{AC}^2 = 39 \cdot 55 \times 39 \cdot 55 = 01564 \cdot 2025$, second ——————

difference of the terms $= \overline{10627 \cdot 2025}$, its square root is, $103 \cdot 08$; therefore, dividing by 2, we get

$$\mathrm{D}n = \frac{103 \cdot 08}{2} = 51 \cdot 54 \text{ feet.}$$

Again, from the equations marked (t), we have

$$\mathrm{D}p = \tfrac{1}{2}\sqrt{2(\mathrm{AD}^2 + \mathrm{DE}^2) - \mathrm{AE}^2},$$

Which formula indicates the following process;

$\mathrm{AD}^2 = 67 \cdot 05 \times 67 \cdot 05 = 4495 \cdot 7025$

$\mathrm{DE}^2 = 50 \times 50 = 2500.$ [rad. sign,

$2(\mathrm{AD}^2 + \mathrm{DE}^2) = 2 \times \overline{6995 \cdot 7025} = 13991 \cdot 405$, first term under the

$\mathrm{AE}^2 = 60 \times 60 = 3600$ second, ——————

difference of the terms $= \overline{10391 \cdot 405}$, its square root is $101 \cdot 94$; therefore, dividing by 2, we obtain

$$\mathrm{D}p = \frac{101 \cdot 94}{2} = 50 \cdot 97 \text{ feet.}$$

Now, $\mathrm{A}n = \frac{1}{2}\mathrm{AC}$, and $\mathrm{A}p = \frac{1}{2}\mathrm{AE}$; that is, $\mathrm{A}n = \dfrac{39 \cdot 55}{2} = 19 \cdot 775$, and $\mathrm{A}p = \dfrac{60}{2} = 30$ feet; therefore, in the triangles ADn and ADp, there are given the sides AD, Dn and An; AD, Dp and Ap; to find the angles ADn and ADp, whose sum is equal to the angle $n\mathrm{D}p$.

To find the angle ADn, we have

$$\text{vers. AD}n=\frac{\text{A}n^2-(\text{AD}-\text{D}n)^2}{2\text{AD}\cdot\text{D}n}$$

From which we derive the following operation.

An=19·775	——————19·775,		
AD=67·05	————Dn=51·54, add.		
86·825	71·315		
51·54	67·05		
35.285	4.265	.. log.	0.629919
	35.285	.. log.	1.547590
	51.54	ar.co. log.	8.287856
	67·05×2=134.1	ar.co. log.	7·872571
nat.vers. 11° 59′ 30″=·02182	....	log.	8.337936,

consequently, the measure of the angle ADn, is 11° 59′ 30.″

To find the angle ADp, we have

$$\text{vers. AD}p=\frac{\text{A}p^2-(\text{AD}-\text{D}p)^2}{2\text{AD}\cdot\text{D}p}$$

Which formula affords the following operation,

Ap=30	——————30		
AD=67·05	———— Dp=50·97 add.		
97·05	80·97		
50·97	67·05 subtract		
46·08	13·92	... log.	1·143639
	46·08	... log.	1·663512
	50·97	ar.co. log.	8·292685
	67·05×2=134·1	ar.co. log.	7·872571
nat. vers. 25° 1′=·09384	......	log.	8·972407;

consequently, the measure of the angle ADp, is 25° 1′. But the angle nDp is equal to the sum of the angles ADn and ADp; that is,

the angle nDp=11° 59′ 30″+25° 1′=37° 0′ 30″,

and from what we have stated respecting the centre of gravity of the plane triangle, we have

$$\text{D}b=\tfrac{2}{3}\text{D}n,\text{ and D}c=\tfrac{2}{3}\text{D}p;$$

$$\text{that is, D}b=\frac{51{\cdot}54\times2}{3}=34{\cdot}36\text{ feet,}$$

$$\text{and D}c=\frac{50{\cdot}97\times2}{3}=33{\cdot}98\text{ feet.}$$

Join bc; then in the triangle bDc, there are given the two sides Db, Dc and the included angle bDc; to find the side bc.

Find φ an angle such, that

$$\tan\varphi=\frac{2\sin.\tfrac{1}{2}b\text{D}c\sqrt{\text{D}b.\text{D}c}}{\text{D}b-\text{D}c}$$

which formula implies the following operation.

$$
\begin{array}{rll}
Db=34.36 \;\ldots\ldots & \log. & 1.536053 \\
Dc=33\cdot 98 \;\ldots\ldots & \log. & 1.531223 \\
Db\cdot Dc=34.36\times 33.93=1167.5528 & \log. & \overline{3.067276} \text{ sum of logs.} \\
\sqrt{Db.Dc}=\sqrt{1167.5528}=34.17 & \log. & 1.533638 \tfrac{1}{2} \text{ sum of logs.} \\
\tfrac{1}{2}\, bDc=\tfrac{1}{2}\,(37^{\circ}\,0'30'')=18^{\circ}\,30'15'' & \log.\ \sin. & 9.501570 \\
\text{constant} & \log. & 0.301030 \\
Db-Dc=34.36-33.98=.38 & \text{ar. co. log.} & 0.420216 \\
\text{hence we have } \varphi=88^{\circ}\,59'47'' & \log.\ \tan. & \overline{11.756454}.
\end{array}
$$

having thus found the value of the angle φ, the value of the side bc, is found by the following very simple equation, viz.

$$bc=(Db-Bc)\ \sec.\ \varphi.$$

Now, the natural secant of φ; that is, the natural secant of $88^{\circ}\,59'47''$ is $57\cdot 082$; consequently, by the preceding equation, we have

$$bc=\cdot 38\times 57\cdot 082=21\cdot 69 \text{ feet.}$$

We have now very nearly attained the object of our research; viz. the point H, in which the line bc is divided into two parts, that are to each other, reciprocally as the areas of the trapezium ABCD, and the triangle ADE.

It has already been shown, that the line cb is two thirds of the line co, and in like manner it may be shown that Ec is two thirds of the line Eo; but the value of Eo, from the equations marked (t), is

$$Eo=\tfrac{1}{2}\sqrt{2\,(AE^2+DE^2)-AD^2}.$$

and the numerical operation indicated by this formula, is as under,

$$
\begin{array}{l}
AE^2=60\times 60=3600 \\
DE^2=50\times 50=2500 \\
2\,(AD^2+DE^2)=2\times \overline{6100}=12200, \text{ first term under the radical} \\
AD^2=67\cdot 05\times 67\cdot 05=\ \underline{4495\cdot 7025}, \text{ second ———————,} \\
\text{difference of the terms}=\ 7704\cdot 2975, \text{ its square root is, } 87\cdot 78;
\end{array}
$$

consequently, by halving, we have

$$Eo=\frac{87\cdot 78}{2}=43\cdot 89 \text{ feet,}$$

and we have just stated that Ec is two thirds of Eo; that is

$$Ec=\frac{43\cdot 99\times 2}{3}=29\cdot 26 \text{ feet.}$$

$$\text{Now } bc=\frac{21.41\times 2}{3}=14.27 \text{ feet.}$$

Hence, by subtraction, we get

$co = 43\cdot 89 - 29\cdot 26 = 14.63,$

and $bo = 21.41 - 14\cdot 27 = 7.14$ nearly, and the side bc, as determined above, is . . . . 21.69 feet.

Therefore, in the small triangle bco, there are given the three sides bc, bo, and co, respectively equal to 21·67, 7.14 and 14.63 feet; to find the angle cbo, which being added to the supplement of the angle abc, gives the angle cbG.

To find the angle cbo, we have

$$\text{vers. } cbo = \frac{co^2 - (bc - bo)^2}{2bc \cdot bo},$$

and the numerical operation afforded by this expression, is as follows, viz.

$co = 14\cdot 63$ ———————— $14\cdot 63$

$bc = 21.69$ ———————— $bo = 7.14$, add

36·32   21·77,

7.14   21.69, subtract

29·18   .08 . . . . log. 8.903090

29·18 . . . . log. 1.465085

21·69 . ar. co. log. 8·663740

7.14 × 2 = 14.28 . ar. co. log. 8·845272

nat. vers. 7° 2′ 30″ = 0.00754 . . . . log. 7·877187,

consequently, the angle $cbG = (180° - abc) + cbo$; that is,

$cbG = (180° - 72°\ 11') + 7°\ 2'\ 30'' = 114°\ 51'\ 30''.$

Then in the triangle Gbc, there are given the two sides bc and bG, equal respectively to 21.69 and 6.43 feet, and the included angle cbG equal to 114° 51′ 30″; to find the side Gc which joins the centre of gravity of the trapezium ABCD to that of the triangle ADE.

Find φ an angle such, that

$$\tan.\ \varphi = \frac{2 \sin. \frac{1}{2} cbG \sqrt{bc \cdot bG}}{bc - bG},$$

and the numerical process indicated by this theorem is as follows, viz.

$bc = 21\cdot 69$ . . . . log. 1·336260

$bG = 6\cdot 43$ . . . . log. 0·808211

$bc \cdot bG = 21\cdot 69 \times 6\cdot 43 = 139\cdot 4667$ log. 2·144471 sum of logs.

$\sqrt{bc \cdot bG} = \sqrt{139\cdot 4667} = 11\cdot 81$ . . . log. 1·072235 ½ sum of logs.

$\frac{1}{2} cbG = \frac{1}{2}(114° 51' 30'') = 57° 25' 45''$ log. sin. 9·925686

constant log. 0·301030

$bc - bG = 21\cdot 69 - 6\cdot 43 = 15\cdot 26$ ar. co. log. 8.816445

hence we have $\varphi = 52°\ 31'\ 26''$ log. tan. 10·115396.

Having thus determined the value of the angle φ, the value of the side GC, is found by the following very simple equation, viz.

$$GC = (bc - bG) \sec. \varphi.$$

Now, the natural secant of φ, that is, the natural secant of $52° 31' 26''$ is $1 \cdot 643$; consequently, by the above equation, we have

$$GC = 15.26 \times 1.643 = 25.08 \text{ feet.}$$

Therefore the distance between the centre of gravity of the trapezium ABCD, and that of the triangle ADE, is $25 \cdot 08$ feet, and the centre of gravity of the polygon divides that distance into two parts, that are to each other reciprocally as the areas of the trapezium ABCD, and the triangle ADE.

Now, the three sides of the triangle ADE are $67 \cdot 05$, 60 and 50 feet; consequently, its area is determined by the following process, the rule for which is given in every book on the subject of Mensuration extant.

AD =	67·05		
AE =	60		
DE =	50		
AD + AE + DE =	177·05,		
$\frac{1}{2}$(AD + AE + DE) =	88·525,	. . . . log.	1·947065,
88·525 — 67·05 =	21·475,	. . . . log.	1·331933,
88·525 — 60 =	28·525,	. . . . log.	1.455226,
88·525 — 50 =	38·525,	. . . . log.	1.585742,
		log.	6·319966, sum of the logs.
natural number, or area = 1445·38		log.	3·159983, $\frac{1}{2}$ sum of the logs.

Therefore the area of the triangle ADE, is 1445·38 square feet; and we have already found that the area of the trapezium ABCD, is 292·794 + 719·778 = 1012.572 square feet.

Hence, for the distances GH and CH, we have

$$1445 \cdot 38 + 1012 \cdot 572 : 25 \cdot 08 :: 1445 \cdot 38 : 14 \cdot 75 \text{ feet nearly,}$$
$$1445 \cdot 38 + 1012 \cdot 572 : 25 \cdot 08 :: 1012 \cdot 572 : 10 \cdot 33 \text{ feet.}$$

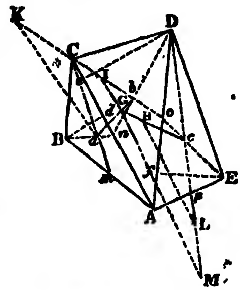
The calculation of this example is exceedingly laborious, nevertheless it involves nothing difficult; it merely presupposes a knowledge of the three cases of plane trigonometry, for without such knowledge the calculation cannot be performed. We are not aware that the labour can be much abridged, but in actual practice it is not necessary to attend to all the particulars of accurate arrangement that are exhibited in the example before us, and consequently the process may be conducted with greater rapidity; but still, every step of the operation must be performed, and all the parts of the figure calculated after the manner exemplified above; because that part which is more immediately the object of

research, is situated at such a distance from the parts that are given, that it cannot be approached without previously determining all the intermediate and circumjacent parts of the figure: hence the immense labour necessary for its determination. It may, however happen, that a different arrangement of data may tend to abbreviate the process, and bring us by fewer steps to the object proposed; but we must here also observe, that in whatever way the question is enunciated, the labour will still be considerable, as the figure is composed of a trapezium and a triangle, whose centres of gravity must be determined separately before the centre of gravity of the polygon can be found.

44. In the example which follows, we shall propose the data differently, and assume some parts as given, which in the present instance we have had to calculate; but even then, the labour will be found to be very great.

EXAMPLE 2. In the five sided figure ABCDE, there are given the sides AB, BC, CD, DE and AE, equal respectively, to 25, 17, 23, 36 and 16 feet, and the diagonals AC and AD, equal respectively, to 38 and 40 feet; at what point in the superficies of the figure does its centre of gravity occur?

CONSTRUCTION. With the lines AB, BC and AC, equal respectively, to the numbers 25, 17 and 38, construct the triangle ABC, and on AC as a base, with the lines AD and CD, equal respectively, to the numbers 40 and 23, construct the triangle ACD; and lastly, on AD as a base, with the lines AE and DE, equal respectively to the numbers 16 and 36, construct the triangle ADE; then is ABCDE the five sided figure whose centre of gravity is proposed to be found.



Bisect the lines AB, AC, AD and AE in the points m, n, o and p, and draw the lines Cm, Bn; Co, Dn; Dp and Eo, cutting each other two and two in the points a, b and c; then shall a, b and c be the centre of gravity of the triangles ABC, ACD and ADE.

Join ab, and let fall the perpendiculars Bd and De; then, if we conceive the magnitudes or areas of the triangles ABC and ACD to be wholly collected in the points a and b, their common centre of gravity, or that of the trapezium ABCD, must subsist in that line, and as we have already shown, divides it into two parts, that are to each other, reciprocally as the areas of the triangles ABC and ACD. But the perpendiculars Bd and De, are as the areas of the triangles ABC and ACD; therefore, the centre of gravity of the trapezium ABCD, divides the line ab into two parts, that are to each other, reciprocally as the perpendiculars Bd and De.

Produce bc to K, making bI and IK respectively equal to the perpendiculars Bd and De; join Ca, and through the point I, draw IG

parallel to Ka, meeting ab in the point G; then is G the common centre of gravity of the triangles ABC and ACD, or the centre of gravity of the trapezium ABCD.

Join Gc; then, if we conceive the magnitudes or areas of the trapezium ABCD and the triangle ADE, to be respectively collected in the points G and c, their common centre of gravity, or that of the whole figure ABCDE must subsist in the line Gc, and divide it into segments that are to each other reciprocally as the areas of the trapezium and triangle, or reciprocally as the products AC (Bd+De) and AD·Ef, where Ef is perpendicular to AD.

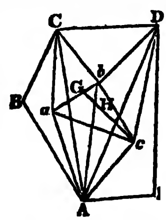
Produce cp to M, making cL and LM respectively proportional to the products AC (Bd+De) and AD.Ef; join MG, and through the point L draw LH parallel to MG, meeting Gc in the point H; then is H the centre of gravity of the figure ABCDE, which was proposed to be found; and if GH and cH be taken in the compasses, and applied to the same scale as that from which the figure was constructed, they will be found to indicate 4·49 and 9·88 feet respectively.

Numerical Calculation.

In order to render the numerical operation more distinct, we shall renew the diagram, using such lines only in its construction, as are absolutely necessary for connecting the parts to which the process of calculation has to be applied.

Let ABCDE, be the five sided figure, whose sides AB, BC, CD, DE and AE, with the diagonals AC and AD are given, and whose centre of gravity H is proposed to be found.

Find the points a, b and c, the centres of gravity of the constituent triangles ABC, ACD and ADE; join ab, bc and ac, and from the points A, C and D draw the lines Aa, Ab and Ac; Ca and Cb; Db and Dc; and if G be the centre of gravity of the trapezium ABCD, join cG, and let cG be divided in H, so that



GH : cH :: area Δ ADE : area trap. ABCD,

then is the point H the place of the centre of gravity sought.

Now, it is obvious, that in order to determine the position of the point H, we must first determine the sides of the triangle abc by which it is included, and also the magnitude of the line Gc in which it is situated; for which purpose, by referring to the formulæ, class (s), we have

$$\mathrm{A}a=\tfrac{1}{3}\sqrt{2\,(\mathrm{AB}^2+\mathrm{AC}^2)-\mathrm{BC}^2},$$
$$\mathrm{A}b=\tfrac{1}{3}\sqrt{2\,(\mathrm{AC}^2+\mathrm{AD}^2)-\mathrm{CD}^2},$$
$$\mathrm{A}c=\tfrac{1}{3}\sqrt{2\,(\mathrm{AD}^2+\mathrm{AE}^2)-\mathrm{DE}^2},$$
$$\mathrm{C}a=\tfrac{1}{3}\sqrt{2\,(\mathrm{AC}^2+\mathrm{BC}^2)-\mathrm{AB}^2},$$
$$\mathrm{C}b=\tfrac{1}{3}\sqrt{2\,(\mathrm{AC}^2+\mathrm{CD}^2)-\mathrm{AD}^2},$$
$$\mathrm{D}b=\tfrac{1}{3}\sqrt{2\,(\mathrm{AD}^2+\mathrm{CD}^2)-\mathrm{AC}^2},$$
$$\mathrm{D}c=\tfrac{1}{3}\sqrt{2\,(\mathrm{AD}^2+\mathrm{DE}^2)-\mathrm{AE}^2},$$

or, by actually substituting the numerical values of the several lines in these equations, we obtain as follows:

$$\begin{aligned}
\text{A}a &= \tfrac{1}{3}\sqrt{2(625+1444)-289} &&= 20\cdot 68 \text{ feet}\\
\text{A}b &= \tfrac{1}{3}\sqrt{2(1444+1600)-529} &&= 24\cdot 85 \text{ —,}\\
\text{AC} &= \tfrac{1}{3}\sqrt{2(1600+256)-1296} &&= 16\cdot 38 \text{ —,}\\
\text{C}a &= \tfrac{1}{3}\sqrt{2(1444+289)-\ 625} &&= 17.77 \text{ —,}\\
\text{C}b &= \tfrac{1}{3}\sqrt{2(1444+529)-1600} &&= 16\cdot 14 \text{ —,}\\
\text{D}b &= \tfrac{1}{3}\sqrt{2(1600+529)-1444} &&= 17\cdot 68 \text{ —,}\\
\text{DC} &= \tfrac{1}{3}\sqrt{2(1600+1296)-256} &&= 24\cdot 80 \text{ —.}
\end{aligned}$$

Consequently, from the numerical values of these several lines as we have determined them above, the values of the angles $a\text{A}b$, $b\text{AC}$ and $a\text{AC}$, and of course, the lines ab, bc and ac can easily be found. The several angles concerned in the computation, will have their values determined by the reduction of the following class of formulæ, which for the sake of brevity, have been so modified as to bring out the results in terms of the versed sines; and in order that the expressions may be arranged in one line, we shall lay aside the fractional form, and adopt the conventional character which indicates the operation of division; hence we shall have

$$\begin{aligned}
\text{vers. } a\text{AC} &= \{ac^2-(\text{AC}-a\text{A})^2\} \div 2\text{AC}.a\text{A},\\
\text{vers. } b\text{AC} &= \{bc^2-(\text{AC}-b\text{A})^2\} \div 2\text{AC}.b\text{A},\\
\text{vers. } b\text{AD} &= \{b\text{D}^2-(\text{AD}-b\text{A})^2\} \div 2\text{AD}.b\text{A},\\
\text{vers. } c\text{AD} &= \{c\text{D}^2-(\text{AD}-c\text{A})^2\} \div 2\text{AD}.c\text{A}.
\end{aligned}$$

These formulæ are beautifully adapted for logarithmic computation, and the method of so applying them has been sufficiently exemplified in the solution of the preceding example; we shall not therefore, in this case, continue the logarithmic process, but simply indicate the steps of reduction, by substituting the numerical values of the several lines, according as they are combined in the four foregoing equations.

$$\begin{aligned}
&\{17.77^2-(38-20.68)^2\} \div 2\times 38\times 20.68 = .01005 = \text{nat. vers. } 8^\circ\ 8',\\
&\{16.14^2-(38-24.85)^2\} \div 2\times 38\times 24.85 = .04637 = \text{nat. vers. } 17\ 31,\\
&\{17.68^2-(40-24.85)^2\} \div 2\times 40\times 24.85 = .04178 = \text{nat. vers. } 16\ 37,\\
&\{24.80^2-(40-16.38)^2\} \div 2\times 40\times 16.38 = .04360 = \text{nat. vers. } 14\ 54.
\end{aligned}$$

Hence then, the angles $a\text{A}b$, $b\text{AC}$, and $a\text{AC}$, are found to be respectively as follows.

$$\begin{aligned}
a\text{A}b &= 8^\circ\ 8'+17^\circ\ 31' = 25^\circ\ 39',\\
b\text{AC} &= 16\ \ 37+14\ \ 54 = 31\ \ 31,\\
a\text{AC} &= 25\ \ 39+31\ \ 31 = 57\ \ 10.
\end{aligned}$$

Consequently, the three sides ab, bc and ac, have their values expressed by the three following equations, viz.

$$ab=\sqrt{a\text{A}^2+b\text{A}^2-2a\text{A}\cdot b\text{A}\cos.\,a\text{A}b},$$
$$bc=\sqrt{b\text{A}^2+c\text{A}^2-2b\text{A}\cdot c\text{A}\cos.\,b\text{A}c},$$
$$ac=\sqrt{a\text{A}^2+c\text{A}^2-2a\text{A}\cdot c\text{A}\cos.\,a\text{A}c}.$$

Or, by substituting the numerical values of the sides and angles, according to the combinations in the above formulæ, we obtain

$$ab=\sqrt{20.68^2+24.85^2-2\times 20.68\times 24.85\times\cos.\,25°39'}=10.89 \text{ feet},$$
$$bc=\sqrt{24.85^2+16.38^2-2\times 24.85\times 16.38\times\cos.\,31°31'}=13.86 \text{ ——},$$
$$ac=\sqrt{20.68^2+16.38^2-2\times 20.68\times 16.38\times\cos.\,57°10'}=18.13 \text{ ——}.$$

Therefore, in the triangle abc, we have given, the three sides ab, bc and ac, respectively equal to $10\cdot 89$, $13\cdot 86$ and $18\cdot 13$ feet; to find the angle abc, the versed sine of which is expressed by the following theorem, viz.

$$\text{vers. } abc=\{ac^2-(bc-ab)^2\}\div 2bc.ab.$$

Or, by substituting the numerical values of the three sides, we shall have

$\{18.13^2-(13.86-10.89)^2\}\div 2\times 13.86\times 10.89=1.05964=$ nat. ver. $95°\ 25'$.

Now, by our fundamental principle stated in the first proposition, the line ab is divided in the point G, into two parts, that are to each other, reciprocally as the masses or magnitudes of the bodies applied at a and b; but these masses in the present instance are represented by the areas of the triangles ABC and ACD, which are supposed to be concentrated in the points a and b; therefore, the line ab, is divided in the point G, into two parts, that are to each other reciprocally as the areas of the triangles ABC and ACD, that is,

$$b\text{G} : a\text{G} :: \text{area } \Delta\,\text{ABC} : \text{area } \Delta\,\text{ACD}.$$

Put $s=$ the semiperimeter of the triangle ABC,
and $s'=$ ———————————————— ACD;

then the writers on mensuration have shewn, that the areas of the triangles ABC and ACD, are represented by the following equations, viz.

$$\text{area } \Delta\,\text{ABC}=\sqrt{s\cdot(s-\text{AB})\,(s-\text{BC})\,(s-\text{AC})},$$
$$\text{area } \Delta\,\text{ACD}=\sqrt{s'.(s'-\text{AC})\,(s'-\text{CD})\,(s'-\text{AD})};$$

$$\text{but } s=\tfrac{1}{2}(25+17+38)=40\ ,$$
$$s'=\tfrac{1}{2}(38+23+40)=50.5;$$

therefore, by substituting the numerical values of the sides, in the foregoing expressions for the areas, we obtain

area Δ ABC $=\sqrt{40 \times (40-25)\times(40-17)\times(40-38)}=166.132$ [square feet,

area Δ ACD $=\sqrt{50.5\times(50.5-38)\times(50.5-23)\times(50.5-40)}=426.934$ [square feet.

consequently, by compounding the above analogy, we shall have

area Δ ABC + area Δ ACD : bG + aG :: area Δ ABC : bG ;

that is, $166\cdot132+426\cdot934 : 10\cdot89 :: 166.132 : 3.05$ feet.

Then, in the triangle cbG, we have given the sides cb and bG, equal respectively to 13.86 and 3·05 feet, and the contained angle cbG equal to 93° 25′; to find the side cG opposite to the given angle.

Now, it has been shown in a former part of this operation, that with these data the expression for the side opposite to the given angle, is

$$cG=\sqrt{bc^2+bG^2-2bc.bG\cos.cbG};$$

or, by substituting the numerical values, it becomes

$$cG=\sqrt{13\cdot86^2+3\cdot05^2+2\times13\cdot86\times3.05\times\cos.93^\circ 25'}=14.37 \text{ feet.}$$

But the point H, which is the common centre of gravity of the trapezium ABCD, and the triangle ADE, divides the line cG into two parts, which are to each other reciprocally as the areas of these two figures; now, the area of the trapezium ABCD is $166.132+426.934 =593.066$ square feet, and if s'' represent the semiperimeter of the triangle ADE; then, as we have shown above, its area is expressed as follows, viz.

$$\text{areas } \Delta \text{ ADE}=\sqrt{s''\times(s''-\text{AD})\ (s''-\text{DE})\ (s''-\text{AE})};$$

which equation, by substituting the numerical values of the sides, becomes

$$\text{areas } \Delta \text{ ADE}=\sqrt{46\times(46-46)\ (46-36)\ (46-16)}=287.75 \text{ sq. feet;}$$

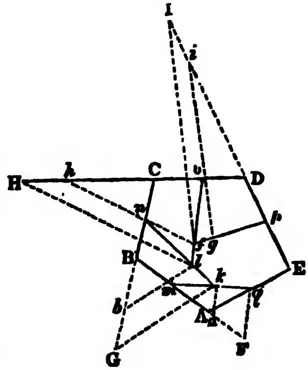
then $593.066+287.75 : 14.37 :: 287\cdot75 :$ GH $=4\cdot49$ feet;

therefore cH $=c$G $-$ GH $=14.37-4.49=9.88$ feet.

45. Such is the method of calculating generally, the position of the centre of gravity of any irregular five sided figure or polygon; but there are certain particular cases, which it is unnecessary here to specify, where the sides and angles are so related to one another, that the computation becomes greatly simplified; and should the figure happen to be regular, the position of the centre of gravity, and that of the centre of magnitude would be the same; in which case, either of them is assignable by the perpendicular bisection of any two sides of the figure, or by the bisection of any two of its angles.

And if we conceive the whole surface of the polygon to be uniformly concentrated in its boundary or perimeter, then the centre of gravity of such boundary may be determined in the following manner, viz.

Let ABCDE be an irregular five sided figure or polygon, and let it be proposed to determine the centre of gravity, on the supposition that the whole surface is uniformly collected in the boundary ABCDE.



Bisect the sides AB, BC, CD, DE and AE in the points m, n, o, p and q; then shall the points of bisection m, n, o, p and q be respectively the centres of gravity of the sides AB, BC, CD, DE and AE: join mq, and conceive the lines AB and AE to be wholly accumulated in the points m and q; then must the common centre of gravity of the lines AB and AE exist in the line mq.

Produce mA to F, making ma and aF respectively proportional to the lines AE and AB, or equal to their halves Aq and Am; join Fq, and through the point a draw ak parallel to Fq, meeting mq in the point k, then shall k be the common centre of gravity of the lines AB and AE. Join nk, and imagine the lines AB and AE to be collected in the point k, which is, as we have shown above, their common centre of gravity; and let the line BC be supposed to produce its effect at the point n; then must the common centre of gravity of the three lines AB, AE and BC occur in the line nk.

Produce nB to G, making nb and bG respectively equal mF and Bn; join Gk, and through the point b draw bl parallel to Gk, meeting nk in the point l; then shall the point l be the common centre of gravity of the three lines AB, BC and AE. Join ol, and conceive the three lines AB, BC and AE, to be wholly collected in the point l, while the line CD is supposed to produce its effect at the point o; then must the common centre of gravity of the four lines AE, AB, BC and CD, occur in the line ol.

Produce oC to H, making oh and hH equal respectively, to nG and oC; join Hl, and through the point h, draw hf parallel to Hl, meeting ol in the point f; then shall the point f be the common centre of gravity of the four lines AE, AB, BC and CD. Join pf, and imagine the four lines AE, AB, BC and CD to be wholly accumulated in the point f, while the line DE is conceived to produce its effect at the point p; then must the common centre of gravity of the five lines AE, AB, BC, CD and DE, or that of the whole perimeter ABCDE, occur in the line pf.

Produce pD to I, making pi and iI respectively equal to oH and pD; join If, and through the point i draw ig parallel to If, meeting pf in the point g; then shall the point g be the common

centre of gravity of the five lines AE, AB, BC, CD and DE, which constitute the boundary of the figure ABCDE.

Note. The division of the distance between any two points or centres, in the reciprocal ratio of the body or mass supposed to be collected at each point, the reader will perceive, has been very extensively employed throughout the whole of the foregoing theory, it constituted our first principle, and, with the exception of the plane triangle, has acted a part more or less conspicuous in every succeeding example. It is founded on the general property of the centre of gravity of a body, or system of bodies, viz.

That the sum of the products of each body into its respective distance from any plane, is equal to the product of the whole system into its distance from the same plane.

We shall not pursue this department of the subject any further, the principle of construction being so obvious, that readers of the most humble capacity may, without difficulty, extend its application to figures of any number of sides; and in the present instance, the numerical calculation, in addition to the examples which we have already exhibited, would extend the subject so far, and give our pages such a complicated appearance, that we prefer dispensing with it altogether, and proceed in the next place to determine the centre of gravity of the triangular pyramid.

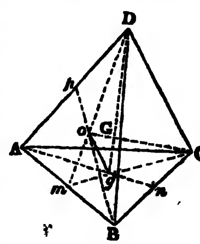
Section Eighth.

OF THE CENTRE OF GRAVITY OF PYRAMIDAL FIGURES.

46. Problem. *To determine the position of the centre of gravity of any triangular pyramid.*

Let the figure ABCD represent the triangular pyramid, the position of whose centre of gravity is required to be ascertained.

Bisect the two edges AB and BC in the points m and n; draw An and Cm, cutting one another in the point g; then is g the position of the centre of gravity of the triangular face ABC; join Dg, then because the pyramid ABCD is composed of triangles, similar and parallel to the triangle ABC, it is manifest that the line Dg must intersect all those triangles in a similar manner, and since it meets the centre of gravity of the triangle ABC, which is the lowermost of the series, it must pass through the centre of gravity of all that are above it; consequently the common centre of gravity of all the triangles composing the



pyramid, or the centre of gravity of the pyramid itself, occurs in the line Dg.

Again, bisect the edge AD in the point p; draw Bp and Dm, cutting each other in the point o; then is o the position of the centre of gravity of the triangular face ADB; join Co, then because the pyramid ABCD, is composed of triangles similar and parallel to the triangle ADB, it is obvious that the line Co must intersect all those triangles in a similar manner; and since it meets the centre of gravity of the triangle ADB, which is the most remote in the series, it must pass through the centre of gravity of all those lying between it and the point D; consequently, the common centre of gravity of all the triangles composing the pyramid, or the centre of gravity of the pyramid itself, must occur in the line Co; and we have also shown that it occurs in the line Dg; but the two lines Dg and Co being both in the plane of the triangle CmD, must intersect one another; consequently, their point of intersection G must necessarily be the place of the centre of gravity of the pyramid ABCD.

Now, because the points g and o are the centres of gravity of the triangles ABC and ADB; it follows that mg is equal to one third of the line mC, and mo equal to one third of mD; therefore, we have

$$mg \quad mC \quad mo : mD :$$

hence, if og be drawn, the lines og and DC shall be parallel to one another, and consequently, the triangles oGg and DGC are similar; hence we have

$$oG : GC \quad og : DC,$$
$$gG : GD \quad og : DC,$$

but og : DC 1 : 3; wherefore, we get

$$oG = \tfrac{1}{3}GC, \text{ or } oG = \tfrac{1}{4}oC,$$
$$\text{and } gG = \tfrac{1}{3}GD, \text{ or } gG = \tfrac{1}{4}gD.$$

Consequently, by subtraction we obtain $GC = \frac{3}{4}oC$, and $GD = \frac{3}{4}gD$; therefore, if the same construction were effected with regard to the other two triangular faces of the pyramid, the same property or law would obtain; hence we infer generally that

In any triangular pyramid, the distance of its centre of gravity from any one of its angles, is three fourths of the straight line connecting that angle with the centre of gravity of its opposite triangular face.

47. In our investigations with regard to the plane triangle, we denoted the three sides AB, BC and AC respectively by the letters d, δ and δ'; we shall therefore, in addition to the notation there employed, denote the other three edges of the pyramid, viz. AD, BD and CD respectively, by the three letters a, b and

Put φ equal to the angle DmC, a known or assignable angle; then, by Plane Trigonometry, we have

$$gD=\sqrt{mD^2+mg^2-2mD.mg\ \cos.\ \varphi};$$

but by the foregoing inference, we get

$$GD=\tfrac{3}{4}gD\text{ ; that is}$$

$$GD=\tfrac{3}{4}\sqrt{mD^2+mg^2-2mD.mg\ \cos.\ \varphi}.$$

Now, in the discussion respecting the plane triangle, it was proved that

$$mD^2=\frac{2AD^2+2BD^2-AB^2}{4},\text{ and}$$

$$mC^2=\frac{2AC^2+2BC^2-AB^2}{4}$$

consequently, by evolution we get

$$mD=\tfrac{1}{2}\sqrt{2AD^2+2BD^2-AB^2},\text{ and}$$

$$mC=\tfrac{1}{2}\sqrt{2AC^2+2BC^2-AB^2};$$

but it has been shown above,

$$\text{that } mg=\tfrac{1}{3}mC\text{; that is,}$$

$$mg=\tfrac{1}{6}\sqrt{2AC^2+2BC^2-AB^2},$$

$$\text{and } mg^2=\frac{2AC^2+2BC^2-AB^2}{36}.$$

Moreover, by Plane Trigonometry we get

$$\cos.\ \varphi=\frac{mD^2+mC^2-DC^2}{2mD.mC};$$

or by substituting the above values of mD^2, mA^2, mD and mC, in this last expression, we obtain

$$\cos.\ \varphi=\frac{AD^2+BD^2+AC^2+BC^2-AB^2-2CD^2}{\sqrt{2AD^2+2BD^2-AB^2}\times\sqrt{2AC^2+2BC^2-AB^2}}.$$

Consequently, by adopting the literal representatives of the edges of the pyramid, and substituting in the expression for GD, the values of mD^2, mg^2, mD, mg and cos. φ; we shall finally obtain

$$GD=\tfrac{1}{4}\sqrt{3(a^2+b^2+c^2)-(d^2+\delta^2+\delta'^2)}$$

The property implied by this equation is, that when all the edges of a triangular pyramid are given, or assignable by calculation.

The distance of the vertex of the pyramid from the centre of gravity, is equal to one fourth of the square root of the difference that arises, by substracting the sum of the squares of the three sides of the base, from the three times the sum of the squares of the edges meeting in the vertex.

Consequently, if each angle of the pyramid be severally considered as the vertex, and AG and BG be drawn; then the preceding property will afford the four following equations, viz.

When the vertex is at A, we get,

$$\left.\begin{array}{l}1.\ \text{AG}=\tfrac{1}{4}\sqrt{3(a^2+d^2\times\delta'^2)-(b^2+c^2+\delta^2)}.\\ \quad\text{When the vertex is at B, we get,}\\ 2.\ \text{BG}=\tfrac{1}{4}\sqrt{3(b^2+d^2+\delta^2)-(a^2+c^2+\delta'^2)}.\\ \quad\text{When the vertex is at C, we get,}\\ 3.\ \text{CG}=\tfrac{1}{4}\sqrt{3(c^2+\delta^2+\delta'^2)-(a^2+b^2+d^2)}.\\ \quad\text{When the vertex is at D, we get}\\ 4.\ \text{DG}=\tfrac{1}{4}\sqrt{3(a^2+b^2+c)-(d^2+\delta^2+\delta'^2)}.\end{array}\right\}\quad(\text{A})$$

Hence, it is evident, that from these four equations, the distance of the centre of gravity of any triangular pyramid, from any one of its angles, can easily be found, in terms of its six edges, and we may here observe that, in proposing examples for illustration, we are not at liberty to assume the magnitudes of the six edges of the figure at random; for it is obvious that one or other of them must have its magnitude dependent on the other five, and the inclination of the planes in which they are situated; consequently, in order that a solution may be possible, the data must be so related, that the six edges of the pyramid can always be ascertained.

48. The following numerical examples will suffice to exemplify the importance of the foregoing remark.

EXAMPLE 1. In the triangular pyramid ABCD, there are given the five edges AB, BC, AC, AD and CD, equal, respectively, to 18, 16, 22, 26 and 30 inches, and the angle in the plane DAB, which is subtended by the side DB, the unknown edge of the pyramid is 78° 30′; what is the distance between the centre of gravity of the figure and the angle C?

Here then, in the first place, we have to determine the line BD which subtends the given angle DAB, and joins the extremities of the given sides AB and AD; for which purpose,

Find φ, an angle such, that

$$\tan.\ \varphi=\frac{2\sin.\ \tfrac{1}{2}\text{DAB}\sqrt{\text{AD}\cdot\text{AB}}}{\text{AD}-\text{AB}};\ \text{that is,}$$

$$\tan.\ \varphi=\frac{2\times\cdot 63271\times\sqrt{26\times 18}}{26-18}=3\cdot 42188=\text{nat. tan. } 73^\circ\ 42'\ 35''$$

Then, having found the value of the angle φ, the value of the side BD, is found by the following very concise and simple theorem viz.

$$BD=(AD-AB)\ \sec.\ \varphi.$$

Now, the natural secant of φ, or the natural secant of 73° 42′ 35″ is 3·565; therefore, by substituting the numerical values of AD, AB and sec. φ, the foregoing theorem gives

$$BD=(26-18)\times 3{\cdot}565=28{\cdot}52 \text{ inches.}$$

Having thus determined the magnitude of the side BD, the distance CG will be found from No. 3 of the preceding equations, in the following manner:

$$\left.\begin{array}{l} c^2=30\times 30=900 \\ \delta^2=16\times 16=256 \\ \delta'^2=22\times 22=484 \end{array}\right\}=1640, \text{ sum of the three squares,}$$

3, multiply

4920, three times the sum of the squares.

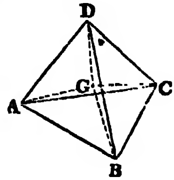
$$\left.\begin{array}{l} a^2=26\times 26=676 \\ b^2=28{\cdot}52\times 28{\cdot}52=813{\cdot}3904 \\ d^2=18\times 18 \quad =324 \end{array}\right\}=1813{\cdot}3904, \text{ sum of the three squares;}$$

consequently, by subtraction, evolution and division, we obtain for the required distance,

$$CG=\tfrac{1}{4}\sqrt{4920-1813{\cdot}3904}=13{\cdot}93 \text{ inches.}$$

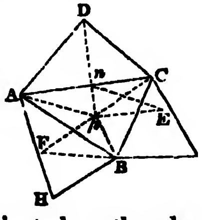
It is further manifest, that if the distances AG, BG and DG had been required by the question, they would have been found from the equations 1, 2 and 4, exactly in the same manner as we have determined the distance CG above.

EXAMPLE 2. In the triangular pyramid ABCD, there are given, the three edges AB, AC and AD, equal respectively, to 50, 58 and 44 feet, and the contained angles BAC and DAC, equal respectively, to 38 and 42 degrees; and, moreover, the angle which the face of the pyramid DAC makes with the base BAC, is 62° 44′; what are the distances of G, the centre of gravity of the figure, from the angles A, B, C and D?



Here follow the projection and development of the pyramid ABCD, on the plane of the base ABC.

Draw the lines AB, AC and AD, respectively equal to 50, 58 and 44 feet, and in such a manner, that the angles BAC and DAC, shall be respectively equal to 38 and 42 degrees; from the point D on AC, let fall the perpendicular Dn, and at the point n make the angle CnE equal to the complement of 62° 44′; make nE equal to nD, and from the E draw Ep, meeting Dn produced perpendicularly in the point p; join Ap, Bp and Cp, then shall the figure ABCp represent the pyramid ABCD, projected on the plane



of the base ABC, with the consequent diminution of the projected lines Ap, Bp and Cp, and the projected angles pAB, pAC, pBA, pCA, pBC and pCB; all of which are obviously diminished in the projection, while the angles ApB, ApC and BpC, situated round p, the place of the projected vertex, are manifestly increased, their sum in the projection being equal to a whole circumference, or 360 degrees.

At the point p, in the projected line Bp, erect the perpendicular pF, make pF equal to pE and join FB; then at the points A and B, in the line AB, with the distances AD and BF, describe arcs, cutting one another in the point H; join AH and BH, then shall the angle BAH, be the other plane angle, which, with the given angles BAC and DAC, constitutes the solid angle at the point A, and the triangle BAH is the third face of the pyramid, corresponding to the plane BAD, or to its projection BAp.

Again, at the points B and C in the line BC, with the distances BH and CD, describe arcs cutting one another in the point I; join BI and CI, then shall the triangle BCI be the fourth face of the pyramid, corresponding to the plane BDC, or to its projection BCp.

Hence, the figure ADCIBH, is the developement of the pyramid ABCD on the plane of the base, where all the parts of the figure, both lines and angles, are delineated in their proper dimensions, and, consequently, the three unknown edges BC, BD and DC, of the pyramid ABCD, which correspond to the three lines BH, BC and CD in the developement, are determinable by measurement to a very great degree of accuracy.

The use of the foregoing projection and developement is to guide the operator in the numerical determination of the edge BD in the pyramid, which corresponds to the line BF or BH in the developed figure. The two edges BC and DC, by reason of their connexion with the given parts, can be found directly without any subsidiary calculation, but the edge BD being differently related to the parts which are known, requires a different process to assign its value.

If we conceive the point A to be placed at the centre of a sphere, the three planes BAC, DAC and BAD, which constitute the solid angle at the point A, would, if produced, intersect the spheric surface, forming thereon a spherical triangle, whose sides would be the correct measures of the plane angles BAC, DAC and BAD, and whose angles would measure the inclinations of the planes which form the solid angle at the centre of the sphere.

By this method of considering the subject, we have first to determine the angle BAD, which corresponds to BAH in the developement, and from the angle BAD or BAH thus determined, the edge BD or side BH can be found, exactly in the same manner as the edges BC and DC.

Now, from the manner in which the data are proposed, it is obvious that the determination of the angle BAD requires the solution of that case of a spherical triangle, where

two sides and the contained angle are given, to find the other side.

If the three sides of the spherical triangle, or, which is the same thing, the three angles of the pyramid BAC, DAC and BAD, be respectively represented by the letters x, y and z, while the inclination of the planes BAC and DAC, or the angle contained by the sides x and y, is represented by the letter z; then the writers on Spherical Trigonometry have shown, that

$$\cos. z = \cos. x \cos. y + \sin. x \sin. y \cos. z.$$

In which equation z is the required angle, and x, y and z, those which are given in the question.

But we propose to show that the side or edge BD, or its equivalent BF or BH, is determinable from the foregoing projection and developement of the pyramid, without having recourse directly to spherical considerations, and the method of determining it is as follows.

Consider the line AD as the radius of a circle, which, for the sake of convenience, put equal to unity, then is nD$=n$E, the sine of the angle DAC$=y$, to that radius, and An its cosine. But by the construction, the angle pnE$=z$, is the angle of inclination of the two planes BAC and DAC; therefore, by Plane Trigonometry, we have

$$p\text{E} = p\text{F} = \sin. y \sin z,$$

$$\text{and } np = \sin. y \cos. z;$$

$$\text{now } np \div \text{A}n = \tan. n\text{A}p = \tan. \varphi; \text{ that is,}$$

$$\tan. \varphi = \tan. y \cos. z.$$

But the angle BA$p=(x-\varphi)$, and A$p^2=$An^2+np^2; that is

$$\text{A}p^2 = \cos^2. y + \sin^2. y \cos^2. z,$$

or because $\cos^2. z = 1 - \sin^2. z$, we get

$$\text{A}p^2 = 1 - \sin^2. y \sin^2. z;$$

consequently, by evolution, we have

$$\text{A}p = \sqrt{1 - \sin^2. y \sin^2. z}.$$

Then, in the triangle BAp, there are given, the sides AB and Ap, respectively equal to d and $\sqrt{1-\sin^2. y \sin^2. z}$, and the included BA$p$ equal to $(x-\varphi)$, to find the square of the side Bp, which by Plane Trigonometry, is

$$\text{B}p^2 = \text{AB}^2 + \text{A}p^2 - 2\text{AB} . \text{A}p \cos. (x-\varphi);$$

that is,

$$\text{B}p^2 = d\{d - 2\cos. (x-\varphi)\sqrt{1-\sin^2. y \sin^2. z}\} + 1 - \sin^2. y \sin^2. z,$$

to which add the square of pF, viz. $\sin^2. y \sin^2. z$, and we get

$$\text{BF}^2 = \text{BD}^2 = d\{d - 2\cos. (x-\varphi)\sqrt{1-\sin^2. y \sin^2. z}\} + 1;$$

hence, by evolution, we have

$$\text{BF} = b = \sqrt{d\{d - 2\cos. (x-\varphi)\sqrt{1-\sin^2. y \sin^2. z}\} + 1}.$$

In the solution of the question, however, this last step is unnecessary, for by referring to the theorems class (A), it will be seen that the squares of the several edges of the figure enter the expression, and not the first power, as the above equation would imply, consequently, the step immediately preceding that marked (B) is what must be employed in the present instance, but the ultimate form may also frequently be of use both in this and other inquiries, for which reason we have thought proper to exhibit it. The squares of the other two unknown edges are expressed as below, viz.

$$c^2 = a^2 + \delta'^2 - 2\,a\delta' \cos. y,$$
$$\delta^2 = d^2 + \delta'^2 - 2\,d\delta' \cos. x,$$

and if we employ the theorem for the value of cos. z, formerly alluded to, we shall obtain

$$b^2 = a^2 + d^2 - 2\,ad\,(\cos. x \cos. y + \sin. x \sin. y \cos. z).$$

For practical purposes, this last value of b^2 is preferable to that derived from the developed figure, as it precludes the necessity of finding the subsidiary angle φ. We shall therefore employ this expression, and proceed forthwith to the resolution of the question.

To find the value of b^2, we have

$x = 38°$	log. cos. 9·896532,	. . . .	log. sin.	9·789342
$y = 42$	log. cos. 9·871073,	. . . .	log. sin.	9·825511
nat. num. 0·58560	log. 9·767605,	$z = 62° 44'$	log. cos.	9·660991
0·18873		nat. num. 0·18873	log.	9·275844

$$-0{\cdot}77433 \times 2 \times 44 \times 50 = -3407.052$$
$$\left.\begin{array}{r} a^2 = 44 \times 44 = +1936 \\ d^2 = 50 \times 50 = +2500 \end{array}\right\}\text{ add.}$$
hence we have $b^2 = +1029{\cdot}948.$

To find the value of c^2 we have

$$\left.\begin{array}{r} -0{\cdot}74314 \times 2 \times 44 \times 58 = -3792.986 \\ a^2 = 44 \times 44 = +1936 \\ \delta'^2 = 58 \times 58 = +3364 \end{array}\right\}\text{ add.}$$
hence we have $c^2 = +1507{\cdot}014,$

To find the value of δ^2, we have

$$\left.\begin{array}{r} -0{\cdot}78801 \times 2 \times 50 \times 58 = -4570.458 \\ d^2 = 50 \times 50 = +2500 \\ \delta'^2 = 58 \times 58 = +3364 \end{array}\right\}\text{ add.}$$
hence we have $\delta^2 = +1293.542.$

Having thus determined the squares of the three edges BD, CD and BC, we have now to substitute them, together with the squares of AB, AC and AD, according as their literal representatives are combined in the equations class (A), and we shall have for the required distances

$$AG=\tfrac{1}{4}\sqrt{3(1936+2500+3364)-(1029{\cdot}948+1507{\cdot}014+1293{\cdot}542)}=34{\cdot}97 \text{ feet},$$
$$BG=\tfrac{1}{4}\sqrt{3(1029{\cdot}948+2500+1293{\cdot}542)-(1936+1507{\cdot}014+3364)}=33{\cdot}23 \text{ feet},$$
$$CG=\tfrac{1}{4}\sqrt{3(1507{\cdot}014+1293{\cdot}542+3364)-(1936+1029.948+2500)}=28{\cdot}34 \text{ feet},$$
$$DG=\tfrac{1}{4}\sqrt{3(1936+1029{\cdot}948+1507{\cdot}014)-(2500+1293{\cdot}542+3364)}=19{\cdot}78 \text{ feet}.$$

Other examples, involving different combinations of data, and requiring different modes of solution, might easily be proposed, but our limits forbid us to expatiate further on this case.

49. If the four equations in class (A) be squared and added together, we shall obtain the following expression:

$$AG^2+BG^2+CG^2+DG^2=\tfrac{1}{4}(a^2+b^2+c^2+d^2+\delta^2+\delta'^2), \text{ that is}$$

In any triangular pyramid, the sum of the squares of its six edges is equal to four times the sum of the squares of the distances of the centre of gravity from each angle of the figure.

This property of the triangular pyramid is evidently similar to that which we deduced for the plane triangle, but as we there observed, it is of very little practical utility, and although it is exceedingly curious, it does not seem to merit a particular illustration as being applicable to any useful purpose.

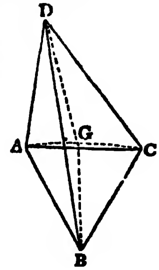
If the sides of the base AB, BC and AC are equal, then the equations class (A) will give for the respective distances the following expressions, viz.

$$\left.\begin{aligned}
&1.\ AG=\tfrac{1}{4}\sqrt{3a^2+5d^2-(b^2+c^2)},\\
&2.\ BG=\tfrac{1}{4}\sqrt{3b^2+5d^2-(a^2+c^2)},\\
&3.\ CG=\tfrac{1}{4}\sqrt{3c^2+5d^2-(a^2+b^2)},\\
&4.\ DG=\tfrac{1}{4}\sqrt{3(a^2+b^2+c^2-d^2)}.
\end{aligned}\right\} \quad (B)$$

Equations analogous to these would flow from the supposition of the three sides of any of the faces being equal, but it is presumed that the class here given, which is applicable to the equilateral base, will be found sufficient for the reader's practice.

50. The following numerical example will suffice to exemplify the mode of applying these equations to practical purposes, and if the given quantities are so selected as to preclude the necessity of any subsidiary calculation, the application is sufficiently simple.

EXAMPLE. In the triangular pyramid ABCD, there are given the two edges AD and CD, equal, respectively, to 32 and 40 feet, and the angle ADB, contained between the edge AD, and the unknown edge BD, equal to 35 degrees; what is the distance of the centre of gravity from each angle of the pyramid, supposing the side of the equilateral base ABC to be 25 feet?



In the plane triangle ADB, there are given the two sides AD and AB, with the angle ADB, to find the side BD.

By Plane Trigonometry, we have

$$\text{AB} : \text{AD} :: \sin.\ \text{ADB} : \sin.\ \text{ABD};$$
$$\text{that is } 25 : 32 :: \sin.\ 35^\circ : \sin.\ 47^\circ\ 14';$$
$$\text{and again } \sin.\ 35^\circ : \sin.\ (47^\circ\ 14' + 35) :: 25 : 43 \cdot 18 = \text{BD}$$

Here then, we have given, according to our literal notation, $a=32$; $b=43 \cdot 18$; $c=40$, and $d=25$ feet; consequently, by substituting these numbers, as is indicated in the equations, class (B), we shall have the several distances expressed as follows: viz.

$$\text{AG} = \tfrac{1}{4}\sqrt{1024 \times 3 + 625 \times 5 - (1864 \cdot 5124 + 1600)} = 13 \cdot 04 \text{ feet},$$
$$\text{BG} = \tfrac{1}{4}\sqrt{1864 \cdot 5124 \times 3 + 625 \times 5 - (1024 + 1600)} = 19 \cdot 515 \text{ —},$$
$$\text{CG} = \tfrac{1}{4}\sqrt{1600 \times 3 + 625 \times 5 - (1024 + 1864 \cdot 5124)} = 17 \cdot 74 \text{ —},$$
$$\text{DG} = \tfrac{1}{4}\sqrt{3\,(1024 + 1864 \cdot 5124 + 1600 - 625)} = 28 \cdot 34 \text{ —},$$

51. If the edges of the pyramid which terminate in the vertex become equal to one another, while the sides of the base admit of all possible magnitudes; the equations, class (A), will give for the respective distances, the following expressions: viz.

$$\left.\begin{aligned} \text{AG} &= \tfrac{1}{4}\sqrt{3\,(d^2 + \delta'^2) + (a^2 - \delta^2)}, \\ \text{BG} &= \tfrac{1}{4}\sqrt{3\,(d^2 + \delta^2) + (a^2 - \delta'^2)}, \\ \text{CG} &= \tfrac{1}{4}\sqrt{3\,(\delta^2 + \delta'^2) + (a^2 - d^2)}, \\ \text{DG} &= \tfrac{1}{4}\sqrt{9a^2 - (d^2 + \delta^2 + \delta'^2)}. \end{aligned}\right\} \quad \text{(c)}$$

Equations analogous to these, would manifestly flow from the supposition of any three of the edges, terminating at the same angle being equal, but it is needless to exhibit the several classes of equations in the above class.

The following example will be sufficient for the illustration of the equations in the above class.

EXAMPLE. The sides of the base of a triangular pyramid are respectively 12, 18 and 24 feet, and the three edges terminating in the vertex are each 21 feet; what is the distance of the centre of gravity from each angle of the pyramid?

In this example there are given $d=12$; $\delta=18$; $\delta'=24$, and $a=21$ feet; let these numbers be situated for their representatives in the several equations of class (C), and the distances of the centre of gravity from the four corners of the pyramid, will be expressed as follows: viz.

$$\begin{aligned}
AG&=\tfrac{1}{4}\sqrt{3(144+576)+(441-324)} = 11\cdot93 \text{ feet},\\
BG&=\tfrac{1}{4}\sqrt{3(144+324)+(441-576)} = 8\cdot905 \text{ —},\\
CG&=\tfrac{1}{4}\sqrt{3(324+576)+(441-144)} = 13\cdot69 \text{ —},\\
DG&=\tfrac{1}{4}\sqrt{441\times9-(144+324+576)} = 13\cdot52 \text{ —}.
\end{aligned}$$

53. If the three sides of the base, and the three edges of the pyramid, that terminate in the vertex, are each equal among themselves, the equations, class (A), will give for the respective distances the following expressions:

$$\left.\begin{aligned}
AG=BG=CG&=\tfrac{1}{4}\sqrt{a^2+5d^2}\ ,\\
DG&=\tfrac{1}{4}\sqrt{3(3a^2-d^2)}.
\end{aligned}\right\} \qquad (D)$$

Where we may observe that a represents the edges terminating in the vertex, and d the side of the base.

Example. In a triangular pyramid, whose sides are equal isosceles triangles, and whose base is an equilateral plane, there are given the sides of the base, and the edges that terminate in the vertex, equal, respectively, to 36 and 56 feet; required the position of the centre of gravity with respect to the angles of the figure?

Here we have given $d=36$, and $a=56$; let these numbers be substituted in the equations, class (D), and we have

$$\begin{aligned}
AG=BG=CG&=\tfrac{1}{4}\sqrt{3136+1296\times5}=24\cdot52 \text{ feet}\\
DG&=\tfrac{1}{4}\sqrt{3(3136\times3-1296)}=22\cdot52 \text{ feet}.
\end{aligned}$$

From the results of this example, it may be inferred that, because the three equal edges that terminate in the vertex of the figure, are greater than the three equal sides of the base, the centre of gravity is nearer to the vertex than it is to the angles at the base; but if the sides of the base were greater than the edges, terminating in the vertex, the contrary would be the case.

54. If all the six edges of the figure are equal to one another, the equation, expressing the distance of the centre of gravity from each of its angles, becomes

$$AG=BG=CG=DG=\tfrac{a}{4}\sqrt{6}; \qquad (E)$$

where a represents the edge of the figure, which in this case is the regular tetraëdron; for which the centre of gravity and the centre of magnitude are the same.

Example. The edge of a tetraëdron is 32 inches; at what distance from each of its angles is the point where its centre of gravity occurs?

Here, from the equation, we get

$$AG = 8\sqrt{6} = 19 \cdot 6 \text{ inches nearly.}$$

55. The problem which we have just resolved, may be considered as a very important one, the principle unfolded in its solution, being applicable to the determination of the centre of gravity of any pyramid whatever; for a pyramid whose base is any polygon,

Will have its centre of gravity in the line drawn from the vertex to the centre of gravity of the base, and at the distance of three fourths of its length from the vertex.

56. If the base of the pyramid be a regular polygon, and the axis of the figure perpendicular to the centre of magnitude, then

The centre of gravity is in the axis of the figure, at three fourths of its length, distant from the vertex; and the same thing holds for the centre of gravity of a right cone.

57. And if the axis of the pyramid, or cone, be oblique to the plane of the base, the general principle will still obtain; that is, the straight line which joins the vertex with the centre of gravity of the base, will pass through the centre of the figure, but in this case the numerical calculation will, on account of the obliquity, be a little more intricate.

NOTE. The cases which we have hitherto considered, have all been resolved by the application of the very simplest principles, of elementary geometry, and the algebraical formulæ arising from those principles, so far as they have been employed in the calculations, have in no instance involved equations exceeding the second degree; which circumstance, more than any other, has precluded the necessity of applying to the higher analysis for the solution of any one example that has been proposed in elucidation of our subject.

In the problems which follow, however, the figures being more or less bounded by curve lines, have usually been resolved by the principles appertaining to the fluxional calculus, or by some other method analogous to it; but since it does not accord with our plan, to treat the subject in such an abstruse and refined manner, we shall deem it necessary in continuing the centre of gravity to adopt the results obtained by other writers, and exemplify their application to practice, by selecting such examples as the nature of the subject may seem to require.

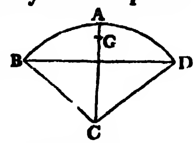
The first problem that suggests itself according to our plan thus modified, is that in which we are required to find the centre of gravity of a circular arc.

SECTION NINTH.

OF THE CENTRE OE GRAVITY OF CURVILINEAR FIGURES.

58. PROBLEM. *To determine the centre of gravity of a circular arc.*

Let BAD be the circular arc, whose centre of gravity G is required to be found.



Bisect the arc BAD in the point A by the straight line AC passing through C the centre of the circle; then, because the particles that compose the arc BAD are symmetrically disposed with respect to the line AC; it follows from our opening scholium, that the straight line AC passes through the centre of gravity of the arc BAD, and the writers on mechanics have shown, that if

a=BAD, the length of the arc in parts of the radius,
c=BD, the chord of the arc,
r=CA, CB or CD, the radius of the circle, and
δ=CG, the distance between C, the centre of the circle, and G the centre of gravity of the arc BAD; then

$$a\delta=cr; \qquad \text{(F)}$$

which equation being reduced to an analogy by the rules of algebra, becomes

$$a : r :: c : \delta;$$

and this analogy evidently implies, that

The distance of the centre of gravity of a circular arc from the centre of the circle is a fourth proportional to the length of the arc, the radius of the circle, and the chord of the arc.

Take the following examples for illustration.

EXAMPLE 1. The radius of a circle is 56 inches, and the chord of an arc cut off from the circle is 42 inches; at what distance from the centre of the circle is the centre of gravity of the arc cut off?

Here we have half the chord of the arc equal to 21 inches; consequently, by Plane Trigonometry, we have

$$56 : 21 :: \text{rad} : 0{\cdot}375 = \text{nat. sin. } 22^\circ\, 1'\, 28'';$$

hence the number of degrees in the arc BAD$=22^\circ\ 1'\ 28''\times 2=44^\circ\, 2'\, 56''$; then, by a table of circular arcs, we get

44° 0′ 0″	length of the arc to radius (1)	0·7679449
2 0	————————	5818
56	————————	2715
44 2 56	————————	0·7687982;

therefore, the length of the arc to radius 56 is

$$0\cdot7687982 \times 56 = 43\cdot0527 \text{ inches very nearly;}$$

consequently, by the preceding analogy, we obtain

$$43\cdot0527 : 56 :: 42 : 54\cdot64 \text{ inches nearly.}$$

Hence, the position of the centre of gravity G, is in the line AC bisecting the arc in the point A, and passing through C the centre of the circle, and at the distance of 54·64 inches from the centre C.

Example 2. The chord of an arc is 102 inches, and its height or versed sine is 28 inches; at what distance from the centre of the circle is the centre of gravity of the arc?

In this example, if in addition to our former notation, we put v to denote the versed sine or height of the arc; then the writers on Mensuration have shown that

$$a = \tfrac{1}{3}\{4\sqrt{c^2+4v^2} - c\},$$

$$\text{and } r = (c^2 + 4v^2) \div 8v.$$

Let the numbers given in the example be substituted for their representatives in these equations, and we shall have

$$a = \tfrac{1}{3}\{4\sqrt{102^2 + 4 \times 28^2}) - 102\} = 121\cdot15 \text{ inches,}$$

$$r = (102^2 + 4 \times 28^2) \div 8 \times 28 = 60\cdot4464 \text{ inches.}$$

Hence, by our analogy, we obtain

$$121\cdot15 : 604464 :: 102 : 50\cdot9 \text{ inches very nearly;}$$

consequently, the centre of gravity of the arc is distant
from the centre of the circle, 50·9 inches,
from the middle or crown of the arc, 9·5464 inches,
and from the middle of the chord, 18·4536 inches.

When $a = 60°$; then $c = r$, and the equation marked (F) is transformed into

$$a\delta = r^2,$$

which being converted into an analogy, is

$$a : r :: r : \delta; \text{ that is,}$$

the radius, is a mean proportional betwixt the arc, and the distance of the centre of gravity from the centre of the circles.

When $a=90°$; then $c=r\sqrt{2}$; and the equation marked (F), is transformed into

$$ad=r\sqrt{2};$$

and because the length of a quadrantal arc, of a circle whose radius is r, is equal to 1·5708 times the radius, we obtain, by substitution

$$\delta=0\cdot9\,r \text{ very nearly.}$$

When $a=180°$; then $c=2r$, and the equation marked (F), is transformed into

$$a\delta=2\,r^2,$$

or by substituting for the arc a its value $3\cdot1416r$, we obtain

$$1\cdot5708\delta=r$$

or dividing both sides by 1·5708, we get

$$\delta=0\cdot6366\,r,$$

When $a=360°$; then $c=0$; consequently, the equation marked (F), becomes

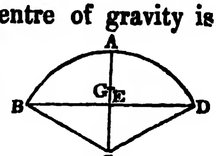
$$a\delta=0,$$

but since the product of the terms on one side of the equation becomes equal to nothing, it is evident that one or other of the terms themselves must be nothing; now the arc a cannot be the evanescent quantity, for it is, according to the supposition, equal to the whole circumference; consequently δ, the term with which it is combined, must vanish, and then, the centre of gravity coincides with the centre of the circle.

The preceding class of derived equations are too simple to require examples for illustration, we shall therefore proceed to the next problem in the order of simplicity.

60. PROBLEM. *To determine the centre of gravity of a circular segment.*

Let BADE be the circular segment, whose centre of gravity is required to be determined.



Bisect the chord BD perpendicularly in the point E by the straight line AE, which passes through C, the centre of the circle.

Then, because all lines which can be drawn in the segment BADE, parallel to the chord BD, are bisected by the line AE, these bisected lines, considered two and two, are symmetrical with respect to the line AE, and consequently, by our scholium, AE passes through the centre of gravity of the segment BADE.

Therefore, if the parts of the figure be denoted as in the case of the circular arc, the writers on Fluxions have shown, that

$$6r\delta\,(a = \sin.\, a) = c^3. \qquad \text{(H)}$$

Here however we must observe, that a and sin. a are supposed to be reduced to the proposed radius r, and not considered as corresponding to the tabular radius unity; and furthermore, the expression $a - \sin.\, a$ is to be employed when the segment is less than a semicircle, but when the segment exceeds a semicircle, the expression $a + \sin.\, a$ comes into use.

61. This being premised, we proceed to illustrate the application of the formula, by the resolution of the following appropriate examples.

Example 1. The chord of a circular segment is 224, and the radius of the circle 113 inches; what is the distance between the centre of the circle and the centre of gravity of the segment?

Here, we have half the chord of the segment equal to 112 inches, consequently, by Plane Trigonometry, we get

$$113 : 112 :: \sin.\, 90^\circ\,(1) : 0.99115 = \text{nat. sin. } 82^\circ\, 22';$$

hence, the number of degrees in the arc of the segment is

$$82^\circ\, 22' \times 2 = 164^\circ\, 44';$$

therefore, by a table of the lengths of circular arcs, we have

$$\begin{array}{lll} \text{arc } 164^\circ\, 44' & \text{to radius } 1 & = 2.8751391 \\ \text{sine } 164\ \ 44 & \text{———} & = \underline{0.2633115}, \text{ subtract,} \\ \text{hence } (a - \sin.\, a) & \text{———} & = 2.6118276; \end{array}$$

consequently, $(a - \sin.\, a)$ to radius $(r = 113)$ is $2.6118276 \times 813 = 295.1365188$;

then substituting this number for $(a - \sin.\, a)$, and 113 for r in the preceding equation, and it becomes by contracting the decimal,

$$200102.56\,\delta = c^3;$$

$$\text{Now } c^3 = 224 \times 224 \times 224 = 11239424;$$

consequently, by division, we have

$$\delta = \frac{11239424}{200102.56} = 50.62 \text{ inches very nearly,}$$

for the distance between the centre of the circle and the centre of gravity of the segment.

Example 2. The height or versed sine of a circular segment is 128, and the diameter of its base 224 inches; what is the distance between the centre of the circle and the centre of gravity of the segment?

It was shown in the solution of the second example, under equation (D), that if v represent the height or versed sine of the arc, c the chord or diameter joining its extremities, and r the radius of the circle; then

$$r=(c^2+4v^2)\div 8v;$$

let this value of r be substituted for it in the equation marked (F), and it becomes

$$3\delta(c^2+4v^2)(a\mp\sin. a)=4c^3v; \qquad (G)$$

and moreover, by Plane Trigonometry, we have

$$\frac{(c^2+4v^2)}{8v} : \frac{c}{2} :: \sin. 90^\circ (1) : \sin. \tfrac{1}{2}a=\frac{4cv}{(c^2+4v^2)};$$

consequently, by substituting the numbers proposed in the example, the expression for sin. $\frac{1}{2}a$ as found above, becomes

$$\sin. \tfrac{1}{2}a=\frac{4\times 224\times 128}{224^2+4\times 128^2}=0\cdot 99115=\text{nat. sin. } 82^\circ\ 22'.$$

In this case, however, the segment exceeds a semicircle; consequently, $\frac{1}{2}a$ exceeds 90° or a quadrant; that is $a^\circ=360^\circ-2(82^\circ\ 22')=195^\circ\ 16'$; therefore, by a table of the lengths of circular arcs we have

$$\begin{aligned} &\text{arc } 195^\circ\ 16' \text{ to radius } 1=3\cdot 4080463\\ &\text{sine } 195\ 16 \text{———————}=0\cdot 2633115 \text{ add}\\ &\text{hence } (a+\sin. a)\text{———————}=\overline{3\cdot 6713578};\end{aligned}$$

consequently, $(a+\sin. a)$ to radius r, is $3\cdot 6713578\times r$; but $r=(c^2+4v^2)\div 8v$

$$\text{that is, } r=\frac{224^2+4\times 128^2}{8\times 128}=113$$

$$\text{therefore, } 3\cdot 6713578\times 113=414\cdot 8634314,$$
$$\text{or, } (a+\sin. a)=414\cdot 8634314;$$

then, by substituting the numerical values of the letters c and v in equation (G), it becomes

$$3\times(\overline{224^2+4\times 128^2})\times 414\cdot 8634314\times\delta=4\times 224^3\times 128$$

or by actually expanding the expressions and contracting the decimal we obtain

$$144014032\cdot 12\delta=5305008128$$

consequently, by division, we get

$$\delta=\frac{5305008128}{144014032\cdot 12}=36\cdot 8 \text{ inches very nearly, for the distance}$$

between the centre of the circle, and the centre of gravity of the segment.

62. We may, however, remark that the results obtained in this way, are only approximative by reason of the rule which we have adopted for determining the segmental area; but it is presumed, that the degree of accuracy thus arrived at, will, in all practical cases, be found sufficient for the attainment of every useful purpose.

When the chord $c=2r$, the segment becomes a semicircle, in which case the arc $a=3\cdot1416r$, and sin. $a=o$; therefore, the equation marked (G) assumes the following form; viz.

$$3r\delta\{3.1416r \mp 0\}=4r^3$$

which again, by further reduction, gives

$$3\delta=1\cdot27323r$$

or dividing by 3, it is

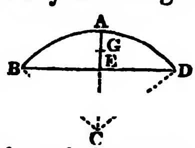
$$\delta=0\cdot42441r.$$

This case is too simple and obvious to require any illustration, we therefore pass it over and go on to another

PROBLEM. *To determine the centre of gravity of a circular sector.*

63. Let the figure BADC, be the circular sector, whose centre of gravity is proposed to be determined.

Join BD, which bisect perpendicularly in the point E by the straight line AC, cutting the arc in A and passing through C, the centre of the circle.



Then, because the parts of the figure, viz. the sectors BAC and DAC, are symmetrically disposed with respect to the line AC, it follows from our scholium that AC passes through the centre of gravity of the sector ABCD.

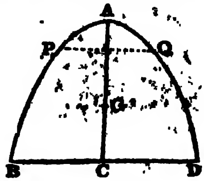
Therefore, the notation remaining, we shall have the distance of the centre of gravity of the sector from the centre of the circle, expressed by the following equation: viz.

$$3a\delta=2cr. \qquad (I)$$

The value of δ in this equation, is manifestly two thirds of its value for the circular arc; consequently, the mode of procedure in the one case will be the same as in the other, with the exception of taking two-thirds of the result; for which reason we deem it unnecessary, to propose any examples in this place for the purpose of illustration; we therefore proceed.

PROBLEM. *To determine the centre of gravity of any parabola.*

64. Let BADC, be the parabola whose centre of gravity is proposed to be determined.



Join BD, which bisect perpendicularly in the point C, by the straight line CA, passing through A, the vertex of the figure.

Then, because the parts of the figure BAC and DAC are similar by the nature of the curve, all lines drawn parallel to BD must be bisected by the line AC, and because the point C is the centre of gravity of the line BD, AC must pass through the centre of gravity of all the lines drawn parallel to BD.

Now, the parabola BADC, is obviously composed of an infinite number of straight lines, drawn parallel to BD and bisected by the absciss AC; consequently, the common centre of gravity of all the constituent lines must be the centre of gravity of the figure; hence the centre of gravity of the parabola BADC, is situated in the absciss or axis AC.

Let a = AC, the given absciss or part of the axis,
p = PQ, the parameter of the axis,
y = BC, any ordinate,
and n = any exponent of the absciss and parameter.

Then, the general equation expressing the relation between the absciss and ordinates of any parabola, is

$$a^n = p^{n-1}y.$$

Therefore, if δ = AG, the distance between the centre of gravity and vertex of the figure; then the foregoing equation gives

$$\delta = (n+1)\,a \div (n+2).$$

Now, in the common, or Apollonian parabola, $n = \frac{1}{2}$; that is, the equation to the common parabola is

$$ap = y^2, \text{ or } y = \sqrt{ap};$$

which equation implies, that

The ordinate of the common parabola is a mean proportional between the absciss and the parameter of the axis.

Consequently, by substituting $\frac{1}{2}$ for n in the preceding value of δ, we have

$$\delta = \tfrac{3}{5}\,a. \qquad (\text{K})$$

That is, in the common or Apollonian parabola.

The position of the centre of gravity is in the axis of the figure, and at the distance of three fifths of the absciss from the vertex.

EXAMPLE. If the absciss of a parabola is 60 inches or 5 feet; what is the distance of the centre of gravity from the vertex?

Here, by the above equation we get

$$\delta = \frac{60 \times 3}{5} = 36 \text{ inches or } 3 \text{ feet.}$$

65. If the figure under consideration be the common semiparabola; then the distance of the centre of gravity from the vertex, is expressed by the following equations: viz.

When the absciss and ordinate are given,

$$\delta = \tfrac{1}{40}\sqrt{576a^2 + 225y^2}. \qquad \text{(L)}$$

When the absciss and parameter of the axis are given

$$\delta = \tfrac{1}{40}\sqrt{a\{576\,a + 225\,p\}}. \qquad \text{(M)}$$

These equations express the distance between the vertex and the centre of gravity, but they do not determine the direction of that distance, by reason of the figure not being symmetrically disposed with respect to the absciss AC; the method of finding the direction of the line in which the centre of gravity of the semiparabola is situated may, however, be easily inferred from the following consideration: viz.

If we conceive the semiparabolas BAC and DAC, to be wholly concentrated in their respective centres of gravity; then, it is manifest, that their common centre of gravity, or that of the parabola BADC, must occur in the line which connects the centres of gravity of the semiparabolas BAC and DAC, and because these semiparabolas are equal to one another in magnitude, their common centre of gravity must bisect the connecting line.

Now, we have already shown, that the centre of gravity of the parabola BADC is in the absciss or axis AC; consequently, the line which connects the centres of gravity of the semiparabolas BAC and DAC, is bisected by the absciss or axis AC, and because the semiparabolas are symmetrically disposed with respect to the axis AC, the straight line which joins their centres of gravity, is bisected perpendicularly by AC; from which it appears that

The centre of gravity of any semiparabola, occurs in the ordinate to the axis, passing through the centre of gravity of the whole parabola.

Hence we infer, that

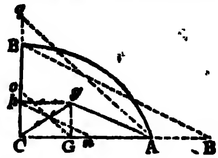
The distance between the centres of gravity of the parabola and semiparabola; and the respective distances of their centres of gravity from the vertex, constitute the three sides of a right angled triangle, of which the distance between the vertex and centre of gravity of the semiparabola is the hypothenuse.

Consequently, the distance between the centres of gravity of the parabola and semiparabola is found from the equations (K) and (L), to be

$$\sqrt{\frac{576a^2+225y^2}{1600}-\frac{9a^2}{25}}=\tfrac{3}{8}y=\tfrac{3}{8}\text{BC}.$$

66. From which, and equation (K), we deduce the following very simple construction.

Let BAC be the semiparabola whose centre of gravity g is proposed to be determined.



Produce the axis CA, and ordinate CB, to the points r and q; making cn and nr, equal respectively to the numbers 3 and 5; and co and oq, equal respectively the numbers 2 and 3.

Join rB, and through the point n, draw nm parallel to rB, meeting the ordinate BC in the point m.

Again, join qA, and through the point o, draw oG parallel to qA, meeting the axis or absciss AC in the point G, which is the centre of gravity of the parabola.

Then, through the point m draw mg parallel to the axis AC, and through G draw Gg parallel to the ordinate CB; then shall g, their point of intersection, be the centre of gravity of the semiparabola BAC.

Put $\delta'=cg$, the distance between the centre of gravity of the semiparabola BAC, and C the origin of the co-ordinates; then we have

When the absciss and ordinate are given,

$$\delta'=\tfrac{1}{40}\sqrt{256\,a^2+225\,y^2} \qquad \text{(N)}$$

When the absciss and parameter of the axis are given,

$$\delta'=\tfrac{1}{40}\sqrt{a\,\{256a+225\,p\}} \qquad \text{(O)}$$

From these two equations, in combination with those marked (L) and (M), the distances cg and Ag can be computed, and thence also, can the position of the line AG be ascertained.

And moreover, if φ represent the angle CAg, contained between the axis or absciss AC, and the line Ag, in which the centre of gravity of the semiparabola is situated; then we have

When the absciss AC and ordinate BC are given,

$$\tan.\ \varphi=5y\div 8a. \qquad \text{(P)}$$

When the absciss AC and p, the parameter of the axis are given,

$$\tan\ \varphi=5\sqrt{ap}\div 8a. \qquad \text{(Q)}$$

And from each of these equations, according as the ordinate, or parameter of the axis is given, the direction of the line Ag can easily be assigned.

The following numerical example will elucidate the whole procedure.

EXAMPLE. The absciss of a semiparabola is 56 inches, and its corresponding ordinate 22 inches; at what distance is the centre of gravity situated from the vertex, and what is the direction of the line in which it is situated with respect to the absciss?

In this example we have given the absciss $a=56$, and the ordinate $y=22$, and the equations which fulfil the conditions of the question are those marked (L) and (P); in which, if we substitute the given numerical values of a and y, we get

$$\delta = \text{A}g = \tfrac{1}{40}\sqrt{576 \times 56^2 + 225 \times 22^2} = 34.6 \text{ inches very nearly,}$$

for the distance from the vertex of the figure.

Then to find the direction, it is

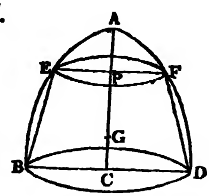
$$\tan. \varphi = \frac{5 \times 22}{8 \times 56} = 0.24554 = \text{nat. tan. } 13^\circ\, 47'\, 43'';$$

consequently, the direction which the line, where the centre of gravity of the semiparabola is situated, makes with the absciss, is an angle of $13^\circ\, 47'\, 43''$.

And exactly after the same manner would the distance and inclination be obtained from the equations (M) and (Q), when the parameter is given.

67. PROBLEM. If the parabola BAD, be supposed to revolve around the axis AC, which remains fixed, it will by such a revolution generate the solid BADC, a figure, which the writers on Mensuration have denominated

A parabolic conoid, or paraboloid.



Now, since the figure BADC is a solid of revolution, it is obviously symmetrical with respect to its axis: consequently, by our scholium, the axis AC passes through the centre of gravity of the solid BADC.

Therefore, the notation remaining as in the case of the parabolic surface the general equation for the curve, is

$$a^n = p^{n-1} y.$$

And the expression for the value of AG, the distance between the vertex and the centre of gravity of the figure, is

$$\delta = (2n+1)a \div (2n+2).$$

Now, in the common or Apollonian parabola, the exponent $n=\frac{1}{2}$; therefore, by substitution, the foregoing equation becomes

$$\delta = \tfrac{2}{3}\, a. \qquad \text{(R)}$$

This very simple equation implies, that

The distance between the vertex and the centre of gravity of a parabolic conoid, is equal to two thirds of the axis.

EXAMPLE. The axis of a parabolic conoid is 108 inches; what is the distance between the vertex and the centre of gravity of the figure?

Here, by equation (R), we have

$$\text{AG}=\delta=\frac{108\times 2}{3}=72 \text{ inches.}$$

68. When the exponent n is equal to unity, the generating surface BAC becomes a triangle, and the conoid BAD is transformed into a cone, the equation expressing the distance AC being

$$\delta=\tfrac{3}{4}a. \qquad \text{(s)}$$

This corresponds to an inference drawn from the case of the triangular pyramid treated of in the foregoing pages, but here the theorem is much more easily deduced, and more rigorously established.

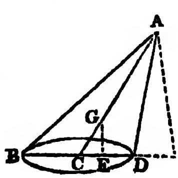
EXAMPLE 1. The axis or altitude of an upright cone, is 234 inches; at what distance from the vertex is the centre of gravity?

Here, by equation (s), we have

$$\text{AG}=\frac{234\times 3}{\ }=175.5 \text{ inches, the distance sought.}$$

EXAMPLE 2. The diameter of the base of a scaline cone, whose axis is inclined to the plane of the base in an angle of 56° 24′, is 48 inches; what is the distance of the centre of gravity from the vertex; supposing the slant side of the cone to measure 52 inches?

At the point C, the centre of the base, make the angle DCA equal to 56° 24′, and at the point D, the extremity of the diameter BD, with the distance DA equal to 52 inches, the slant side of the cone, describe an arc cutting the oblique axis CA in the point A, and join BA and DA; then is BADC the scaline cone, whose centre of gravity C is proposed to be determined.



Produce the diameter of the base BD to the point F, making CE and EF proportional to the numbers 1 and 3; join FA, and through the point E, draw EG parallel to FA, meeting AC in the point G; then shall G be the centre of gravity of the scaline cone BAD.

By Plane Trigonometry, we have

AD : DC :: sin. ACD : sin. CAD; that is,

$52 : 24 :: \sin. 56^\circ 24' : \sin. 22^\circ 36' 29''$,

then $56^\circ 24' + 22^\circ 36' 29'' = 79^\circ 0' 29'' = 180^\circ - \text{ADC}$ therefore,

$\sin. 56^\circ 24' : \sin. 79^\circ 0' 29'' :: 52 : 61.29$ inches nearly, for the length of AC, the axis of the scaline cone.

Then, by equation (S), we have

$$\text{AG} = \delta = \frac{61.29 \times 3}{4} = 45{\cdot}97 \text{ inches very nearly,}$$

for the distance between A the vertex, and G the centre of gravity of the scaline cone BAD.

PROBLEM. *To find the centre of gravity of the frustum of a parabolic conoid or paraboloid, when the distance between the ends of the frustum and the diameters of those ends are given.*

69. In the parabolic conoid BADC, (see the diagram next preceding), if a cutting plane parallel to the base, pass through P, any point in the axis AC; then, the figure BEFD thus cut off, is by the writers on Geometry denominated

the frustum of a parabolic conoid, or of a paraboloid.

And which being a solid of revolution, is manifestly symmetrical with respect to its axis or height PC; consequently by our scholium, PC passes through the centre of gravity of the frustum BEFD.

Put $R = \text{BC}$, the radius of the greater end,
$r = \text{EP}$, the radius of the lesser end,
$h = \text{PC}$, the axis or height of the frustum,
and $\delta = \text{PG}$, the distance between the centre of gravity and the centre of the lesser end.

Then, the expression involving the distance of G, the centre of gravity, from P the centre of the lesser end, is

$$3\delta\,(R^2 + r^2) = h(2R^2 + r^2);$$

or dividing both sides of the equation, by $3\,(R^2 + r^2)$, the expression for the distance PG becomes

$$\delta = h\,(2R^2 + r^2) \div 3\,(R^2 + r^2). \qquad \text{(T)}$$

70. *The practical rule which this equation affords, is as follows.*

RULE.—*To twice the square of the radius of the greater end, add the square of the radius of the lesser end, and multiply the sum by the height; then, divide the product by three times the sum of the square of the radii of the two ends, and the quotient will be the distance between the centre of the lesser end, and the centre of gravity of the frustum.*

EXAMPLE 1. The distance between the ends of the frustum of a parabolic conoid is 36 inches, and the diameters of the two ends are respectively 256 and 144 inches, at what distance from the centre of the lesser end is the centre of gravity of the frustum?

In this example there are given, $h=36$; $R=128$, and $r=72$ inches; therefore, by the equation (T), or by the rule deduced from it, we have

$$\begin{aligned} 2R^2 &= 2\times 128\times 128 = 32768, \\ r^2 &= \phantom{2\times{}} 72\times\ 72 = 5184, \\ h\,(2R^2+r^2) &= 36\times 37952 = 1366272 \text{ dividend,} \end{aligned}$$

$$\begin{aligned} R^2 &= 128\times 128 = 16384 \\ r^2 &= 72\times\ 72 = 5184 \\ 3\,(R^2+r^2) &= 3\times 21568 = 64704 \text{ divisor;} \end{aligned}$$

therefore, by division, we have

$$PG=\delta=\frac{1366272}{64704}=21.12 \text{ inches very nearly.}$$

EXAMPLE 2. From a parabolic conoid, whose axis is 84 inches, a frustum is cut off at the distance of 22 inches from the vertex; what is the distance between the centre of the lesser end, and the centre of gravity of the frustum, supposing the parameter of the generating parabola to be 16 inches?

Here from the equation of the curve, we have

$$\begin{aligned} R^2 &= 84\times 16 = 1344, \\ r^2 &= 22\times 16 = 352, \end{aligned}$$

and moreover $h=84-22=62$ inches for the height of the frustum; therefore by the rule we get

$$\begin{aligned} h\,(2R^2+r^2) &= 62\,(2\times 1344+352) = 188480 \text{ dividend,} \\ 3\,(R^2+r^2) &= 3\,(1344+352) = 5088 \text{ divisor;} \end{aligned}$$

consequently, by division, we have

$$PG=\delta=\frac{188480}{5088}=37.04 \text{ inches very nearly.}$$

71. PROBLEM. *To find at what distance from the centre of the lesser end, is the centre of gravity of the conic frustum, when its height and the diameters of the two ends are given.*

If the generating surface BEFD be a trapezoid, the solid produced by its revolution round the axis PC is obviously the conic frustum BEFD, which being symmetrical with respect to the axis, must, according to our scholium, have the axis PC passing through

its centre of gravity; consequently, the notation remaining as above, the expression involving the distance between the centre of gravity, and the centre of the lesser end of the frustum, is

$$4\delta(R^2+Rr+r^2)=h(3R^2+2Rr+r^2),$$

or dividing both sides of the equation by $4(R^2+Rr+r^2)$, the expression for the distance PG becomes

$$\delta=h(3R^2+2Rr+r^2)\div 4(R^2+Rr+r^2). \qquad (U)$$

And the practical rule derived from this expression, may be enunciated in the following manner.

72. Rule.—*To the sum of the squares of the radii of the two ends, add their product, then multiply the sum by 4 and reserve the result for a divisor. To three times the square of the radius of the greater end, add the square of the radius of the lesser end, together with twice the product of the radii, and multiply the sum by the height of the frustum for a dividend.*

Then, divide the dividend by the reserved divisor, and the quotient will express the distance between the centre of magnitude of the lesser end, and the centre of gravity of the frustum.

Example. The diameters of the two ends of a conic frustum, whose height is 26 inches, are equal respectively to 20 and 34 inches; at what distance from the centre of magnitude of the lesser end, is the centre of gravity of the frustum?

In this example there are given, $h=26$; $R=17$, and $r=10$ inches; then, by equation (U), or the rule derived from it, we have

$$\left.\begin{aligned} R^2&=17\times 17=289 \\ r^2&=10\times 10=100 \\ Rr^2&=17\times 10=170 \end{aligned}\right\}, \text{ add.}$$

$$4(R^2+Rr+r^2)=4\times \overline{559}=2236 \text{ divisor,}$$

$$\left.\begin{aligned} 3R&=3\times 289=867 \\ r^2&=10\times 10=100 \\ 2Rr&=2\times 170=340 \end{aligned}\right\} \text{ add.}$$

$$h(3R^2+2Rr+r^2)=26\times \overline{1307}=33982 \text{ dividend;}$$

consequently, by division, we have

$$PG=\delta=\frac{33982}{2236}=15.19 \text{ inches nearly.}$$

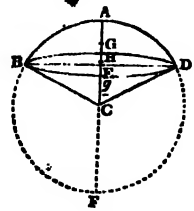
Since the mode of procedure for any other example, would be the same as that exhibited above, it is needless to extend the illustration in this place, as no useful variety of examples can be proposed.

73. Problem. *To find the centre of gravity of a spheric segment.*

Put r=CA=CB=CD, the radius of the sphere ADFD,

v=AE, the height or versed sine of the segment BAD,

and δ=AG, the distance between the centre of gravity and the vertex of the segment.



Then, since the solid BADE is symmetrically disposed with respect to the versed sine AE, it follows from our scholium, that AE passes through the centre of gravity of the spheric segment BADE, and the expression involving the distance of the vertex from the centre of gravity, is

$$\delta\,(12r-4v)=v(8r-3v).$$

or dividing both sides of the equation by $(12r-4v)$, the expression for the distance AG becomes

$$\delta=v\,(8\,r-3\,v)\div(12\,r-4\,v). \qquad \text{(v)}$$

The practical rule derived from this equation, may be expressed in words at length, as follows:

74. Rule. *From twelve times the radius of the sphere subtract four times the versed sine or height of the segment, and reserve the remainder for a divisor.*

Then from eight times the radius of the sphere subtract three times the height or versed sine; multiply the remainder by the versed sine, and divide the product by the reserved divisor; then the quotient will express the distance between the vertex and the centre of gravity of the spheric segment.

Example 1. The diameter of a sphere is 288 inches, and the height of a segment thereof is 24 inches; what is the distance between the vertex and the centre of gravity of the segment?

Here we have given, $r=144$, and $v=24$ inches; hence, by the rule we get

$$\begin{aligned} 12r &= 12\times 144=1728 \\ 4v &= 4\times 24=96 \text{ subtract,} \\ (12r-4v) &= 1632, \text{ reserved divisor.} \\ 8r &= 8\times 144=1152 \\ 3v &= 3\times 24=72 \text{ subtract,} \\ v(8r-3v) &= 24\times 1080=25920 \text{ dividend;} \end{aligned}$$

consequently, by division, we have

$\text{AG}=\delta=\dfrac{25920}{2236}=15\cdot 89$ inches nearly, for the distance between the centre of gravity and the vertex of the segment.

75. If d represent the distance between the centre of the sphere, and the centre of gravity of the segment; then we have

$$d=\overline{12\,r\,(r-v)+3v^2}\div\overline{(12\,r-4\,v)}. \qquad \text{(w)}$$

EXAMPLE 2. The data remaining as in the last example, required the distance of the centre of the sphere from the centre of gravity of the segment.

In this example there are given, $r=144$, and $v=24$ inches; let these numbers be substituted for r and v in the preceding equation (w), and it becomes

$$d=\frac{\sqrt{12\times 144\,(144-24)+3\times 24\times 24}}{12\times 144-4\times 24}=128\cdot 12 \text{ inches nearly.}$$

When $v=r$, the segment is a hemisphere, and the equation marked (v) is transformed into

$$\delta=\tfrac{5}{8}\,r;$$

or if we refer the centre of gravity to the centre of the sphere the equation marked (w) gives

$$d=\tfrac{3}{8}\,r.$$

These two equations require no illustration, we therefore pass them over, and proceed with the determination of the centre of gravity of a spheric sector.

76. PROBLEM. *To determine the centre of gravity of a spheric sector.*

The spheric sector BADC, (see the preceding diagram), is manifestly a solid of revolution, being generated by the motion of the circular sector BADC about the radius or axis AC; it is composed of the cone BCD and the spheric segment BAD, and is evidently symmetrical with respect to the radius or axis AC; consequently, by our scholium, the radius AC passes through the centre of gravity of the sector BADC, or the common centre of gravity of the cone BCD and the segment BAD, of which the spheric sector BADC is composed.

Put $r=$ AC $=$ BC $=$ DC, the radius of the sphere BADF,
$v=$ AE, the height or versed sine of the segment BAD,
$\delta=$ AH, the distance between the middle of the base, and the centre of gravity of the sector BADC.

Then is $r-v=$ EC, the altitude of the cone;

consequently, the centre of gravity of the cone BCD, as found by equation (s), is

$$cg=\frac{3\,(r-v)}{4};$$

and the centre of gravity of the spheric segment BAD, as found by equation (w) is

$$Gg=\{12r(r-v)+3v^2\}\div(12r-4v)$$

Therfore, the distance between the centres of gravity of the spheric segment and cone, is

$$cG=\{12\,r\,(r-v)+3v^2\}\div(12r-4v)-\frac{3(r-v)}{4}$$

which by reduction becomes

$$cg=\frac{3r^2}{12r-4v}.$$

Now, if we conceive the segment BAD and the cone BCD, to be wholly collected into the points G and g, their respective centres of gravity, it is evident that their common centre of gravity, or that of the sector BADC, must subsist in the connecting line Gg, and divide it into two parts, that are to each other reciprocally, as the masses of the segment and the cone.

But according to the writers on Mensuration the solidity of the segment

$$\text{is } 1\cdot0472\, v^2\,(3r-v),$$

and the solidity of the cone

$$\text{is } 1\cdot0472\, v\,(2r^2-3rv+v^2);$$

therefore, their joint solidity, or the solidity of the sector,

$$\text{is } 2\cdot0944\, r^2 v;$$

consequently, by our first principle, we have

$$2\cdot0944r^2v : 1\cdot0472v(2r^2-3rv+v^2)::\frac{3r^2}{12r-4v} : \text{GH}=\frac{2r^2-3rv+v^2}{24r-8v};$$

to which if we add $\text{AG}=v\,(8r-3v)\div(12r-4v)$, {*see equation* (V)}, we obtain

$$\text{AH}=\delta=\tfrac{1}{8}\,(2r+3v). \qquad \text{(X)}$$

And the practical rule derived from this equation, is as follows:

77. RULE. *To twice the radius of the sphere, add three times the versed sine, or height of the segment; then take one eighth part of the sum for the distance between the centre of gravity and the centre of the sector's base.*

EXAMPLE. The radius of a sphere is 56 inches, and the versed sine of the segment of the sector's base is 18 inches; required the distance between the centre of the spherical base and the centre of gravity of the sector?

In this example we have given $r=56$ inches, and $v=18$ inches, therefore, by equation (X), or the rule derived from it, we have

$$2r = 2\times56 = 112$$
$$3v = 3\times18 = 54 \text{ add}$$
$$\tfrac{1}{8}(2r+3v) = \tfrac{1}{8}\times166 = 20\cdot75 \text{ inches, for AH,}$$

the distance between the centre of gravity H, and A the centre of the sector's base.

78. The figures which we have illustrated in this treatise on the centre of gravity, are such as generally occur in the practice of mechanics, but there are many others which frequently present themselves, all more or less allied to those which we have now con-

sidered, but of which it is unnecessary to give a detailed solution; the principle of them will be found enumerated in the following summary.

1. The centre of gravity of the surface of a cylinder is the same as the centre of gravity of the parallelogram, made by a plane passing along the axis.

2. The centre of gravity of the surface of a cone, is the same as the centre of gravity of its triangular section.

3. The centre of gravity of the surface of a conic frustum is the same as the centre of gravity of the trapezoid, formed by a plane passing along the axis.

4. The centre of gravity of the surface of a spheric segment is at the middle of its versed sine or height.

5. The equation (*ss*), which answers to the centre of gravity of a conic frustum, is alike applicable to the frustum of any regular pyramid, taking R and r, to denote the greater and lesser sides of their polygonal ends.

6. The equation (*tt*), which answers to the centre of gravity of the segment of a sphere, applies also to the segment of a spheroid, without alteration, and the modified equation for the hemisphere applies to the hemispheroid.

There are several other figures and properties connected with the centre of gravity, the discussion of which does not properly belong to this place, but which will be pointed out, demonstrated and applied, when we come to treat of the composition, resolution and properties of motion.

We shall now proceed to lay before our readers a tabular view of the principal formulæ that have been investigated and applied in the preceeding pages; such an arrangment greatly expedites the process of reference, and serves admirably as a means of comparison between the different results, whether they are considered as instruments of computation, or as links in the chain of philosophical research.

The following is a list of the several equations that have been employed in determining the centres of gravity of the respective figures to which they refer.

Nature of the figure.	Form of the Equations.	Page where found.	Particular remarks and conditions of the data.
Two bodies on a straight lever without weight;	$d=\frac{b\text{D}}{a+b}$ . . . . $\delta=\frac{a\text{D}}{a+b}$	11 12	a and b the masses of the bodies; D the length of the lever, and d, δ the distances of the centre of gravity from the extremities; m denoting the weight of an unit in length of the lever.
straight lever with weight.	$d=\frac{(2b+m\text{D})\text{D}}{2(a+b+md)}$ . . . . $\delta=\frac{(2a+m\text{D})\text{D}}{2(a+b+md)}$	13 13	
Three bodies in a straight line without weight	$x=\frac{bd+m\delta}{a+b+m}$	18	a, b and m the masses of the bodies, d the length of the lever, δ the distance between the bodies a and m, and x the distance of a from the common centre of gravity of the system.
	$\delta=\frac{d}{2m}(a+-b)$ . . . . . $d=\frac{2m\delta}{a+m-b}$	22 23	The common centre of gravity in the middle of the line. a, b, m and δ given to find the length of the line such, that the centre of the system may bisect it.
Three bodies in a straight line with weight.	$x=\frac{(2b+wd)d+2m\delta}{2(a+b+m+wd)}$	31	w, the weight of an unit in length of the lever.

Nature of the figure.	Form of the Equations.	Page where found.	Particular remarks and conditions of the data.
Four bodies in the same straight line without weight.	$x=\left(\frac{m\delta+\delta'+bd}{a+m+n+b}\right)$	34	a, m, n and b, the masses of the bodies; δ, δ' and d the distances of a from m, n and b; x the distance from a to the common centre of gravity of the system.
Three bodies anyhow posited in space; ϕ acute	$AG=\frac{1}{b+c}\sqrt{(b+c)(bd^2+c\delta'^2)-bc\delta^2}$.	38	a, b and c, the bodies; d, δ and δ', the respective distances, and AG the distance between the body a and the common centre of gravity of b and c; ϕ the angle of the figure at b
ϕ obtuse	$AG=\frac{1}{b+c}\sqrt{(b+c)\{bd^2+c(\delta'^2-\delta^2\}+c^2\delta^2}$	38	
ϕ acute	$AH=\frac{1}{a+b+c}\sqrt{(b+c)(bd^2+c\delta'^2)-bc\delta^2}$.	42	a, b and c; d and δ' being the same as above, and AH the distance from a to the common centre of gravity.
ϕ obtuse	$AH=\frac{1}{a+b+c}\sqrt{(b+c)\{bd^2+c(\delta'^2-\delta^2)\}=c^2\delta'^2}$	42	
	$BH=\frac{1}{a+b+c}\sqrt{(a+c)\{ad'^2+c(\delta^2-\delta'^2)\}+c^{2\prime}\delta^2}$	44	BH the distance between b and the common centre of gravity.
	$CH=\frac{1}{a+b+c}\sqrt{(a+b)\{a\delta'^2\mp b(\delta^2-d^2)\}+b^2d}$	44	
	$AH=\frac{d}{a+b+c}\sqrt{b^2+bc+c^2}$ . . .	47	The distances d, δ and δ' equal among themselves.
	$BH=\frac{d}{a+b+c}\sqrt{a^2+ac+c^2}$ . .	47	
	$CH=\frac{d}{a+b+c}\sqrt{a^2+ab+b^2}$ . .	47	
	$AH=BH=CH\frac{d}{3}\sqrt{2}$ . .	49	The three bodies and their distances equal.

Nature of the figure.	Form of the Equations.	Pages where found.	Particular Remarks and conditions of the data.
Triangular figures.	$AH = \frac{1}{3}\sqrt{2(d^2+\delta'^2)-\delta^2}$ $BH = \frac{1}{3}\sqrt{2(d^2+d^2)-\delta'^2}$ $CH = \frac{1}{3}\sqrt{2(\delta^2+\delta'^2)-\delta^2}$	51	d, δ and δ' the sides of the triangle, and AH, BH and CH, the distances of A, B and C from the centre of gravity.
Triangular figures.	$AD = \frac{1}{2}\sqrt{2(d^2+\delta'^2)-\delta^2}$ $BF = \frac{1}{2}\sqrt{2(d^2+\delta^2)-\delta'^2}$ $CE = \frac{1}{2}\sqrt{2(\delta^2+\delta'^2)-d^2}$	51	a, o and δ' the sides of the triangle; AD, BF and CE the distances from the angular points to the middle of the opposite sides.
	$d = \sqrt{2(AH^2+BH^2)-CH^2}$ $\delta = \sqrt{2(BH^2+CH^2)-AH^2}$ $\delta' = \sqrt{2(AH^2+CH^2)-BH^2}$	54	Particulars as above.
	$d^2+\delta^2+\delta'^2 = 3(AH^2+BH^2+CH^2)$	55	
Quadrilateral figures.	$pH\,(2AD+BC) = nH\,(2BC+AD)$	67	AD and BC parallel sides of the figure d, δ and δ' as referred to in the text, (see the diagram p. 66 and the description p. 68.)
	$nH = \frac{(2AD+BC)\sqrt{2(d^2+\delta^2)-\delta'^2}}{6(AD+BC)}$ $pH = \frac{2BC+AD\sqrt{2(d^2+\delta^2)-\delta'_2}}{6(AD+BC)}$	68	
Figure in the form of a trapezoid.	$nH = \frac{(2AD+BC)\sqrt{4d^2-\delta'^2}}{6(AD+BC)}$ $pH = \frac{(2BC+AD)\sqrt{4d^2-\delta'^2}}{6(AD+BC)}$	71	Particulars as stated above.

Nature of the figure.	Form of the Equations.	Pages where found	Particular remarks and conditions of the data.
Triang. pyramid	$AG = \frac{1}{4}\sqrt{3(a^2+d^2+\delta'^2)-(b^2+c^2+\delta^2)}$. $BG = \frac{1}{4}\sqrt{3(b^2+d^2+\delta^2)-(a^2+c^2+\delta'^2)}$. $CG = \frac{1}{4}\sqrt{3(c^2+\delta^2+\delta'^2)-(a^2+b^2+d^2)}$. $DG = \frac{1}{4}\sqrt{3(a^2+b^2+c^2)-(d^2+\delta^2+\delta'^2)}$.	98	d, δ, δ' the sides of the base; a, b and c the edges of the pyramid; and A, B, C and D the vertex in the order of the equations; AG, BG, CG and DG, the respective distances from the vertex to the centre of gravity of the figure.
Triang. pyramid	$AG = \frac{1}{4}\sqrt{3a^2+5d^2-(b^2+c^2)}$ . . $BG = \frac{1}{4}\sqrt{3b^2+5d^2-(a^2+c^2)}$ . . $CG = \frac{1}{4}\sqrt{3c^2+5d^2-(a^2+b^2)}$ . . $DG = \frac{1}{4}\sqrt{3(a^2+b^2+c^2-d^2)}$ . .	103	Sides of the base equal to one another.
Triang. pyramid	$AG = \frac{1}{4}\sqrt{3(d^2+\delta'^2)+(a^2-\delta^2)}$ . . $BG = \frac{1}{4}\sqrt{3(d^2+\delta^2)+(a^2-\delta'^2)}$ . . $CG = \frac{1}{4}\sqrt{3(\delta^2+\delta'^2)+(a^2-d^2)}$ . . $DG = \frac{1}{4}\sqrt{9a^2-(d^2+\delta^2+\delta'^2)}$ . .	104	The edges of the pyramid equal, and the sides of the base of any magnitude.
Triang. pyramid	$AG = BG = CG = \frac{1}{4}\sqrt{a^2+5d^2}$ . . $DG = \frac{1}{4}\sqrt{3(3a^2-d^2)}$ . . .	105	The sides of the base, and the edges of the pyramid equal.
Triang. pyramid	$AG = BG = CG = DG = \frac{a}{4}\sqrt{6}$ . .	105	The six edges of the figure equal among themselves.

Nature of the figure.	Form of the Equations.	Pages where found	Particular remarks and conditions of the data.
Circular arc.	$a\delta = cr$	107	a the length of the arc, c the chord, r the radius and δ the distance betwn. the centre of the circle and centre of gravity of the arc.
Circular segment	$3\delta(c^2+4v^2)(a \mp \sin.\phi) = 4c^3v$.	111	a, c and δ as before, and v the versed sine or height of the segment.
Circular sector.	$3a\delta = 2cr$	112	a, c, δ and r as before for the arc and segment.
General parabol.	$\delta = (n+1)a \div (n+2)$	113	a the absciss, δ the distance between the vertex and the centre of gravity, and n the order or degree of the curve.
Common parabola.	$\delta = \frac{3}{5}a$	113	a and δ as above.
Common semiparabola.	$\left.\begin{array}{l}\delta = \frac{1}{40}\sqrt{576a^2+225y^2} \text{ . . . .} \\ \delta = \frac{1}{4}\sqrt{a\{576a+225p\}} \text{ . . . .}\end{array}\right\}$	114	a, the absciss, y the ordinate, δ the distance between the vertex and centre of gravity, and p the parameter of the axis.
Parabolic conoid, or paraboloid.	$\delta = \frac{2}{3}a$	116	a the axis and δ the distance from the vertex to the centre of gravity.
Right cone.	$\delta = \frac{3}{4}a$.	117	a and δ as above.

Nature of the figure.	Form of the Equation.	Page where found.	Particular remarks and conditions of the data.
Frust. of a paraboloid.	$\delta = h(2R^2+r^2) \div (3R^2+r^2)$ . . .	118	R and r the radii of the ends, h the height, and δ the dist. between the centre of gravity and the centre of the lesser end.
Frust. of a cone.	$\delta = h(3R^2+2Rr+r^2) \div 4(R^2+Rr+r^2)$.	120	R, r, h and δ as before.
Spheric segm.	$\delta = v(8r-3v) \div (12r-4v)$ . . .	121	r the radius, v the versed sine and δ the dist. between the vertex and the centre of gravity
	$\delta = \frac{3}{8}r$	122	The centre of gravity referred to the centre of the sphere.
Spheric sector.	$\delta = \frac{1}{8}(2r+3v)$	123	r the radius of the sphere, v the versed sine of the sector, and δ the distance between the centre of gravity and the middle of the base.

THE END OF THE TREATISE ON THE CENTRE OF GRAVITY.

The Third Treatise,

COMPRISING

THE MECHANICAL POWERS.

I. OF THE LEVER.

INTRODUCTION.

THE Mechanical Powers are those simple machines by which force or motion may be transmitted and modified, both as to its quantity and direction. These machines are reducible to the *lever*, the *cord*, and the *inclined plane;* but for the purpose of keeping within the acknowledged boundaries of scientific arrangement, we shall follow the old and popular distribution, and class them under the names of the *lever*, the *wheel and axle*, the *pulley*, the *wedge*, the *inclined plane*, and the *screw*.

The Mechanical Powers may be considered either statically or dynamically. In the former case they are machines by which forces of determinate quantities and direction are made to balance other forces of other quantities and direction. In the latter case they are considered as the means, by which certain motions of determinate quantity and direction, may be made to produce other motions in other directions and quantities. That case considers simply the phenomena of equilibrium: this investigates the laws of motion as concerned in the application of machines, whether simple or compound. It is, however, the first of these cases which we shall discuss in this volume: Dynamics, or the science of moving bodies, belongs to a subsequent volume of these Mechanics.

The Mechanical Powers are of great antiquity. We are told that Archimedes, at the siege of Syracuse, constructed machines and engines which suddenly raised the Roman galleys into the air, and then let them fall with such violence into the water that

they either sunk, or were dashed to pieces. Powerful levers, to which chains and shears were attached, would do this; and when the power, that overcame the resistance, was suddenly thrown off, the weight that had been raised would be dashed beneath the waves. The defence of his countrymen was unsuccessful, and the philosopher perished by the hand of a barbarian soldier. He has left us, however, as monuments of his consummate genius, two books upon the equilibrium of solid bodies, in which he has demonstrated that

> "When a balance with unequal arms is in equilibrio, by means of two weights in its opposite scales, these weights must be reciprocally proportional to the arms of the balance."

From this general principle, all the properties of the lever and of machines, referable to the lever, might have been deduced as corollaries; but Archimedes contented himself with recording the results of his predecessors, and his premature death has deprived us of those investigations and their consequences, to which his discovery or application of the foregoing principle would have led.

In demonstrating the leading property of the lever, he introduces what must be admitted as a very ancient axiom; that,

> "If the two arms of the balance are equal, the two weights must also be equal when an equilibrium takes place."

Archimedes will lose nothing as a mathematician by assigning this axiom to the age of Abraham, who "weighed to Ephron the silver, four hundred shekels, current money of the merchant, for the field which was in Machpelah and the cave that is therein, which were made sure unto Abraham for the possession of a burying-place by the sons of Heth."*

The use of the inclined plane, the pulley, the wheel and axle, the screw, and the wedge, was well known of old, but it has happened that the Oriental theory of these simple mechanical powers, like historical records concerning Asia and Egypt, where they were first invented and applied, has perished; and we stand indebted to Archimedes alone for all the scientific information of which time has not bereaved us; since, therefore, we cannot assign a more ancient

* Genesis, ch. XXIII. v. 17—20.

authority, to Archimedes we must ascribe the theory of the Mechanical Powers.

Plutarch being a Greek was a great admirer of Archimedes, to whom he attributes various machines, which have not reached our times; and as his discoveries in hydrostatics belong not to this part of our work, we shall merely mention them as efforts of that master-mind which has conferred upon statics the principles and theory of an accurate science.

We cannot contemplate the application of any machine, without considering these three things as involved in its use:—1. The force or resistance to be overcome; 2. the means by which it is to be overcome, and 3. the machine itself applied with these means in overcoming the resistance. The force to be overcome is denominated the *weight*, the equivalent force to sustain or overcome this, is called the *power*, and the machine is the ***instrument*** or artifice which transmits this power, which it expends at a slow rate, and in a more advantageous direction than if it were immediately applied to the weight or resistance. The power may be natural, artificial, or animal; as, the action of wind or running water, the force of steam, or the labour of men or of beasts.

The lever, placed at the head of the mechanical powers, appears to have been the first engine which men used to change the direction of motion and to overcome forces or resistances of different intensities. Thus, a beam or rod is employed as a lever, simply by resting one part on a fulcrum or prop as an axis, which thence becomes the centre of motion to the engine; and there are few machines constructed in which levers of one class or another do not occur. But the most remarkable, and perhaps the most elegant adaptations of the lever, whether straight or bent, will be found in the limbs of animals, the wings of birds, and the fins of fishes.

We must not, however, suppose that the lever, or indeed any one of the mechanical powers, possesses the means of generating force, or is endowed with any innate applicability for saving labour; on the contrary, "in using them in any case, even more labour or bodily exertion is expended than would suffice to do the work without them:"* of what utility, then, are the mechanical powers?—why

* Dr. Arnot's Elements of Physics, vol. i. p. 188. Fifth Edition.

this: they allow a small force to take its time to produce any requisite magnitude of effect, at the expense of additionally overcoming a certain amount of friction or other resistance. To explain this, a man with a handspike or crow-bar, sixty inches in length from the fulcrum or prop, will, with a force of 56 pounds, equipoise a weight of one ton suspended to the other end of the bar; the distance between the load and the fulcrum being only three inches and three quarters of an inch. A man with a crow-bar will move large blocks of granite or marble, or immense logs of wood, that would, perhaps, require twenty men to do the same. True, he takes more time to do this than the twenty men would take, and hence we reason, that whatever is gained in power through a machine, is lost in speed or in time, and *vice versâ*. In fact, no machine can exert more force than has passed into it from some source which has generated that force.

There are three orders of levers, to each of which we have devoted as much attention and space, as its utility and importance appeared to require, and we have reduced the whole theory to a series of consecutive problems, from which we have drawn our rules for the purpose of solving such questions as the subject demanded for its illustration, and for shewing its practical applications. By thus handling the equations of equilibrium of the lever, we have, as in our previous subjects, combined precept with example: and while nothing has been omitted that deserved our attention, every thing has been introduced with which the reader might desire to be acquainted: the novelty of our plan is, however, secondary to its utility. But ample and copious as is our treatise upon the lever, we could only, in this part of our course, treat of its properties in the condition of equilibrium or of balanced rest; and point out, among many cases of its applicability in the arts, a few of the most strikingly useful, as where the weight of cast-iron beams may be determined, when they are made so large as to render it difficult to weigh them; and the weight of large uniform masses such as wood, stone, or metal, when they assume the form of a bended lever.

A. J.

MECHANICAL POWERS.

CONTENTS.

1. OF THE LEVER.

2. OF THE WHEEL AND AXLE.

3. OF THE PULLEY.

4. OF THE INCLINED PLANE.

5. OF THE SCREW.

6. OF THE WEDGE.

OF THE LEVER.

Definitions and Illustrations.

1. A *lever* is an inflexible bar or rod, supported on a prop or *fulcrum*, about which, as a *centre of motion*, it turns freely when acted on by two or more forces applied at different points of its length.

Levers are of two distinct kinds, viz.

1. *Those in which the centre of motion exists between the forces.*
2. *Those in which it does not.*

Of this latter kind there are two varieties, viz.

1. Such as have the *resistance** between the *power* and *centre of motion.*
2. Such as have the *power* between the *centre of motion* and the *resistance.*

And the first of the above kinds of lever, combined with the two varieties of the second kind, constitute what authors have denominated the three *orders of levers;* by this appellation, therefore, shall we distinguish them.

A lever of the first order, has the centre of motion or fulcrum between the power and the resistance.

A lever of the second order, has the resistance between the centre of motion and the power.

A lever of the third order, has the power between the centre of motion and the resistance.

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* The forces that operate on the lever may in general be considered as of two kinds, viz. *active* and *passive* forces. The *active force*, or that which endeavours to produce motion, is called the *power*, and the *passive force*, or that which endeavours to remain at rest, is called the *resistance* or weight.

CASE I.

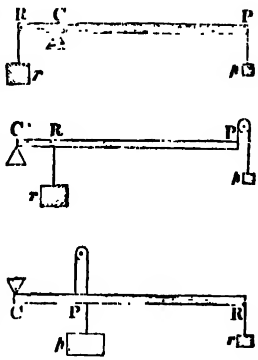
Section First.

WHEN THE CENTRE OF MOTION CORRESPONDS TO THE THREE ORDERS OF LEVERS.

First. *When the lever is considered void of weight or gravity.*

2. The principle on which the theory of the lever depends is simply as follows, that is,

Proposition 1. *When two forces acting in the same plane and in parallel directions keep an inflexible bar or lever (considered without weight or gravity) in equilibrio; these forces are to each other inversely as their distances from the fulcrum or centre of motion.*



Let $\text{D}=\text{CP}$, the distance between the centre of motion and the point where the power (p) is applied;

$d=\text{CR}$, the distance between the centre of motion and the point where the resistance acts;

$p=$ the power,

and $r=$ the resistance;—then by the principle specified above, we have

$$p : r :: d : \text{D};$$

or, by making the product of the mean terms equal to the product of the extremes, we obtain

$$p\text{D}=rd. \qquad (a)$$

This equation expresses the conditions of equilibrium when the weight of the lever is not considered, but when that element is taken into account, the expression becomes more complicated, as will be seen further on.

The equation of equilibrium being thus established, we shall proceed to the developement of the theory which simply consists in resolving the following problems.

3. Problem 1. *Given the magnitude of the power p, its distance from the centre of motion* D, *and the magnitude of the resistance r; to find d its distance from the centre of motion.*

Let both sides of the general equation (a) be divided by the resistance r, and we get

$$d=\frac{p\text{D}}{}$$

Which expression affords the following practical rule.

RULE. *Multiply the magnitude of the power by its distance from the centre of motion, and divide the product by the magnitude of the resistance for its distance from the centre of motion.*

EXAMPLE 1. In a lever of the first order, a power of 7 pounds acting at the distance of 48 inches from the fulcrum or centre of motion, is found to balance a resistance of 84 pounds; at what distance from the centre of motion does the resistance act?

By the rule we have $\frac{48 \times 7}{84} = 4$ inches for the distance sought.

EXAMPLE 2. In a lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the fulcrum or centre of motion, is found to balance a resistance or weight of 112 pounds; at what distance from the centre of motion does this weight act?

By the rule we have $\frac{56 \times 2}{112} = 1$ inch the distance required.

EXAMPLE 3. In a lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the fulcrum or centre of motion, is found to balance a resistance or weight of 16 pounds; at what distance from the centre of motion does the resistance act?

By the rule we have $\frac{256 \times 2}{16} = 32$ inches for the distance required.

4. PROBLEM 2. *Given the magnitude of the weight or resistance r, its distance from the centre of motion d, and the magnitude of the power p; to find D, its distance from the centre of motion.*

Let both sides of the general equation (a) be divided by the power p, and we get

$$D = \frac{rd}{p}.$$

Which expression affords the following practical rule.

RULE. *Multiply the magnitude of the weight or resistance by its distance from the centre of motion, and divide the product by the magnitude of the power for its distance from the centre of motion.*

EXAMPLE 1. In a lever of the first order, a resistance or weight of 84 pounds acting at the distance of 4 inches from the centre of motion, is found to balance a power of 7 pounds; at what distance from the centre of motion does the power act?

By the rule we have $\frac{84 \times 4}{7} = 48$ inches the distance required.

EXAMPLE 2. In a lever of the second order, a resistance or weight of 112 pounds acting at the distance of one inch from the fulcrum or centre of motion, is found to balance a power of 2 pounds; at what distance from the centre of motion is the power applied?

By the rule we have $\frac{112 \times 1}{2} = 56$ inches the distance required.

EXAMPLE 3. In a lever of the third order, a resistance or weight of 16 pounds acting at the distance of 32 inches from the centre of motion, is found to balance a power of 256 pounds; at what distance from the centre of motion is the power applied?

By the rule we have $\frac{16 \times 32}{256} = 2$ inches the distance required.

5. PROBLEM 3. *Given the magnitude of the power p, its distance from the centre of motion D, and the distance of the resistance from the centre of motion d; to find r the magnitude of the resistance.*

Let both sides of the general equation (a) be divided by d, the distance of the weight or resistance from the centre of motion, and we get

$$r = \frac{p\,D}{d}.$$

Which expression affords the following practical rule.

RULE. *Multiply the magnitude of the power by its distance from the fulcrum or centre of motion, and divide the product by the distance of the weight or resistance from the fulcrum, then the quotient will be the magnitude of the resistance sought.*

EXAMPLE 1. In a lever of the first order, a power of 7 pounds acting at the distance of 48 inches from the centre of motion, is found to balance a resistance or weight acting at the distance of 4 inches; what is the magnitude of the resistance?

By the rule we have $\frac{7 \times 48}{4} = 84$ pounds for the resistance sought.

EXAMPLE 2. In a lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the fulcrum or centre of motion, is found to balance a resistance or weight acting at the distance of one inch; what is the magnitude of the resistance?

By the rule we have $\frac{2 \times 56}{1} = 112$ pounds for the resistance sought.

EXAMPLE 3. In a lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the centre of motion, is

found to balance a weight or resistance acting at the distance of 32 inches; what is the magnitude of the resistance?

By the rule we have $\frac{256\times 2}{32}=16$ pounds for the resistance sought.

6. Problem 4. *Given the magnitude of the weight or resistance* r, *its distance from the centre of motion* d, *and the distance of the power from the centre of motion* D; *to find* p *the magnitude of the power.*

Let both sides of the general equation (a) be divided by D, the distance of the power from the centre of motion, and we get

$$p=\frac{rd}{D}.$$

Which expression affords the following practical rule.

Rule. *Multiply the magnitude of the resistance by its distance from the centre of motion, and divide the product by the distance of the power from the fulcrum, then the quotient will be the magnitude of the power sought.*

Example 1. In a lever of the first order, a weight or resistance of 84 pounds acting at the distance of 4 inches from the fulcrum or centre of motion, is found to balance a certain power acting at the distance of 48 inches; what is the magnitude of the power?

By the rule we have $\frac{84\times 4}{48}=7$ pounds for the power sought.

Example 2. In a lever of the second order, a weight or resistance of 112 pounds acting at the distance of one inch from the fulcrum or centre of motion, is found to balance a certain power acting at the distance of 56 inches; what is the magnitude of the power?

By the rule we have $\frac{112\times 1}{56}=2$ pounds for the power sought.

Example 3. In a lever of the third order, a weight or resistance of 16 pounds acting at the distance of 32 inches from the fulcrum or centre of motion, is found to balance a certain power acting at the distance of 2 inches; what is the magnitude of the power?

By the rule we have $\frac{16\times 32}{2}=256$ pounds for the power sought.

Secondly. *When the weight of the lever makes an element in the equilibrated mass.*

7. The four problems preceding, with the rules and examples deduced from them, apply only in the case where the bar or lever on which the forces act is considered to be absolutely inflexible and void of gravity or weight; but in what follows, the weight of the lever is taken into account, and the equilibrium is still supposed to

obtain. Now, the principle by which the effect produced by the weight of the lever is to be estimated, is as follows.

PROPOSITION 2. *If a straight inflexible bar, or lever, be kept in equilibrio by any number of parallel forces acting at different points, the sum of the moments of the forces acting on different sides of the fulcrum or centre of motion are equal.*

Hence, in a lever of the first order, we may consider the weight of each arm as a determinate power applied at the centre of gravity, but when the arms of the lever are uniform, the centre of gravity in each is situated at the middle of its length between the extremity and the centre of motion; therefore, if φ represent the weight of a unit in length, of the same denomination as p and r, the weight of the arm CP will be expressed by $D\varphi$, and its moment when reduced to the centre of gravity is $\frac{1}{2}D^2\phi$; in like manner, the weight of the arm CR is expressed by $d\varphi$, and its moment when reduced to the centre of gravity is $\frac{1}{2}d^2\varphi$; but by equation (a), the moment of the power is pD, and that of the resistance rd; consequently, by the principle stated above, the sum of the moments on each side of the fulcrum or centre of motion must be in equilibrio; that is

$$2pD+D^2\varphi=2rd+d^2\varphi. \qquad (b)$$

This is the equation of equilibrium for a lever of the first order when its own weight is taken into the account; we shall therefore, in the next place, proceed to the developement of its several terms according to the plan pursued in the developement of equation (a)

8. PROBLEM 1. *Given the magnitude of the power p, its distance from the centre of motion D; the magnitude of the resistance r, and φ the weight of a unit of the length of the lever expressed in the same terms as p and r; to find d, the distance from the centre of motion at which the weight or resistance acts.*

Let both sides of the general equation (b), be divided by φ, the weight of a unit of the lever, and we get

$$d^2+\left(\frac{2r}{\varphi}\right)d=D^2+\left(\frac{2p}{\varphi}\right)D,$$

An equation from which the value of d is to be determined; the best method of performing the process is as follows.

Substitute the given quantities or their powers according as they are expressed in the above equation, and a numerical equation will arise, from which the value of the unknown quantity can be obtained by any of the rules employed for the reduction of quadratic equations as delivered by the writers on Algebra.

EXAMPLE 1. In a lever of the first order, a power of 7 pounds acting at the distance of 48 inches from the fulcrum or centre of

motion, is found to balance a weight or resistance of 84 pounds; at what distance from the centre of motion does the resistance act, supposing one inch in length of the lever to weigh one pound?

Here we have $p=7$; $\text{D}=48$; $r=84$ and $\varphi=1$; let these numerical values or their powers be substituted for p, D, r, and φ in the following equation, and it becomes*

$$d^2+168d=2976,$$

complete the square and we have

$$d^2+168d+7056=10032,$$

extract the square root of both sides, and we obtain

$$d+84=100\cdot16-,$$

transpose, and we finally have

$d=16\cdot16$ inches nearly, for the distance required.

10. If we refer to the first example to the first problem of equation (a), where the data as well as the quæsitum are the same as in the present case, we find the resulting distance to be only 4 inches, whereas in this instance it is $16\cdot16$ inches; hence it is manifest, of what importance it must be in practice to consider the effect produced by the weight of the lever itself, since with the same data, in the simple case of equilibrium only, it produces such a difference in the result.

Let us now enquire if the equilibrium actually obtains from the above determination; for which purpose, we must substitute the given numbers, and that determined by the process, in equation (b) according to the combination there indicated, and we have

$$2p\text{D}=2\times7\times48=\ \ 672$$
$$\text{D}^2\varphi=48\times48\times1=\underline{2304}$$

Sum of the moments on one side $=2976$;

$$2rd=2\times84\times16\cdot16=2714\cdot87$$
$$d^2\varphi=16\cdot16\times16.16\times1=\underline{\ 261\cdot13}$$

Sum of the moments on the other $=2976\cdot00$;

hence it is manifest that the equilibrium is perfect.

Example 2. In a lever of the first order, a power of 56 pounds applied at the distance of 60 inches from the centre of motion, is found to equipoise a weight of 2240 pounds; at what distance from the centre of motion is the weight applied, supposing one inch in length of the lever to weigh three pounds?

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* To solve a quadratic equation. Reduce the equation to the form $x^2\pm ax=\pm b$; add the square of half a to both sides; then $x^2\pm ax+\frac{a^2}{4}=b+\frac{a^2}{4}$, whence $x\pm\frac{a}{2}=\pm\sqrt{\left(b+\frac{a^2}{4}\right)}$; and therefore $x=\pm\sqrt{\left(b+\frac{a^2}{4}\right)}\mp\frac{a}{2}$.

Here we have $p=56$; $D=60$; $r=2240$ and $\varphi=3$; consequently, by substitution, the equation becomes

$$d^2+1493\tfrac{1}{3}d=5840,$$

complete the square and we have

$$d^2+1493\tfrac{1}{3}d+\left(\frac{2240}{3}\right)^2=563351\tfrac{1}{9},$$

extract the square root of both sides, and we obtain

$$d+\frac{2240}{3}=750\cdot566,$$

transpose, and we finally have

$d=750\cdot566-746\cdot666=3\cdot9$ inches for the distance sought.

And after the same manner as we proved the equilibrium to obtain in the preceding example, it may be proved to obtain in this, but the method of operation is so obvious, that we think it quite needless to repeat it.

11. PROBLEM 2. *Given the magnitude of the weight or resistance r, its distance from the centre of motion d, the magnitude of the power p, and φ, the weight of a unit of the length of the lever, expressed in the same terms as p and r; to find D, the distance from the centre of motion at which the power acts.*

Since the expressions for D and d in the general equation (b) are symmetrical, it is evident that the quadratic equation involving the values of D must be similar in form to that involving the values of d, it is in fact the same equation, and consequently the second problem just proposed is unnecessary, for its resolution is identified with that of problem 1; but in order to preserve uniformity of system, the equation is again repeated, simply having the terms reversed; thus.

$$D^2+\left(\frac{2p}{\varphi}\right)D=d^2+\left(\frac{2r}{\varphi}\right)d,$$

and the method of performing the operation is in every respect the same as that described in the preceding problem.

EXAMPLE 1. In a lever of the first order, a resistance or weight of 84 pounds acting at the distance of 16·16 inches from the centre of motion, is found to balance a power of 7 pounds; at what distance from the centre of motion does the power act, supposing one inch in length of the lever to weigh one pound?

Here we have $r=84$; $d=16\cdot16-$; $p=7$, and $\varphi=1$; consequently, by substitution, the equation becomes

$$D^2+14D=2976,$$

complete the square and we have

$$D^2+14D+49=3025,$$

extract the square root of both sides and we obtain

$$D+7=55,$$

transpose, and we finally have

$D=48$ inches for the distance required at which the power acts.

EXAMPLE 2. In a lever of the first order, a weight or resistance of 2240 pounds acting at the distance of 3·9 inches from the fulcrum or centre of motion, is found to equipoise a power of 56 pounds; at what distance from the centre of motion does the power act, supposing one inch in length of the lever to weigh 3 pounds?

Here we have $r=2240$; $d=3\cdot9$; $p=56$ and $\varphi=3$; consequently, by substitution, the equation becomes

$$D^2+37\tfrac{1}{3}D=5839\cdot21,$$

complete the square and we have

$$D^2+37\tfrac{1}{3}D+\left(\frac{56}{3}\right)^2=6187\cdot66,$$

extract the root of both sides and we obtain

$$D+18\cdot66=78\cdot66,$$

transpose and we finally have

$D=60$ inches for the distance at which the power acts.

The equilibrium may be proved for both the foregoing examples, after the same manner as we proved it for the first example in the last problem.

12. PROBLEM 3. *Given the magnitude of the power p; its distance from the centre of motion D; the distance of the weight or resistance from the centre of motion d, and φ, the weight of a unit in length of the lever, expressed in the same terms as p and r; to find r the magnitude of the resistance.*

From the general equation of equilibrium (b), by transposition and division we obtain

$$r=\frac{2pD+(D^2-d^2)\varphi}{2d},$$

a simple equation, from which the value of r can easily be ascertained.

The rule in words at length may be expressed as follows.

RULE. *Multiply together the sum of the distances of the power and resistance from the centre of motion, the difference of those distances and the weight of an unit in length of the lever; then, to the product, add twice the power drawn into its distance from the centre of motion, and divide the sum by twice the distance of the resistance, for the magnitude of the resistance sought.*

EXAMPLE 1. In a lever of the first order, a power of 7 pounds acting at the distance of 48 inches from the fulcrum or centre of motion, is found to balance a weight or resistance acting at the distance of 16.16 inches; what is the magnitude of the resistance, supposing one inch in length of the lever to weigh one pound?

By the rule we have

$$48+16\cdot16=64\cdot16,$$
$$48-16\cdot16=31\cdot84;$$
$$\text{then, } 64\cdot16\times31\cdot84=2042\cdot8544$$
$$2\times7\times48=\underline{672}$$
$$2\times16\cdot16=32\cdot32)\overline{2714\cdot8544}(=84 \text{ pounds*}$$

very nearly for the resistance sought.

EXAMPLE 2. In a lever of the first order, a power of 56 pounds acting at the distance of 60 inches from the fulcrum or centre of motion, is found to equipoise a weight or resistance acting at the distance of 3·9 inches; what is the magnitude of the resistance, supposing the weight of one inch in length of the lever to be 3 pounds?

By the rule we have

$$60+3\cdot9=63\cdot9,$$
$$60-3\cdot9=56\cdot1;$$
$$\text{then, } 63\cdot9\times56\cdot1\times3=10754\cdot37$$
$$2\times56\times60=\underline{6720}$$
$$2\times3\cdot9=7\cdot8)\overline{17474\cdot37}(=2240 \text{ pounds very}$$

nearly for the resistance.

13. PROBLEM 4. *Given the magnitude of the weight or resistance r, its distance from the centre of motion d; the distance of the power from the centre of motion* D, *and* φ, *the weight of a unit in length of a lever expressed in terms of p and r; to find p the magnitude of the power.*

From the general equation of equilibrium (*b*), by transposition and division we obtain

$$p=\frac{2rd+(d^2-\text{D}^2)\varphi}{2\text{D}},$$

a simple equation from which the value of p can easily be ascertained, and in which, the second member in the numerator of the fraction on the right hand side of the equation, is additive or subtractive, according as the distance of the weight or resistance from the centre of motion, is greater or less than the distance of the power.

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* The last line of these examples might have been differently expressed; but we have chosen the expression $2\times16.16=32.32)2714.8544(=84$, as a mode of arrangement very often met with among practical writers; and the veriest tyro will know that the expression (=84) means that this quotient is equal to the divisor 32.32 taken eighty-four times out of the dividend 2714.8544. The same observation applies to the previous examples in this Treatise on the Lever.

The rule in words at length may be expressed as follows.

RULE. *Multiply together, the sum of the distances of the resistance and power from the centre of motion, the difference of those distances and the weight of an unit in length of the lever; then, to the product, add twice the weight or resistance drawn into its distance from the centre of motion, and divide the sum by twice the distance of the power, for the magnitude of the power required.*

EXAMPLE 1. In a lever of the first order, a weight or resistance of 84 pounds acting at the distance of 16·16 inches from the fulcrum or centre of motion, is found to balance a certain power acting at the distance of 48 inches; what is the magnitude of the power, supposing one inch in length of the lever to weigh one pound?

By the rule we have

$$\begin{aligned} 16\cdot16+48 &= 64\cdot16, \\ 16\cdot16-48 &= -31\cdot84; \\ \text{then, } 64\cdot16\times-31\cdot84 &= -2042\cdot8544 \\ 2\times84\times16\cdot16 &= \underline{2714\cdot88} \\ 2\times48=96)\ & 672\cdot0256(\text{nearly 7lbs.} \end{aligned}$$

for the power sought.

EXAMPLE 2. In a lever of the first order, a weight or resistance of 2240 pounds acting at the distance of 3·9 inches from the fulcrum or centre of motion, is found to equipoise a certain power acting at the distance of 60 inches; what is the magnitude of the power, supposing one inch in length of the lever to weigh 3 pounds?

By the rule we have

$$\begin{aligned} 3\cdot9+60 &= 63\cdot9, \\ 3\cdot9-60 &= -56\cdot1; \\ \text{then, } 63\cdot9\times-56\cdot1\times3 &= -10754\cdot37 \\ 2\times2240\times3\cdot9 &= \underline{17472\cdot} \\ 2\times60=120)\ & 6717\cdot63(\text{nearly 56lbs.} \end{aligned}$$

for the power sought.

In both the above examples, as will be seen from the operation, the second, or parenthetical member of the numerator of the fraction on the right hand side of the equation, has been used subtractively, because, the distance of the weight or resistance from the centre of motion, is less than the distance of the power.

14. PROBLEM 5. *Given the magnitude of the power p, its distance from the fulcrum or centre of motion D; the magnitude of the resistance r, and its distance from the centre of motion d; to find φ, the weight of an unit in length of the lever.*

From the general equation of equilibrium (*b*), by transposition and division we obtain

$$\varphi = \frac{2\{p\mathrm{D} \backsim rd\}}{d^2 \backsim \mathrm{D}^2},$$

A simple equation, from which the value of φ can easily be ascertained.

The rule in words at length may be expressed as follows.

RULE. *Divide twice the difference of the moments of the power and resistance, by the difference of the squares of their distances from the fulcrum or centre of motion, and the quotient will be the weight of an unit in length of the lever.*

EXAMPLE 1. In a lever of the first order, a power of 7 pounds acting at the distance of 48 inches from the fulcrum or centre of motion, is found to balance a weight or resistance of 84 pounds acting at the distance of 16.16 inches; what is the weight of an unit in length of the lever, and what its whole weight?

By the rule we have

The moment of the resistance $84 \times 16{\cdot}16 = 1357{\cdot}44$
The moment of the power $48 \times 7 = 336$
difference $= 1021{\cdot}44$
2

$$\overline{48+16{\cdot}16} \times \overline{48-16{\cdot}16} = 2042{\cdot}88)\overline{2042{\cdot}88}(\ 1\text{lb.}$$

per inch in length; now, the whole length of the lever is $48 + 16.16 = 64.16$ inches; consequently, its whole weight is 64.16 pounds.

EXAMPLE 2. In a lever of the first order, a power of 56 pounds acting at the distance of 60 inches from the fulcrum or centre of motion, is found to equipoise a weight or resistance of 2240 pounds acting at the distance of 3·9 inches; what is the weight of an unit in length of the lever, and what is its total weight?

By the rule we have

The moment of the resistance $2240 \times 3{\cdot}9 = 8736$
The moment of the power $56 \times 60 = 3360$
difference $= 5376$
2

$$\overline{60+3{\cdot}9} \times \overline{60-3{\cdot}9} = 3584{\cdot}79)\overline{10752}(\text{nearly } 3\text{lbs.}$$

for the weight of one inch in length of the lever; but its whole length is 63·9 inches; therefore, its whole length is 191·7 pounds.

15. From the two foregoing examples it appears, that this problem is applicable to a very important purpose, viz. that of determining the weight of cast iron beams when they are made so large as to render it difficult to weigh them; the equation in its

present form, however, applies only to beams of uniform breadth and depth throughout the length, but it can easily be modified for beams of other shapes by introducing the elements necessary for determining the position of the centre of gravity; a further enquiry into the subject in this place would be foreign to our purpose, we therefore leave it for a more extensive discussion further on, and proceed forthwith to develope the theory of the second order of levers, when the weight of the lever itself is considered.

SECTION SECOND.

OF THE EQUILIBRIUM OF LEVERS OF THE SECOND ORDER.

16. In a lever of the second order, D, according to our notation, is equal to the whole length of the lever; therefore, its whole weight if considered uniform, is represented by $D\varphi$, and its moment when reduced to the centre of gravity is $\frac{1}{2}D^2\varphi$; but by equation (*a*), the moment of the power is pD, and that of the weight or resistance rd; now, by our principle, the moment of the power must be equal to the sum of the moments of the resistance, together with the moment of the weight of the lever; that is

$$2pD = 2rd + D^2\varphi. \qquad (c)$$

This is the equation of equilibrium for a lever of the second order, when its own weight is taken into the account, and the several terms which it involves are separately developed in the following problems.

17. PROBLEM 1. *Given the magnitude of the power p, its distance from the centre of motion* D; *the magnitude of the resistance r, and φ the weight of an unit in length of the lever expressed in the same terms as p and r; to find d, the distance from the centre of motion at which the weight or resistance acts.*

From the general equation of equilibrium (*c*), by transposition and division we obtain

$$d = \frac{D(2p - D\varphi)}{2r},$$

a simple equation, from which the value of *d* can easily be found.

The rule in words at length may be expressed as follows.

RULE. *From twice the magnitude of the power, subtract the whole weight of the lever, and multiply the remainder by the length of the lever, or the distance from the fulcrum or centre of motion at which the power acts; then, divide the product by twice the magnitude of the resistance, for the distance required.*

Example 1. In a lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the fulcrum or centre of motion, is found to balance a weight or resistance of 112 pounds; at what distance from the centre of motion does the resistance act, supposing the whole weight of the lever to be 2 pounds?

By the rule we have

Twice the power = 4 pounds,
Weight of the lever = 2 pounds,

Difference = 2
Length of the lever = 56 inches, multiply

Twice the resistance = 224)112($\frac{1}{2}$ an inch for the distance sought.

Example 2. In a lever of the second order, a power of 56 pounds applied at the distance of 60 inches from the centre of motion, is found to equipoise a weight of 2240 pounds; at what distance from the centre of motion is the weight applied, supposing the lever itself to weigh 25 pounds?

By the rule we have

Twice the power = 112 pounds,
Weight of the lever = 25 pounds,

Difference = 87
Length of the lever = 60 inches, multiply

Twice the resistance = 4480)5220(1·165 inches for the distance sought.

Corol. From the above operations, it may easily be perceived, that, if the whole weight of the lever be equal to twice the power applied, the point where the resistance acts will coincide with the fulcrum, and in that case the power will just support the lever alone, one half of its weight being transferred to the fulcrum, and the other half sustained by the power. But if the whole weight of the lever exceeds twice the power applied, the value of d, or the distance between the centre of motion and the point of resistance, becomes negative, in which case, no equilibrium obtains for a lever of the second order; for, the point of resistance being transferred to the other side of the fulcrum, identifies it with the lever of the first order already considered, from which we conclude, that, when a lever of the second order is kept in equilibrio, the whole weight of the lever itself must either be equal to, or less than, twice the power.

18. Problem 2. *Given the magnitude of the weight or resistance r, its distance from the centre of motion d; the magnitude of the power p, and φ, the weight of an unit in length of the lever, expressed in the same terms as p and r; to find D, the distance from the centre of motion at which the power acts.*

From the general equation of equilibrium (c), by transposition and division we obtain

$$\text{D}^2-\frac{2p}{\varphi}\text{D}=-\frac{2rd}{\varphi},$$

an adfected quadratic equation, from which the value of D is to be determined; the method of performing the reduction is described under equation (b), for the lever of the first order, and need not here be repeated.

EXAMPLE 1. In a lever of the second order, a resistance or weight of 112 pounds acting at the distance of half an inch from the centre of motion, balances a power of 2 pounds; at what distance from the centre of motion does the power act, supposing one inch in length of the lever to weigh the one twenty-eighth part of a pound?

Here we have $r=112$; $d=\frac{1}{2}$; $p=2$, and $\varphi=\frac{1}{28}$; consequently, by substitution, the equation becomes

$$\text{D}^2-112\text{D}=-3136,$$

complete the square and we have

$$\text{D}^2-112\text{D}+3136=0$$

extract the square root of both sides and we get

$$\text{D}-56=0$$

transpose, and we finally have $\text{D}=56$ inches for the distance at which the power acts.

EXAMPLE 2. In a lever of the second order, a resistance or weight of 2240 pounds acting at the distance of 1.165 inches from the centre of motion, balances a power of 56 pounds; at what distance from the centre of motion is the power applied, supposing one inch in length of the lever to weigh five-twelfths of a pound?

Here we have $r=2240$; $d=1.165$; $p=56$, and $\varphi=\frac{5}{12}$; consequently, by substitution, the equation becomes

$$\text{D}^2-268.8\text{D}=-12528,$$

complete the square and we have

$$\text{D}^2-268.8\text{D}+18063.36=5535.36,$$

extract the square root of both sides and we obtain

$$\text{D}-134.4=\pm 74.4,$$

transpose, and we finally have $\text{D}=60$, or 208.8 inches for the distance at which the power acts. Hence it appears, that, in a lever of the second order, there are two points at which the proposed power will balance the resistance,* this does not take place in a lever of the first order, for, although the equation that involves the distance of the point where the power acts from the fulcrum, has two roots; yet, from the form of the equation, one of them is positive and the other negative, and should the negative

* If the square of the power be equal to twice the moment of the resistance, drawn into the weight of an unit in length of the lever, there is only one point where the equilibrium obtains; for then the two roots of the equation are equal, this circumstance occurs in the first example preceding.

root be employed, the lever would then be transformed into that of the second or third order, according as the root so employed, is greater or less than the distance of the point where the resistance acts from the centre of motion.

19. To show that the equilibrium holds for both the values of D above determined, we have only to substitute the numbers given in the example, and those determined by the operation, accordingly as their representatives are combined in the equation marked (c), which equation, as we have elsewhere stated, involves the conditions of equilibrium for a lever of the second order, when the effect produced by the weight of the lever is taken into account.

For the first root or value of D, viz. 60 inches, we have
$$2\times 56\times 60=(2\times 2240\times 1\cdot 165)+(60\times 60\times \tfrac{5}{12})=6720.$$
For the second root or value of D, viz. 208·8 inches, we have $2\times 56\times 208\cdot 8=(2\times 2240\times 1\cdot 165)\times(208\cdot 8\times 208\cdot 8\times \tfrac{5}{12})=23385\cdot 6\cdot$

Hence, it is clear, that in both cases, the equilibrium obtains; and we may further observe, that, if the power be applied at any distance from the centre of motion, greater than 60 and less than 208.8 inches, it will prevail and motion take place.

20. PROBLEM 3. *Given the magnitude of the power p, its distance from the centre of motion* D; *the distance of the weight or resistance from the centre of motion d, and φ, the weight of an unit in length of the lever, expressed in the same terms as p and r; to find r the magnitude of the resistance.*

From the general equation of equilibrium (c), by transposition and division we obtain
$$r=\frac{\text{D}(2p-\text{D}\varphi)}{2d},$$
a simple equation, from which the value of r can easily be determined.

The rule in words at length may be expressed as follows.

RULE. *From twice the magnitude of the power, subtract the whole weight of the lever, and multiply the remainder by the length of the lever, or the distance from the fulcrum or centre of motion at which the power acts; then, the product being divided by twice the distance of the weight or resistance from the centre of motion, will give the magnitude of the resistance sought.*

EXAMPLE 1. In a lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the centre of motion, is found to balance a weight or resistance acting at the distance of half an inch; what is the magnitude of the resistance, supposing the whole weight of the lever to be 2 pounds?

By the rule we have

Twice the power $=$ 4 pounds,
Weight of the lever $=$ 2 pounds,
Difference $=$ 2
Length of the lever $=$ 56 inches, multiply.

Twice the distance of the resistance $=1)112(112$ pounds for the resistance sought.

Example 2. In a lever of the second order, a power of 56 pounds acting at the distance of 60 inches from the centre of motion, is found to equipoise a weight or resistance acting at the distance of 1.165 inches; what is the magnitude of the weight or resistance, supposing the whole weight of the lever to be 25 pounds?

By the rule we have

Twice the power $=$ 112 pounds,
Weight of the lever $=$ 25 pounds,
Difference $=$ 87
Length of the lever $=$ 60 inches, multiply.

Twice the distance of the resistance $=2.33)5220(2240$ pounds very nearly for the resistance sought.

From these two examples it is manifest, that if twice the power is just equal to the weight of the lever, the resistance becomes infinitely small, or rather vanishes altogether; consequently, when this is the case, the power is only sufficient to sustain the lever in equilibrio; but if twice the power is less than the weight of the lever, the resistance becomes negative, in which case the weight of the lever prevails and the equilibrium is destroyed.

21. Problem 4. *Given the magnitude of the weight or resistance r, its distance from the fulcrum or centre of motion d; the distance of the power from the centre of motion* D, *and φ, the weight of an unit in length of the lever, expressed in the same terms as p and r; to find p the magnitude of the power.*

From the general equation of equilibrium (c), by division we obtain

$$p=\frac{(2rd+\text{D}^2\varphi)}{2\text{D}},$$

a simple equation, from which the value of p can easily be determined.

The rule in words at length may be expressed as follows.

Rule. *To twice the moment of the weight or resistance, add twice the moment of the weight of the lever, and divide the sum by twice the distance from the centre of motion, of the point at which the power is applied, and the quotient will give the magnitude of the power required.*

EXAMPLE 1. In a lever of the second order, a weight or resistance of 112 pounds acting at the distance of half an inch from the centre of motion, is found to balance a certain power acting at the distance of 56 inches; what is the magnitude of the power, supposing the whole weight of the lever to be 2 pounds?

By the rule we have

Twice the moment of the resistance, $2 \times 112 \times \frac{1}{2} = 112$,
Twice the moment of the weight of the lever, $2 \times 2 \times 28 = 112$,
Twice the distance of the power $= 112)\overline{224}(2$
pounds, the power sought.

EXAMPLE 2. In a lever of the second order, a weight or resistance of 2240 pounds acting at the distance of $1 \cdot 165$ inches from the fulcrum or centre of motion, is found to equipoise a certain power acting at the distance of 60 inches; what is the magnitude of the power, supposing the whole weight of the lever to be 25 pounds?

By the rule we have

Twice the moment of the resistance, $2 \times 2240 \times 1 \cdot 165 = 5220$
Twice the moment of the weight of the lever, $2 \times 30 \times 25 = 1500$
Twice the distance of the power, $2 \times 60 = 120)\overline{6720}(56$
pounds, the power sought.

In the second example to the last problem, it was shown, that, besides the distance of 60 inches from the centre of motion, there is another point at the distance of 208.8 inches, where the moment of the power is equal to the sum of the moments of the resistance and weight of the lever; let us therefore employ the distance of 208.8 inches, and inquire what must be the magnitude of the power to obtain an equilibrium. In this case, the whole weight of the lever is 87 pounds.

Therefore, by the rule we have

Twice the moment of the resistance, $2 \times 2240 \times 1{\cdot}165 = 5220$
Twice the moment of the weight of
the lever, $2 \times 87 \times 104{\cdot}4 = 18165{\cdot}6$
Twice the distance of the power, $2 \times 208.8 = 417.6)\overline{23385{\cdot}6}(56$
pounds for the power sought, the same as before.

22. PROBLEM 5. *Given the magnitude of the power p, its distance from the fulcrum or centre of motion* D; *the magnitude of the resistance r, and its distance from the centre of motion d; to find φ, the weight of an unit in length of the lever.*

From the general equation of equilibrium (c), by transposition and division we obtain

$$\varphi = \frac{2(p\text{D} - rd)}{\text{D}^2},$$

a simple equation, from which the value of φ can easily be determined.

The rule in words at length may be expressed as follows.

RULE. *Divide twice the difference of the moments of the power and the resistance, by the square of the distance at which the power acts from the centre of motion, and the quotient will be the weight of an unit in length of the lever.*

EXAMPLE 1. In a lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the centre of motion, is found to balance a weight or resistance of 112 pounds acting at the distance of half an inch; what is the weight of one inch of the lever, and what is its whole weight?

By the rule we have

$$\begin{array}{lr} \text{The moment of the power,} & 2\times 56=112 \\ \text{The moment of the resistance,} & 112\times \tfrac{1}{2}= \underline{\ 56} \\ & \text{Difference} = 56 \\ & 2 \\ \text{The square of the distance at which the} & \overline{} \end{array}$$

power acts is $56^2=3136)112(\frac{1}{28}$ of a pound per inch in length; but the whole length of the lever is 56 inches; consequently its whole weight is 2 pounds.

EXAMPLE 2. In a lever of the second order, a power of 56 pounds acting at the distance of 60 inches from the centre of motion, is found to equipoise a weight or resistance of 2240 pounds acting at the distance of 1.165 inches; what is the weight of one inch in length of the lever, and what is its whole weight?

By the rule we have

$$\begin{array}{lr} \text{The moment of the power,} & 56\times 60=3360 \\ \text{The moment of the resistance,} & 2240\times 1\cdot165=\underline{2610} \\ & \text{Difference} = 750 \\ & 2 \\ \text{The square of the distance at which the} & \overline{} \end{array}$$

power acts, $60\times 60=3600)1500(\frac{5}{12}$ of a pound per inch in length; but the whole length is 60 inches; consequently, the whole weight is 25 pounds.

SECTION THIRD.

OF THE EQUILIBRIUM OF LEVERS OF THE THIRD ORDER.

23. Having thus developed separately the values of the several terms which compose the equations of equilibrium for levers of the first and second order, we proceed in the next place to inquire, what are the conditions of equilibrium in a lever of the third order, or that which has the power existing between the centre of motion and the resistance. Now, in a lever of the third order, *d*, according to our notation, being equal to the distance between the centre of motion and the point of resistance, must also be equal to the

length of the lever, the resistance being supposed to act at its extremity; therefore, the whole weight of the lever, if considered uniform, is expressed by $d\varphi$, and its moment when reduced to the centre of gravity is $\frac{1}{2}d^2\varphi$; but by equation (a) the moment of the power is pD, and that of the resistance rd, and by the fundamental principle, the moment of the power, in the case of equilibrium, must be equal to the sum of the moments of the resistance, together with the moment of the weight of the lever; that is,

$$2p\text{D}=2rd+d^2\varphi. \qquad (d)$$

This is the equation which expresses the conditions of equilibrium for a lever of the third order, it is symmetrical with the general equation for a lever of the second order, and the several members, or terms of which it is composed, are separately developed in the following problems.

24. Problem 1. *Given the magnitude of the power p, its distance from the centre of motion* D; *the magnitude of the resistance r, and φ, the weight of an unit in length of the lever expressed in the same terms as p and r; to find d, the distance from the centre of motion at which the weight or resistance acts.*

From the general equation of equilibrium (d), by division we obtain

$$d^2+\left(\frac{2r}{\varphi}\right)d=\frac{2p\text{D}}{\varphi},$$

an adfected quadratic equation, from which the value of d is to be determined, by the method described under equation (b), for the lever of the first order.

Example 1. In a lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the fulcrum or centre of motion, is found to balance a weight or resistance of 16 pounds; at what distance from the centre of motion does the resistance act, supposing one inch in length of the lever to weigh half a pound?

Here we have $p=256$; $\text{D}=2$; $r=16$, and $\varphi=\frac{1}{2}$; consequently by substitution, the equation becomes

$$d^2+64d=2048,$$

complete the square and we have

$$d^2+64d+1024=3072,$$

extract the square root of both sides and we get

$$d+32=\pm 55.42,$$

transpose, and we finally obtain

$d=23.42$, or -87.42 inches, for the distance from the centre of motion at which the weight or resistance acts. It is easy to perceive, that the positive value 23.42 only, can obtain in a lever of the third order, for if the negative value -87.42 were adopted, it would transfer the point of resistance beyond the centre of motion, and the lever would consequently coincide with that of the first order.

EXAMPLE 2. In a lever of the third order, a power of 2240 pounds acting at the distance of 1.165 inches from the centre of motion, is found to equipoise a weight or resistance of 56 pounds; at what distance from the centre of motion does the resistance act, supposing one inch in length of the lever to weigh five-twelfths of a pound?

Here we have $p=2240$; $\text{D}=1.165$; $r=56$, and $\varphi=\frac{5}{12}$; consequently, by substitution, the equation becomes

$$d^2+268.8d=12528,$$

complete the square and we have

$$d^2+268.8d+18063.36=30591.36,$$

extract the root of both sides and we obtain

$$d+134.4=\pm 174.89,$$

transpose, and we finally obtain

$d=40.49$, or -309.29 inches, for the distance sought; but as we observed in the last example, the positive value only can be adopted in a lever of the third order, the negative value belonging to one of the first.

25. PROBLEM 2. *Given the magnitude of the weight or resistance r, its distance from the centre of motion d; the magnitude of the power p, and φ, the weight of an unit in length of the lever, expressed in the same terms as p and r; to find* D, *the distance from the centre of motion at which the power acts.*

From the general equation of equilibrium (d), by division we obtain

$$\text{D}=\frac{d(2r+d\varphi)}{2p},$$

a simple equation, from which the value of D can easily be found.

The rule in words at length may be expressed as follows,

RULE. *To twice the magnitude of the resistance, add the whole weight of the lever, multiply the sum by the length of the lever, or the distance from the centre of motion at which the weight acts; then divide the product by twice the power for the distance sought.*

EXAMPLE 1. In a lever of the third order, a resistance or weight of 16 pounds acting at the distance of 23.42 inches from the centre of motion, is found to balance a power of 256 pounds; at what distance from the centre of motion is the power applied, supposing one inch in length of the lever to weigh half a pound?

By the rule we have

Twice the resistance = 32 pounds,
Weight of the lever = 11·71+ pounds,
Sum = 43·71+
Length of the lever = 23·42+ inches, multiply.
Twice the power = 512)1024 (2 inches, the distance sought.

EXAMPLE 2. In a lever of the third order, a resistance or weight of 56 pounds acting at the distance of 40·49 inches from the fulcrum or centre of motion, is found to equipoise a power of 2240 pounds; at what distance from the centre of motion is the power applied, supposing one inch in length of the lever to weigh five-twelfths of a pound?

By the rule we have

Twice the resistance = 112 pounds,
Weight of the lever = 16·79+ pounds,
Sum = 128·79+
Length of the lever = 40·49+ inches, multiply.
Twice the power=4480) 5220 (1·165 inches the distance sought.

26. PROBLEM 3. *Given the magnitude of the power p, its distance from the centre of motion* D; *the distance of the weight or resistance from the centre of motion d, and φ, the weight of an unit in length of the lever, expressed in the same terms as p and r; to find r, the magnitude of the resistance.*

From the general equation of equilibrium (d), by transposition and division we obtain

$$r=\frac{(2p\mathrm{D}-d^2\varphi)}{2d}$$

a simple equation, from which the value of r can easily be determined.

The rule in words at length may be expressed as follows.

RULE. *From twice the moment of the power, subtract twice the moment of the weight of the lever, and divide the remainder by twice the distance from the centre of motion at which the resistance acts, and the quotient will be the magnitude of the resistance sought.*

EXAMPLE 1. In a lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the centre of motion, is found to balance a weight or resistance, acting at the distance of 23·42 + inches; what is the magnitude of the resistance, supposing the lever to be 11·71+ pounds?

By the rule we have

Twice the moment of the power, 256 × 2 × 2 = 1024,
Twice the moment of the lever, 11·71 × 11·71 × 2 = 274.26,
Twice the distance of the resistance, 23·42 × 2 = 46·84) 749·74(16 pounds, the resistance sought.

EXAMPLE 2. In a lever of the third order, a power of 2240 pounds acting at the distance of 1.165 inches from the centre of motion, is found to equipoise a weight or resistance acting at the

distance of 40.49+ inches; what is the magnitude of the resistance, supposing the weight of the lever to be 16.79 pounds?

By the rule we have

Twice the moment of the power, $2240 \times 1.165 \times 2 = 5220$,
Twice the moment of the lever, $16.79 \times 20.245 \times 2 = \underline{\ 679.82}$
Twice the distance of the resistance, $40.49 \times 2 = 80.98$)4540.18(56
pounds, the resistance sought.

It is obvious from these two examples, that if the moment of the power does not exceed the moment of the weight of the lever, it must either be equal to it or less; if equal, the resistance vanishes, and in that case, the power is just sufficient to sustain the lever in equilibrio; but if less, the resistance becomes negative and the weight of the lever prevails.

27. Problem 4. *Given the magnitude of the resistance* r, *its distance from the centre of motion* d; *the distance of the power from the centre of motion* D, *and* φ, *the weight of an unit in length of the lever, expressed in the same terms as* p *and* r; *to find* p *the magnitude of the power.*

From the general equation of equilibrium (d), by division we obtain

$$p = \frac{d(2r + d\varphi)}{2D},$$

a simple equation, from which the value of p can easily be found.

The rule in words at length may be expressed as follows.

Rule. *To twice the magnitude of the resistance, add the whole weight of the lever, multiply the same by the length of the lever, or distance from the centre of motion at which the resistance acts; then divide the product by twice the distance of the power from the centre of motion, and the quotient will be the magnitude of the power required.*

Example 1. In a lever of the third order, a resistance or weight of 16 pounds, acting at the distance of 23.42+ inches from the centre of motion, is found to balance a certain power acting at the distance of 2 inches; what is the magnitude of the power, supposing the whole weight of the lever to be 11.71+pounds?

By the rule we have

Twice the resistance = 32 pounds,
Weight of the lever = 11.71 pounds,
Sum = 43.71
Length of the lever = 23.42 inches, multiply.
Twice the distance of the power = 4)1024(256 pounds for the power required.

EXAMPLE 2. In a lever of the third order, a resistance or weight of 56 pounds, acting at the distance of 40.49 inches from the centre of motion, is found to equipoise a certain power acting at the distance of 1.165 inches; what is the magnitude of the power, supposing the weight of the lever to be 16.79 pounds?

By the rule we have

Twice the resistance = 112 pounds,
Weight of the lever = 16.79 pounds,
Sum = 128.79
Length of the lever = 40.49+ inches, multiply.
Twice the distance of the power=2·33) 5220 (2240 pounds for the power required.

28. PROBLEM 5. *Given the magnitude of the power p, its distance from the centre of motion* D; *the magnitude of the resistance r, and its distance from the centre of motion d; to find φ, the weight of an unit in length of the lever.*

From the general equation of equilibrium (*d*), by transposition and division we obtain

$$\varphi=\frac{2(p\text{D}-rd)}{d^2},$$

a simple equation from which the value of φ can easily be found.

The rule in words at length may be expressed as follows.

RULE. *Divide twice the difference of the moments of the power and the resistance, by the square of the distance from the centre of motion at which the weight or resistance acts, and the quotient will be the weight of an unit in length of the lever.*

EXAMPLE 1. In a lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the centre of motion, is found to balance a weight or resistance of 16 pounds acting at the distance of 23.42 inches; what is the weight of an inch in length of the lever, and what is its whole weight?

By the rule we have

The moment of the power, $2\times256 = 512$
The moment of the resistance, $23.42\times16 = 374.72$
Difference = 137.28

The square of the distance at which the resistance acts, 23.42^2 } $=549.12)274.56(\frac{1}{2}$ of a pound, the weight of one inch in length of the lever; but the whole length of the lever is 23.42 inches; consequently, its whole weight is 11.71 pounds.

EXAMPLE 2. In a lever of the third order, a power of 2240 pounds acting at the distance of 1.165 inches from the centre of

motion, is found to equipoise a resistance of 56 pounds acting at the distance of 40.49 inches; what is the weight of an inch in length of the lever, and what is its whole weight?

By the rule we have

The moment of the power, $2240 \times 1.165 = 2610$
The moment of the resistance, $56 \times 40.49 = 2267.44$
Difference $= 342.56$
2

The square of the distance at which the resistance acts, 40.49^2 $\Big\} = 1639.44)\ 685.12(\frac{5}{12}$ of a pound very nearly, the weight of one inch in length of the lever; but the whole length of the lever is 40.49 inches; consequently, its whole weight is 16.79 pounds.

29. Having thus determined the value of each quantity in terms of the rest for the three different orders of levers, both when the effect produced by the weight of the lever itself is taken into the account, and when it is not; we shall in the next place, collect the theorems together, and present at one view, all that is necessary in the calculation of the simple lever.

The equations of equilibrium are,

1. For the three different orders of levers considered without weight, $p\text{D} = rd$, (a),
2. For a lever of the first order — when the weight is taken into account, $2p\text{D} + \text{D}^2\varphi = 2rd + d^2\varphi$, (b),
3. For a lever of the second order — when the weight is taken into account, $2p\text{D} = 2rd + \text{D}^2\varphi$, (c),
4. For a lever of the third order — when the weight is taken into account, $2p\text{D} = 2rd + d^2\varphi$, (d),

Corol. From these four equations the whole of the foregoing theory has been derived; they include every condition of equilibrium, and of course every thing that is necessary to be known respecting the simple lever, when it is considered straight and inflexible, and subjected to the action of parallel forces in the same plane; but when this is not the case, other elements must be introduced agreeably to the principles laid down in the following pages.

Section Fourth.

OF BENDED LEVERS, BOTH IN THE CASE OF PERFECT LEVITY, AND ALSO WHEN THEY ARE SUPPOSED TO POSSESS WEIGHT.

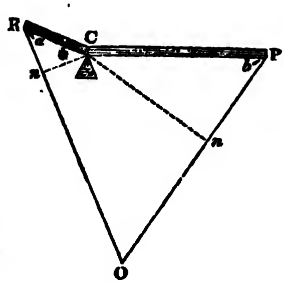
First, *when the bended lever is supposed to possess perfect inflexibility and levity.*

Having in the foregoing pages endeavoured to establish the theory of equilibration for the simple straight lever, both in the case of perfect levity, and also when it is supposed to possess

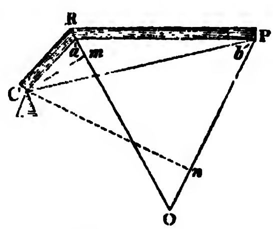
weight; we now proceed to consider the equilibrated conditions of the *bended lever* under the same circumstances of perfect inflexibility and levity, and also when the statical effect of its own weight is taken into the account.

The principle on which the equilibrium of the bended lever depends is simply as follows.

PROPOSITION. *If two forces acting in different directions in the same plane, sustain a bended lever in equilibrio, these forces are to each other, inversely as the straight lines drawn from the fulcrum or centre of motion, perpendicularly to the directions in which they act.*



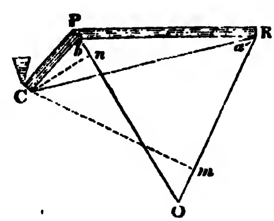
Let the straight lines PO and RO concurring in the point O, in all the figures, represent the directions of the power p, acting at the point P, and the resistance r acting at the point R.* From the point C, the fulcrum or centre of motion, draw the straight lines Cm and Cn perpendicular to the lines of direction RO PO, and draw PC, RC, lines joining the extremities of the lever in the second and third figures. Then, agreeably to the notation adopted for the straight lever,



Put D = PC, the distance between the point P where the power p is applied, and C the fulcrum or centre of motion;

d = RC, the distance between the fulcrum or centre of motion C, and the point R, where the resistance r acts;

and moreover,



Let a = the angle CRO contained between the straight line RO, the direction of the resistance, and CR the straight line joining the point of resistance R, with the fulcrum or centre of motion C;

b = the angle CPO contained between the straight line PO, the direction of the power, and PC the straight line joining the centre of motion C with the point P where the power is applied.

* If the forces applied at the points P and R operate as dead weights, the lines of their direction must be parallel to each other and coincident with the direction of gravity; but if they act as living forces, such as may be excited by machinery or animal strength, the lines of direction RO and PO may attain any degree of obliquity whatever; but whether they are parallel or inclined, the principles and conditions of equilibrium remain unaltered.

Then by Plane Trigonometry we have

$$\text{rad.} : \sin. a :: \text{RC} : \text{C}m = \frac{\text{RC} \sin. a}{\text{rad.}},$$

$$\text{rad.} : \sin. b :: \text{PC} : \text{C}n = \frac{\text{PC} \sin. b}{\text{rad.}}$$

If for RC and PC we substitute d and D, and make rad. $= 1$, we get $\text{C}m = d \sin. a$, and $\text{C}n = \text{D} \sin. b$; then, by the principle enunciated above, for the conditions of equilibrium in the bended lever, we have the following analogy, viz.

$$p : r :: d \sin. a : \text{D} \sin. b,$$

or by equating the products of the mean and extreme terms, it becomes

$$p\text{D} \sin. b = rd \sin. a. \qquad (e)$$

This is the equilibrated expression for the bended lever, on the supposition of perfect inflexibility and levity, and the values of the several factors composing the equation are determined in terms of the rest, by the resolution of the following problems.

32. PROBLEM 20. *Given the magnitude of the power p, its distance from the fulcrum or centre of motion* D; *the magnitude of the resistance r, with the angles of direction a and b;* to find d, the distance between the point of resistance and the centre of motion.*

Let both sides of equation (e) be divided by $r \sin. a$, the factors with which the required term is combined; and we obtain

$$d = \frac{p\text{D} \sin. b}{r \sin. a}.$$

The practical rule expressed in words at length is as follows.

RULE. *Multiply the magnitude of the power by its distance from the centre of motion, and again by the natural sine of the angle of its direction; then, divide the product by the magnitude of the resistance, drawn into the natural sine of its angle of direction, and the quotient will give the distance required.*

EXAMPLE 1. In a bended lever of the first order, a power of 7 pounds, acting at the distance of 48 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a resistance of 84 pounds; at what distance from the centre of motion is the resistance applied, its angle of direction being 18° 52′?

$$\text{nat. sin. } 42° 35' = .67666,$$
$$\text{nat. sin. } 18° 52' = .32337;$$

then, by the rule, we have $\frac{7 \times 48 \times .67666}{84 \times .32337} = \frac{2.70664}{0.32337} = 8.37$ inches, the distance required.

* The angles CRO and CPO described in the notation, may, for the sake of brevity, be designated the *angles of direction*, CRO being the direction of the resistance, and CPO the direction of the power.

This question may be resolved a little differently, for since by the arithmetic of sines, $\frac{1}{\sin. a} = \text{cosec.}\, a$, it is obvious that sin. a may be withdrawn from the denominator of the fraction, if cosec. a be substituted in the numerator; the equation expressing the value of d will then become

$$d = \frac{pD}{r}(\sin. b \ \text{cosec.}\ a).$$

And the operation is indicated in the following manner, the natural cosecant of 18° 52′ being 3·09246,

$$\frac{7 \times 48 \times \cdot 67666 \times 3 \cdot 09246}{84} = 8 \cdot 37 \text{ inches, the same as before.}$$

The process wrought out at length by this latter method would be very laborious, because of the number of places in the decimal fractions, but when it is considered with what facility decimal operations can be abbreviated, this objection is of little consequence; but the greatest quantity of labour would probably be avoided by making use of logarithms, especially where the numbers are large. Similar remarks will apply to most of the following examples on the bended lever, they have been rendered necessary by reason of the introduction of trigonometrical quantities to the equation of equilibrium, which in the case of oblique action could not well be avoided.

Example 2. In a bended lever of the second order, a power of 2 pounds, acting at the distance of 56 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a resistance of 112 pounds; at what distance from the centre of motion does the resistance act, supposing its angle of direction to be 38° 24′?

$$\text{nat. sin. } 42^\circ\, 35' = \cdot 67666,$$
$$\text{nat. sin. } 38\ \ 24 = \cdot 62115;$$

then, by the rule, we have $\frac{2 \times 56 \times \cdot 67666}{112 \times \cdot 62115} = \frac{\cdot 67666}{\cdot 62115} = 1 \cdot 09$ inches, the distance required.

Example 3. In a bended lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the fulcrum or centre of motion and directed in an angle of 38° 24′, is found to balance a resistance of 16 pounds; at what distance from the centre of motion is the resistance applied, its angle of direction being 42° 35′?

$$\text{nat. sin. } 38^\circ\, 24' = \cdot 62115,$$
$$\text{nat. sin. } 42\ \ 35 = \cdot 67666,$$

then by the rule we have $\frac{256 \times 2 \times \cdot 62115}{16 \times \cdot 67666} = \frac{19 \cdot 8768}{\cdot 67666} = 29 \cdot 37$ inches, the distance required.

33. By comparing the results of these three examples with the corresponding ones for the straight lever under problem first, the following particulars present themselves, viz.

1. If the angle of direction of the power exceeds that of the resistance or weight, an equilibrium will not obtain unless the point of resistance be removed further from the centre of motion; all other things remaining as in the straight lever.

2. If the angle of direction of the power is less than that of the resistance, an equilibrium will not obtain, unless the point of resistance be brought nearer to the centre of motion; all other things remaining as in the straight lever.

3. If the angles of direction of the power and resistance are equal, an equilibrium obtains under the same circumstances as in the straight lever. These inferences are evident from the construction of the eqûation, and they are alike applicable to the three different orders of levers.

34. PROBLEM 21. *Given the magnitude of the resistance r, its distance from the centre of motion d; the magnitude of the power p; with the angles of direction a and b; to find* D, *the distance between the centre of motion and the point where the power is applied.*

The factors with which the required term is combined in equation (e), are p sin. b; let both sides of the equation be divided by this product and we obtain

$$D=\frac{rd\sin.a}{p\sin.b}.$$

From this equation it is manifest, that if the values of d, p and r are constant, and the same as they are given for the corresponding examples under the second problem for the straight lever, the value of D will vary according to the magnitude of the angles of direction, these being the elements that disturb the equilibrium in the bended lever.

The practical rule which the preceding equation affords may be expressed as follows.

RULE. *Multiply the magnitude of the resistance by its distance from the centre of motion, and again by the natural sine of the angle of its direction; then divide the product by the magnitude of the power, drawn into the natural sine of the angle of its direction, and the quotient will give the distance required.*

EXAMPLE 1. In a bended lever of the first order, a weight or resistance of 84 pounds, acting at the distance of 4 inches from the centre of motion, with an angle of direction equal to 18° 52′, is found to balance a power of 7 pounds; at what distance from centre of motion is the power applied, supposing the angle of its direction to be 42° 35′?

$$\text{nat. sin. } 18^\circ\, 52' = \cdot 32337,$$
$$\text{nat. sin. } 42\ \ 35 = \cdot 67666;$$

then, by the rule we have $\frac{84 \times 4 \times .32337}{7 \times \cdot 67666} = \frac{15 \cdot 5216}{\cdot 67662} = 22{\cdot}94$ inches the distance sought.

The distance for the straight lever is 48 inches, thereby giving a difference of 25.06 inches in the distance of the power from the centre of motion; but when we consider how much the angle of direction of the power exceeds that of the resistance, the difference, great as it is, is not to be wondered at, for if the power should act at right angles to the arm of the lever to which it is applied, while the angle of direction of the resistance vanishes, the distance of the power, the distance of the resistance, and the resistance would all vanish together, the equilibrium obtaining simply by the power resting in a state of quiescence on the centre of motion.

EXAMPLE 2. In a bended lever of the second order, a weight or resistance of 112 pounds, acting at the distance of one inch from the centre of motion, with an angle of direction equal to 38° 24′, is found to balance a power of 2 pounds; at what distance from the centre of motion is the power applied, supposing the angle of its direction to be 42° 35′?

$$\text{nat. sin. } 38^\circ\, 24' = .62115,$$
$$\text{nat. sin. } 42\ \ 35 = .67666;$$

then, by the rule we have $\frac{112 \times 1 \times .62115}{2 \times .67666} = \frac{34.7844}{.67666} = 5.14$ inches, the distance required.

The distance for the straight lever is 56 inches, giving a difference of only 4.6 inches in the distance from the centre of motion for the bended lever; now, the angles of direction differ only by 4° 11′, and we have stated in the third inference to the last problem, that if the angles of direction are equal, an equilibrium will obtain with the same data both for straight and bended levers.

EXAMPLE 3. In a bended lever of the third order, a weight or resistance of 16 pounds, acting at the distance of 32 inches from the fulcrum or centre of motion, with an angle of direction equal to 42° 35′, is found to balance a power of 256 pounds; at what distance from the centre of motion is the power applied, supposing the angle of its direction to 38° 24′?

$$\text{nat. sin. } 42^\circ\, 35' = .67666,$$
$$\text{nat. sin. } 38^\circ\, 24' = .62115;$$

then, by the rule we have $\frac{16 \times 32 \times .67666}{256 \times .62115} = \frac{1.35332}{.62115} = 2.18$ inches, the distance required.

The distance for the straight lever is only 2 inches; consequently, the distance in this case for the bended lever is increasing,

but the reason is obvious; for since the angle of direction of the resistance exceeds that of the power, an equilibrium cannot obtain until the power be so far removed from the centre of motion, as to counterbalance the effect produced by the difference of the angles of direction.

35. Problem 22. *Given the magnitude of the power p, its distance from the centre of motion* D, *and the distance of the resistance from the centre of motion d, with the angles of direction a and b; to find r, the magnitude of the resistance.*

In the general equation of equilibrium (e), the required term in this case, is combined with the factors d sin. a blended together as a simple product; let both sides of the equation be divided by that product, and we obtain

$$r=\frac{p\mathrm{D}\sin. b}{d\sin. a},$$

an equation, to which a similar remark, as that made for the equation in problem 21, will apply.

The practical rule in words at length may be expressed as follows.

Rule. *Multiply the magnitude of the power by its distance from the centre of motion, and again by the natural sine of the angle of its direction; then, divide the product by the distance of the resistance from the centre of motion drawn into the natural sine of the angle of its direction, and the quotient will give the magnitude of the resistance sought.*

Example 1. In a bended lever of the first order, a power of 7 pounds, acting at the distance of 48 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a certain weight acting at the distance of 4 inches; what is the magnitude of the weight, supposing the angle of its direction to be 18° 52′?

$$\text{nat. sin. } 42^\circ\, 35' = .67666,$$
$$\text{nat. sin. } 18\;\;52 = .32337;$$

then, by the rule, we have $\dfrac{7\times 48\times .67666}{4\times .32337}=\dfrac{56.83944}{.32337}=175.77$ pounds, the weight required.

36. The weight or the resistance, in the corresponding example under the third problem for the straight lever, is 84 pounds, in this case, therefore, there is a difference of 91.77 pounds, although the power and the arms of the lever remain the same; the difference consequently proceeds from the magnitude of the angles of direction, and by examining the construction of the equation we soon discover the conditions on which the variation depends; for since the power p, its distance D, and the distance of the resistance d are

constant, the factor $\frac{pD}{d}$ is constant also; hence the variable element is $\frac{\sin. b}{\sin. a}$, and accordingly as sin. b is greater, equal to, or less than sin. a, the element of variation is greater, equal to, or less than unity, and of course the resistance for the bended lever is accordingly greater, equal to, or less than the resistance for the straight lever, all other circumstances being the same; these remarks apply to the two following examples of this problem, and similar remarks apply to problem 23 following.

EXAMPLE 2. In a bended lever of the second order, a power of 2 pounds, acting at the distance of 56 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a weight or resistance acting at the distance of one inch; what is the magnitude of the resistance, supposing the angle of its direction to be 38° 24′?

$$\text{nat. sin. } 42^\circ\,35' = .67666,$$
$$\text{nat. sin. } 38\;\;24 = .62115;$$

then, by the rule, we have $\frac{2\times 56\times .67666}{1\times .62115}=\frac{75.78592}{.62115}=121.52$ pounds, the resistance sought.

The resistance for the corresponding example in the straight lever is 112 pounds, being a difference of 9.52 pounds, produced by a difference of 4° 11′ in the angles of direction.

EXAMPLE 3. In a bended lever of the third order, a power of 256 pounds, acting at the distance of 2 inches from the fulcrum or centre of motion, and directed in an angle of 38° 24′, is found to balance a certain resistance acting at the distance of 32 inches; what is the magnitude of the resistance, supposing the angle of its direction to be 42° 35′?

$$\text{nat. sin. } 38^\circ\,24' = .62115,$$
$$\text{nat. sin. } 42^\circ\,35' = .67666;$$

then, by the rule, we have $\frac{256\times 2\times .62115}{32\times .67666}=\frac{9.9384}{.67666}=14.69$ pounds, the resistance required.

The corresponding resistance for the straight lever is 16 pounds, being a difference of 1.31 pounds, occasioned by a difference of 4° 11′ in the angles of direction; but the resistance in this case is less in the bended lever than it is in the sraight one; whence we infer, that according as the angle of direction of the resistance exceeds, or falls short of that of the power, a lever of the bended form can be made to act with more or less advantage than a straight one; this circumstance may be useful in constructive mechanics, but at the same time it must not be forgotten, that the arms of the lever are not the same in both cases, it being the distances from the centre of motion at which the forces are applied, where the equality obtains.

37. PROBLEM 23. *Given the magnitude of the resistance r, its distance from the centre of motion d, and the distance of the power from the centre of motion* D, *with the angles of direction a and b; to find p, the magnitude of the power.*

Let both sides of the general equation of equilibrium (e) be divided by the product D sin. b, and it becomes

$$p=\frac{rd\ \text{sin.}\ a}{\text{D sin.}\ b},$$

an equation in which the constant and variable factors are respectively $\frac{rd}{\text{D}}$, and $\frac{\text{sin.}\ a}{\text{sin.}\ b}$, the constant factor denoting the conditions of equilibrium for the straight lever, while the variable element produces the modifications necessary for equilibrating the bended one; the same remark applies to the three equations immediately preceding.

The practical rule may be expressed in words at length as follows.

RULE. *Multiply the magnitude of the resistance by its distance from the fulcrum or centre of motion, and again by the natural sine of the angle of its direction; then divide the product by the distance at which the power is applied drawn into the natural sine of the angle of its direction, and the quotient will give the magnitude of the power required.*

EXAMPLE 1. In a bended lever of the first order, a resistance of 84 pounds, acting at the distance of 4 inches from the fulcrum or centre of motion, and directed in an angle of 18° 52′, is found to balance a certain power acting at the distance of 48 inches; what is the magnitude of the power, its angle of direction being 42° 35′?

$$\text{nat. sin. } 18^\circ\ 52' = .32337,$$
$$\text{nat. sin. } 42\ \ 35 = .67666;$$

then, by the rule we have $\frac{84\times4\times.32337}{48\times.67666}=\frac{2.26359}{.67666}=3.34$ pounds, the power required.

The power for the corresponding example in problem 4 for the straight lever is 7 pounds; hence again, the effect of the oblique action is manifest.

EXAMPLE 2. In a bended lever of the second order, a weight or resistance of 112 pounds, acting at the distance of one inch from the fulcrum or centre of motion, and directed in an angle of 38° 24′, is found to balance a certain power acting at the distance of 56 inches; what is the magnitude of the power, supposing its angle of direction to be 42° 35′?

$$\text{nat. sin. } 38^\circ\ 24' = \cdot62115,$$
$$\text{nat. sin. } 42\ \ 35 = \cdot67666;$$

then, by the rule we have $\frac{112\times1\times\cdot62115}{56\times\cdot67666}=\frac{1\cdot2423}{.67666}=1.84$ pounds, for the power required.

The power for the straight lever is 2 pounds, giving a difference of only 0.16 of a pound; the approximate equality, as may be learned from former remarks, arising from the small difference that obtains in the angles of direction.

Example 3. In a bended lever of the third order, a weight or resistance of 16 pounds, acting at the distance of 32 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a certain power acting at the distance of 2 inches; what is the magnitude of the power, supposing the angle of its direction to be 38° 24′?

$$\text{nat. sin. } 42^\circ\,35' = .67666,$$
$$\text{nat. sin. } 38\ \ 24 = .62115;$$

then, by the rule we have $\dfrac{16 \times 32 \times .6766}{2 \times .62115} = \dfrac{173.22496}{.62115} = 278.88$ pounds, the power required.

The corresponding power for the straight lever is 256 pounds, being 22.88 pounds less than what is required to maintain the equilibrium in the bended lever, when the angle of direction of the resistance exceeds that of the power by 4° 11′.

The four foregoing problems have their equivalents in the case of the straight lever, but the two which follow, having reference to the angles of direction, were not there considered.

38. Problem 24. *Given the magnitude of the power p, its distance from the fulcrum or centre of motion D; the magnitude of the resistance r, its distance from the centre of motion d, and b the angle of direction of the power; to find a the angle of direction of the resistance.*

Let both sides of the general equation of equilibrium (e), be divided by rd, and we obtain

$$\sin. a = \frac{pD \sin. b}{rd}.$$

The practical rule afforded by this expression is as follows.

Rule. *Multiply the magnitude of the power by its distance from the centre of motion, and again by the natural sine of the angle of its direction; then, divide the product by the resistance drawn into its distance from the centre of motion, and the quotient will be the natural sine of the angle of direction sought.*

Example 1. In a bended lever of the first order, a power of 7 pounds, acting at the distance of 48 inches from the centre of motion and directed in an angle of 42° 35′, is found to balance a weight or resistance of 84 pounds acting at the distance of 8.37 inches; what is the angle of direction at which the resistance acts?

nat. sin. $42^\circ\ 35' = .67666$:

then, by the rule we have $\frac{7 \times 48 \times .67666}{84 \times 8.37} = \frac{2.70664}{8.37} = .32337$, the natural sine of $18^\circ\ 52'$; therefore, if the resistance be inclined in an angle of $18^\circ\ 52'$ to the arm of the lever to which it is applied, it will maintain an equilibrium with the power inclined in an angle of $42^\circ\ 35'$ with the other arm; and this will hold in a lever of this order, whatever may be the degree of bending or the inclination of the arms.

EXAMPLE 2. In a bended lever of the second order, a power of 2 pounds acting at the distance of 56 inches from the centre of motion, and directed in an angle of $42^\circ\ 35'$, is found to balance a weight or resistance of 112 pounds acting at the distance of 1.09 inches; what is the angle of direction?

nat. sin. $42^\circ\ 35' = .67666$:

then, by the rule we have $\frac{2 \times 56 \times .67666}{112 \times 1.09} = \frac{.67666}{1.09} = .62115$, the natural sine of $38^\circ\ 24'$.

In the preceding example, it was observed, that the equilibrium would obtain with the same angles of direction, whatever might be the inclination of the arms of the lever to one another; this is obvious, for in a lever of the first order, the lengths of the arms are themselves the distances of the forces from the centre of motion, consequently the quantity of inflexion does not alter the distances; this however is not the case with a lever of the second order, for every variation of the inclination, produces a corresponding variation in the distance of the power from the centre of motion, the arms of the lever remaining the same; therefore, in this case, the same angles of directien will not maintain an equilibrium in every position of the arms.

EXAMPLE 3. In a bended lever of the third order, a power of 256 pounds acting at the distance of 2 inches from the centre of motion and directed in an angle of $38^\circ\ 24'$, is found to balance a weight or resistance of 16 pounds acting at the distance of 29.37 inches; what is the angle of direction?

nat. sin. $38^\circ\ 24' = .62115$;

then, by the rule we have $\frac{256 \times 2 \times .62115}{16 \times 29.37} = \frac{19.8768}{29.37} = .67666$, the natural sine of $42^\circ\ 35'$.

Similar observations to those for a lever of the second order will apply here, it being evident, that every change in the inclination of the arms to one another, must produce a corresponding change in the distance of the resistance from the centre of motion.

39. PROBLEM 25. *Given the magnitude of the resistance r, its distance from the centre of motion d; the magnitude of the power p, its distance from the centre of motion* D, *and a the angle of direction of the resistance; to find b, the angle of direction of the power.*

Let both sides of the general equation of equilibrium(e) be divided by pD, and it becomes

$$\sin. b = \frac{rd \sin. a}{p\mathrm{D}},$$

The practical rule which this equation affords, is as follows.

RULE. *Multiply the magnitude of the resistance, by its distance from the centre of motion, and again by the natural sine of the angle of its direction; then divide the product by the magnitude of the power drawn into its distance from the centre of motion, and the quotient will be the natural sine of the angle of direction sought.*

EXAMPLE 1. In a bended lever of the first order, a weight or resistance of 84 pounds, acting at the distance of 4 inches from the centre of motion, and directed in an angle of 18° 52′, is found to balance a power of 7 pounds, acting at the distance of 22.94 inches; what is the angle of direction in which the power is exerted?

$$\text{nat. sin. } 18^\circ\, 52' = .32337;$$

then, by the rule we have $\frac{84 \times 4 \times .32337}{7 \times 22.94} = \frac{15.52176}{22.94} = .67666$, the natural sine of 42° 35′. (See the remarks for the examples in the last problem.)

EXAMPLE 2. In a bended lever of the second order, a weight or resistance of 112 pounds, acting at the distance of one inch from the centre of motion, and directed in an angle of 38° 24′, is found to balance a power of 2 pounds acting at the distance of 51.4 inches; what is the angle of direction?

$$\text{nat. sin. } 38^\circ\, 24' = .62115;$$

then, by the rule we have $\frac{112 \times 1 \times .62115}{2 \times 51.4} = \frac{34.7844}{51.4} = .67666$, the natural sine of 42° 35′. (See the remarks for the examples in the last problem.)

EXAMPLE 3. In a bended lever of the third order, a weight or resistance of 16 pounds, acting at the distance of 32 inches from the centre of motion, and directed in an angle of 42° 35′, is found to balance a power of 256 pounds, acting at the distance of 2.18 inches, what is the angle of direction?

$$\text{nat. sin. } 42^\circ\,35' = .67666:$$

then, by the rule we have $\frac{16 \times 32 \times .67666}{256 \times 2.16} = \frac{1.75332}{.2.16} = .62115$, the natural sine of $38^\circ\ 24'$.

Thus have we determined separately, the values of the several factors that compose the equation of equilibrium for the bended lever, when it is considered as being destitute of weight; we shall therefore, in the next place, proceed to develope the theory when the statical effect produced by the weight of the lever is taken into account.

40. Secondly. *When the statical effect produced by the weight of the lever is taken into the account.*

In order to investigate this case, let φ as before, represent the weight of an unit in length of the lever; then, for a bended lever of the first order, where the lengths of the arms are respectively represented by d and D, the weights will accordingly become $d\varphi$ and $\mathrm{D}\varphi$; but the whole effect produced by the weight of each arm, is equivalent to a new force equal to half that weight applied at the extremity; consequently, in the case of an equilibrium for a bended lever of the first order, we have, according to our fundamental principle

$$p + \tfrac{1}{2}\mathrm{D}\varphi : r + \tfrac{1}{2}d\varphi :: d \sin. a : \mathrm{D} \sin. b,$$

or by throwing the analogy into an equation by multiplying the extremes and means, we obtain

$$(2p\mathrm{D} + \mathrm{D}^2\varphi) \sin. b = (2rd + d^2\varphi) \sin. a. \qquad (f)$$

This is the equilibrated equation for a bended lever of the first order, when the effect produced by means of its own weight is taken into the account, it differs from the corresponding expression for the straight lever, in so far only, as it is influenced by introducing the trigonometrical values of the angles of direction, and the manner of ascertaining the separate values of the several constituent factors will differ a little also; therefore, in order to continue our plan, we shall go on to determine the value of each factor in terms of the rest, and that the subject may possess a little variety, it becomes expedient to propose a new set of examples and resolve them in a variety of ways.

41. PROBLEM 26. *Given the magnitude of the power p, its distance from the centre of motion* D; *the magnitude of the resistance r, and φ the weight of an unit in length of the lever with a and b the angles of direction; to find d, the distance from the fulcrum at which the resistance acts.*

Let both sides of the general equation of equilibrium *(f)*, be divided by sin. a, and it becomes

$$d^2\varphi + 2rd = (2p\mathrm{D} + \mathrm{D}^2\varphi) \sin. b \text{ cosec. } a,$$

complete the square, and we obtain

$$4d^2\varphi^2+8rd\varphi+4r^2=(8p\text{D}\varphi+4\text{D}^2\varphi^2)\sin. b \operatorname{cosec}. a+4r^2,$$

extract the square root of both sides of this equation, and it becomes

$$2d\varphi+2r=2\sqrt{(2p\text{D}\varphi+\text{D}^2\varphi^2)\sin. b \operatorname{cosec}. a+r^2},$$

finally, transpose the term $2r$ and divide by the coefficient 2φ, and there results

$$d=\frac{1}{\varphi}\left\{\sqrt{\text{D}\varphi(2p+\text{D}\varphi)\sin. b \operatorname{cosec}. a+r^2}-r\right\}.$$

expressions of this kind, arising from the resolution of an adfected quadratic equation, are somewhat difficult to reduce to words; nevertheless, a practical rule in words at length may be rendered intelligible in the following manner.

Rule. *To twice the magnitude of the power, add the weight of that arm of the lever to which it is applied; then, multiply together, the sum, the weight of the arm of the lever on which the power acts, the natural sine of the angle of direction of the power, and the natural cosecant of the angle of direction of the resistance, and to the product add the square of the resistance; then, from the square root of this sum, subtract the magnitude of the resistance, and divide the remainder by the weight of an unit in length of the lever, and the quotient will be the length of the arm, or the distance from the centre of motion at which the weight or resistance acts.*

Example 1. In a bended lever of the first order, a power of 156 pounds, applied at the distance of 8 feet from the fulcrum and directed in an angle of 75° 32′, is found to sustain in equilibrio, a load of 556 pounds, directed in an angle of 84° 40′; at what distance from the fulcrum is the load applied, supposing one foot of the lever to weigh 10 pounds?

$$\text{nat. sin. } 75^\circ\, 32' = .96829,$$
$$\text{nat. cosec. } 84\ \ 40 = 1.00435,$$

twice the magnitude of the power $156\times2=312$ pounds,
weight of the arm of the lever on which the power acts $10\times8=80$ lbs.

$$d=\tfrac{1}{10}\left\{\sqrt{\overline{312+80}\times80\times.96829\times1.00435+556^2}-556\right\}=2.678$$

feet, the distance sought.

The operation by the preceding rule is merely indicated in this process, the work being purposely omitted to save room; it would however, be better to substitute the given quantities at once in the original equation, for in that case, the result is found directly by any of the rules laid down for the resolution of adfected quadratic equations in books of Algebra. The same example resolved by this method is as follows:

Substitute the numbers proposed in the example for their representatives in equation *(f)*, and we get, by reversing the equation and dividing by sin. a,

$$d^2+111.2d=313.6 \text{ sin. } b \text{ cosec. } a,$$

but 313.6 sin. b cosec. $a=313.6\times.96829\times\cdot1.00435=304.976$;

consequently we have

$$d^2+111.2d=304.976,$$

complete the square and it becomes

$$d^2+111.2d+3091.36=3396.336,$$

extract the square root of both sides of the equation, and we get

$$d+55.6=58.28,$$

therefore by transposition we obtain

$d=2.68$ feet, the same as before.

But the same thing may be performed analytically in the following manner.

Let x = the distance or length of the arm required in feet, then is $5x$ = half its weight; consequently, by our fundamental principle it is,

$$156+40 : 556+5x :: x \text{ sin. } 84^\circ\ 40' : 8 \text{ sin. } 75^\circ\ 32',$$

and by converting the analogy into an equation by multiplying the extremes and means, we obtain

$$(5x^2+556x) \text{ sin. } 84^\circ\ 40'=1568 \text{ sin. } 75^\circ\ 32'$$

divide by 5 sin. 84° 40′, and it becomes

$$x^2+111.2x=304\cdot976,$$

the very same equation as we reduced above, having x in place of d.

Example 2. In a bended lever of the first order, a power of 212 pounds, applied at the distance of 14 feet from the centre of motion, and directed in an angle of 86° 50′, is required to sustain in equilibrio or balance a load of 8556 pounds directed in an angle of 60° 25′; what length must the arm of the lever be to which the load is applied, that the equilibrium may just obtain, supposing that the lever is uniform, and that one foot of its length weighs 26 pounds?

nat. sin. 86° 50′ = ·99847,

nat. cosec. 60 25 = 1·14989,

twice the magnitude of the power $212\times2=424$ pounds,

weight of the arm on which the power acts $26\times14=364$ pounds.

Then by the rule we have

$$d=\tfrac{1}{26}\left\{\sqrt{\overline{424+364}\times364\times\cdot99847\times1\cdot14989+8556^2}-8556\right\}=$$

·73 of a foot, the distance sought.

In this example, since the numbers are large, it will be found convenient in some parts of the operation to employ logarithms, as follows.

$424+364=788$	log.	2·896526
364	log.	2·561101
86° 50′	log. sin.	9·999336
60 25	log. cosec.	10·060661
natural number 329324	log.	5·517624
$8556^2=73205136$		

sum $=73534460$, the square root of which is 8575·22; consequently, $d=(8575.22-8556)\div 26=.73$ of a foot, or $8\frac{1}{2}$ inches very nearly.

Substitute the proposed numbers in equation (f), and it becomes by reversing the sides,

$$(26d^2+17112d)\ \sin. 60°\ 25'=11032\ \sin. 86°\ 50',$$

or, dividing both sides by 26 sin. 60° 25′, we obtain

$$d^2+\frac{8556}{13}d=487.167,$$

complete the square, and it becomes

$$d^2+\frac{8556}{13}d+\left(\frac{4278}{13}\right)^2=108778.788,$$

extract the square root of both sides of the equation, and we get

$$d+\frac{4278}{13}=329.816,$$

therefore, by transposition we obtain

$d=0.73$ of a foot, the same as before.

Analytically, thus,

Let $x=$ the distance, or length of the arm required in feet, then is $13x=$ half its weight; consequently, by our fundamental principle, it is

$$212+182 : 8556+13x :: x\ \sin. 60°\ 25' : 14\ \sin. 86°\ 50',$$

by multiplying the extremes and means, we get

$$(13x^2+8556x)\ \sin. 60°\ 25'=5516\ \sin. 86°\ 50',$$

and finally, dividing by 13 sin. 60° 25′, it becomes

$$x^2+\frac{8556}{13}x=487\cdot 167,$$

the same equation as before, having x in place of d.

43. Problem 27. *Given the magnitude of the resistance r, its distance from the centre of motion d; the magnitude of the power p, and φ, the weight of an unit in length of the lever, with a and b, the angles of direction; to find* D, *the distance from the fulcrum or centre of motion at which the power acts.*

Let both sides of the general equation of equilibrium (f) be divided by sin. b, and it becomes

$$\varphi D^2+2pD=(2rd+d^2\varphi)\ \sin. a\ \operatorname{cosec.} b,$$

complete the square, and we obtain,

$$4\varphi^2 D^2+8pD\varphi+4p^2=(8rd\varphi+4d^2\varphi^2)\ \text{sin.}\ a\ \text{cosec.}\ b+4p^2,$$

extract the square root of both sides of this equation, and it becomes

$$2\varphi D+2p=2\sqrt{(2rd\varphi+d^2\varphi^2)\ \text{sin.}\ a\ \text{cosec.}\ b+p^2},$$

transpose the term $2p$, and divide both sides by 2φ, the co-efficient of D, and we finally obtain

$$D=\frac{1}{\varphi}\left\{\sqrt{d\varphi(2r+d\varphi)\ \text{sin.}\ a\ \text{cosec.}\ b+p^2}-p\right\}.$$

This equation is perfectly symmetrical with that which we obtained for the value of d in the preceding problem, and consequently the rule will be expressed very nearly in the same words, as follows.

RULE. *To twice the magnitude of the resistance, add the weight of that arm of the lever to which it is applied; then, multiply together, the sum, the weight of the arm of the lever on which the resistance acts, the natural sine of the angle of direction of the resistance, and the natural cosecant of the angle of direction of the power, and to the product add the square of the power; then, from the square root of this sum, subtract the magnitude of the power, and divide the remainder by the weight of an unit in length of the lever, and the quotient will be the length of the arm, or the distance from the fulcrum or centre of motion at which the power is applied.*

EXAMPLE 1. In a bended lever of the first order, a force of 124 pounds, directed in an angle of 80° 4′, is found to counterpoise a load of 2240 pounds, acting at the distance of $2\frac{1}{2}$ feet from the fulcrum or centre of motion, and directed in an angle of 74° 15′; at what distance from the centre of motion is the power applied, supposing one foot of the lever to weigh 18 pounds?

nat. sin. 74° 15′ = 0·96246,

nat. cosec. 80 4 = 1·01522,

twice the magnitude of the resistance, $2240\times 2=4480$ lbs.

weight of the arm on which the resistance acts, $18\times 2\frac{1}{2}=45$ lbs.

then, by the rule, we have

$$D=\left\{\frac{1}{18}\sqrt{4480+45\times 45\times .96246\times 1.01522+124^2}-124\right\}$$

= 18.83 feet, the distance sought.

Substitute the proposed numbers for their representatives in equation (f), and it becomes

$$(18D^2+248D)\ \text{sin.}\ 80°\ 4'=11312.5\ \text{sin.}\ 74°\ 15',$$

or, dividing both sides by 18 sin. 80° 4′, we obtain

$$D^2+\frac{124}{9}D=614.083,$$

complete the square, and it becomes

$$D^2+\frac{124}{9}D+\left(\frac{62}{9}\right)^2=661.54,$$

extract the root of both sides of the equation, and we get

$$D+\frac{62}{9}=25.72.$$

therefore, by transposition we obtain

$D=18.83$ feet, the same as before.

ANALYTICALLY.

Let x=the distance, or length of the arm required, then is $18x$=its whole weight, and consequently,

$9x$=half its weight;

therefore, by our fundamental principle, we have

$$124+9x : 2240+22.5 :: 2.5 \sin. 74^\circ 15' : x \sin. 80^\circ 4',$$

by multiplying the extremes and means, we have

$$(9x^2+124x) \sin. 80^\circ 4'=5656.25 \sin. 74^\circ 15',$$

divide by $9 \sin. 80^\circ 4'$, and we finally obtain

$$x^2+\frac{124}{9}x=614.083,$$

the identical equation obtained above by the method of substitution, having x in place of D.

EXAMPLE 2. In a bended lever of the first order, a force or power of 3 tons acting at right angles to the end of the lever, is found to equipoise a load of 480 tons, applied at the distance of 2 feet from the centre of motion, and directed in an angle of $86^\circ 28'$; at what distance from the centre of motion is the power applied, supposing one foot in length of the lever to weigh 2cwt.?

nat. sin. $86^\circ 28'=0.9981$,

nat. cosec. $90\ \ 0=1$,

twice the magnitude of the resistance $480\times 2=960$ tons,

weight of the arm on which the resistance acts. $1\times 2=0\cdot 2$ tons;

then, by the rule we have

$D=10\left\{\sqrt{960+0\cdot 2\times 0\cdot 2\times 0.9981+3^2}-3\right\}=111\cdot 66$ feet, the distance sought.

Substitute the given numbers for their representatives in equation (f), and it becomes

$$D^2+60D=19204 \sin. 86^\circ 28',$$

the cosecant of 90° being equal to the radius or unity, has no effect in this case, and therefore it disappears from the equation, but the natural sine of $86^\circ 28'$ is 0.9981; hence, our equation is

$$D^2+60D=19167\cdot 5124,$$

complete the square, and it becomes

$$D^2+60D+30^2=20067.5124,$$

extract the root of both sides of the equation, and we get

$$D+30=141\cdot 66,$$

therefore, by transposition we have
$D = 111 \cdot 66$ feet, the same as before.

ANALYTICALLY.

Let x = the distance, or length of the arm required, then is $\frac{1}{20}x$ = half its weight; consequently by the fundamental principle, we have

$$3+\tfrac{x}{20} : 480+\tfrac{1}{10} :: 2 \sin. 86^\circ 28' : x,$$

or by making the product of the mean terms, equal to the product of the extremes, it is

$$x^2+60x = 19204 \sin. 86^\circ 28',$$

but the natural sine of $86^\circ 28'$ is 0.9981, as has already been shewn above; therefore, we have

$$x^2+60x = 19167.5124,$$

the same equation as before, having x in place of D.

In this example, we have supposed the power to be acting at right angles to the lever; if therefore, we regard the resistance as acting in a direction parallel to that of the power, the inclination of the arms of the lever to each other can easily be assigned, for it is equal to $180^\circ-(90-a)$; and the equilibrium would obtain on a straight lever, if its arms were respectively to each other as D to $d \sin. a$.

44. PROBLEM 28. *Given the magnitude of the power p; its distance from the centre of motion* D; *the distance of the weight or resistance from the centre of motion d, and* φ, *the weight of an unit in length of the lever, with a and b, the angles of direction; to find r the magnitude of the resistance.*

Let both sides of the general equation of equilibrium *(f)*, be divided by sin. a, and it becomes

$$2rd+d^2\varphi = (2p\text{D}+\text{D}^2\varphi) \sin. b. \operatorname{cosec.} a,$$

transpose the term $d^2\varphi$ and divide by $2d$, and we obtain

$$r = \frac{\text{D}(2p+\text{D}\varphi) \sin. b \operatorname{cosec.} a - d^2\varphi}{2d}.$$

The practical rule which this equation affords, is as follows.

RULE. *To twice the magnitude of the power, add the weight of that arm of the lever to which it is applied; then, multiply together, the sum, the length of the arm, or distance of the power from the centre of motion, the natural sine of the angle of direction of the power, and the natural cosecant of the angle of direction of the weight or resistance; from the product subtract the distance of the resistance from the fulcrum, drawn into the weight of the arm on which it acts; then, divide the remainder by twice the length of that arm of the lever to which the resistance is applied, and the quotient will be the magnitude of the resistance required.*

EXAMPLE 1. In a bended lever of the first order, a power of 112 pounds applied at the distance of 35 feet from the fulcrum, and directed in an angle of 66°, is found to balance a certain resistance acting at the distance of 3 feet, and directed in an angle of 77°; what is the magnitude of the resistance, supposing that one foot in length of the lever weighs 6 pounds?

$$\text{nat. sin. } 66° = \cdot 91355,$$
$$\text{nat. cosec. } 77 = 1 \cdot 9263,$$

twice the magnitude of the power $\cdot 112 \times 2 = 224$ pounds,
weight of the arm on which it acts $6 \times 35 = 210$ pounds,
the length of the arm at which the resistance acts $=3$ feet, its weight $=3 \times 6 = 18$ pounds, then, $18 \times 3 = 54$;

therefore, by the rule we have

$$r = \frac{\overline{224+210} \times 35 \times \cdot 91355 \times 1.0263 - 54}{2 \times 3} = 2364.63 \text{ pounds, the}$$

resistance sought.

Let the several data be substituted for their representatives in equation (f), and it becomes

$$(6r + 54) \sin. 77° = 15190 \sin. 66°,$$

or dividing both sides by sin. 77° we get

$$6r + 54 = 15190 \sin. 66° \text{ cosec. } 77°,$$

then transposing 54 and dividing by 6 the coefficient of r, we have

$$r = \frac{15790 \times 0.91355 \times 1.6263 - 54}{6} = 2364.63 \text{ pounds, the same as}$$

before.

ANALYTICALLY.

Let x = the resistance required, then we have

$$112 + 105 : x + 9 :: 3 \sin. 77° : 35 \sin. 66°,$$

or by making the product of the mean terms, equal to the product of the extremes, we have

$$3x \sin. 77° + 27 \sin. 77° = 7595 \sin. 66°,$$

transpose the term 27 sin. 77°, and divide by 3 sin. 77°, and we get

$$x = (7595 \sin. 66° \text{ cosec. } 77° - 27) \div 3,$$

which, by employing the numerical values of sin. 66° and cosec. 77° gives $x = 2364.63$ pounds, the same as before.

EXAMPLE 2. In a bended lever of the first order, a power of 12 tons acting at the distance of 20 feet from the centre of motion, and directed in an angle of 86° 10′, is found to equipoise a certain resistance acting at the distance of 43 feet, and directed in an angle of 86° 10′; what is the magnitude of the resistance, supposing one foot of the lever to weigh 112 pounds or $\frac{1}{20}$ of a ton?

By the third inference to the twentieth problem, we learn, that when the power and resistance are inclined at the same angle to their respective arms of the lever, an equilibrium obtains under the same conditions as if the lever were straight; this is also obvious

from equation (f), for since the angle a is equal to the angle b, sin. a must also be equal to sin. b; therefore, by expunging the equal factors, the equation reduces to

$$2p\text{D}+\text{D}^2\varphi=2rd+d^2\varphi,$$

the very same as the equation of equilibrium for a straight lever of the first order, but this coincidence of form in the equation of equilibrium for the straight and the bended lever, in the case of equal angles of direction, has no influence on the rule which we have given for this problem, the operation becomes simplified, in so far as we know, that the product of sin. b by cosec. a is equal to unity, but the rule remains the same.

nat. sin. 86° 10′= ·99776,
nat. cosec. 86 10 =1·00224,

twice the magnitude of the power $12\times 2 = 24$ tons,
weight of the arm on which it acts $\frac{1}{20}\times 20 = 1$ ton,
length of the arm at which the resistance acts=43 feet, its weight $=\frac{43}{20}=2\cdot 15$ tons; therefore, we get

$$r=\frac{\overline{24+1}\times 20\times\cdot 99776\times 1\cdot 00224-92\cdot 45}{86}=4\cdot 74 \text{ tons nearly,}$$

the resistance sought.

Substitute the data in the equation immdediately above, and it becomes

$$86r+92\cdot 45=500,$$

transpose 92·45, and divide by 86, and we have

$$r=\frac{500-92\cdot 45}{86}=4\cdot 74 \text{ tons, the same as before.}$$

Analytically.

Let x=the magnitude of the resistance required;
then, by the fundamental principle of equilibrium, we have

$$12+.5 : x+1.075 :: 43 : 20,$$

and by making the product of the mean terms equal to the product of the extremes, we get

$$43x+46.225=250,$$

transpose 46.225, and divide by 43, and we have

$$x=\frac{250-46.225}{43}=4.74 \text{ tons, as it ought to be.}$$

In the method by substitution, and also in the analytical method, we have omitted the angles of direction altogether, purposely to show, that since they are equal, they have no influence whatever on the result, that coming out the same by the three methods; but it must not be forgotten, that when the angles of direction are of different magnitudes, they must be retained in the general equation.

45. Problem 29. *Given the magnitude of the resistance r, its distance from the fulcrum d; the distance of the power from*

the fulcrum D, *and* φ, *the weight of an unit in length of the lever, with a and b, the angles of direction; to find p, the magnitude of the power.*

Let both sides of the general equation of equilibrium (f) be divided by sin. b, and it becomes

$$2p\text{D}+\text{D}^2\varphi=(2rd+d^2\varphi)\ \text{sin.}\ a\ \text{cosec.}\ b,$$

transpose the term $\text{D}^2\varphi$, and divide by 2D, and we obtain

$$p=\frac{d(2r+d\varphi)\ \text{sin.}\ a\ \text{cosec.}\ b-\text{D}^2\varphi}{2\text{D}},$$

an equation from which the value of p is easily determined.

The practical rule may be expressed as follows.

Rule. *To twice the magnitude of the resistance, add the weight of that arm of the lever to which it is applied; then multiply together, the sum, the length of the arm, or distance of the resistance from the centre of motion, the natural sine of the angle of direction of the resistance, and the natural cosecant of the angle of direction of the power; from the product subtract the distance of the power from the fulcrum, drawn into the weight of that distance; then, divide the remainder by twice the length of the arm at which the power is applied, and the quotient will give the magnitude of the power sought.*

Example 1. In a bended lever of the first order, a weight or resistance of 2240 pounds, acting at the distance of 7 feet from the centre of motion, and directed in an angle of 60°, is found to equipoise a certain power acting at the distance of 21 feet, and directed in an angle of 65°; what is the magnitude of the power, supposing the weight of one foot in length of the lever to be 20 pounds?

$$\text{nat. sin. } 60°= \cdot 86603,$$
$$\text{nat. cosec. } 65 =1\cdot 10338,$$

twice the magnitude of the resistance $2240\times 2=4480$ pounds,
weight of the arm on which it acts $20\times 7=140$ pounds,
length of the arm on which the power acts $=21$ feet, its weight $=420$ pounds; therefore, by the rule we have

$$p=\frac{4480+140\times 7\times\cdot 86603\times 1\cdot 10338-420\times 21}{21\times 2}=525\cdot 77\text{lbs.}$$

the power sought.

Substitute the proposed data for their representatives in the general equation (f), and we get

$$(42p+8820)\ \text{sin. } 65°=32340\ \text{sin. } 60°,$$

divide by sin. 65°, and we obtain

$$42p+8820=32340\ \text{sin. } 60°\ \text{cosec. } 65°;$$

transpose 8820 and divide by 42 the co-efficient of p, and we have

$$p=\frac{32340\times .86603\times 1.10338-8820}{42}=525.77 \text{ pounds, the}$$

same as above.

ANALYTICALLY.

Let $x=$ the power required, then, by our fundamental principle, we have

$$x+210 : 2310 :: 7 \text{ sin. } 60^\circ : 21 \text{ sin. } 65^\circ,$$

or, by converting the analogy into an equation, we obtain

$$21(x+210)\text{ sin. } 65^\circ=16170 \text{ sin. } 60,$$

divide both sides of the equation by 21 sin. 65°, and it becomes

$$x+210=770 \text{ sin. } 60^\circ \text{ cosec. } 65^\circ,$$

transpose the term 210, and we have

$x=770\times\cdot86603\times1\cdot10338-210=525\cdot77$ lbs. as it ought to be.

EXAMPLE 2. In a bended lever of the first order, what power, acting at the distance of 40 feet from the fulcrum, and directed in an angle of 73° 49′, will counterpoise a load of 180 tons acting at the distance of 2 feet, and directed in an angle of 86° 56′, supposing one foot of the lever to weigh 5 cwt. or $\frac{1}{4}$ of a ton?

nat. sin. 86° 56′ = .99857,
nat. cosec. 73 49 = 1.04134,

twice the magnitude of the resistance $180\times2=360$ tons,
weight of the arm on which it acts $2\times\frac{1}{4}=\frac{1}{2}$ of a ton,
length of the arm on which the power acts $=40$ feet, its weight $=10$ tons; therefore, by the rule, we have

$$p=\frac{360+\cdot5\times2\times\cdot99857\times1\cdot04134-400}{40\times2}=4\cdot37 \text{ tons, the weight}$$

required.

Substitute the given numbers in the general equation (*f*), and it becomes

$$(80p+400)\text{ sin. } 73^\circ 49'=721 \text{ sin. } 86^\circ 56',$$

divide both sides by 80 sin. 73° 49′ and we obtain,

$$p+5=9\cdot0125 \text{ sin. } 86^\circ 56' \text{ cosec. } 73^\circ 49';$$

transpose 5 and introduce the numerical values of sin. 86° 56′ and cosec. 73° 49′, and we have

$p=9\cdot0125\times\cdot99857\times1\cdot04134-5=4\cdot37$ tons, the same as above.

ANALYTICALLY.

Let $x=$ the power required, then by our principle, we have

$$(x+5) : 180+\tfrac{1}{4} :: 2 \text{ sin. } 86^\circ 56' : 40 \text{ sin. } 73^\circ 49',$$

or, by making the product of the mean terms, equal to the product of the extremes, we get

$$40(x+5)\text{ sin. } 73^\circ 49'=360\cdot5 \text{ sin. } 86^\circ 56',$$

and dividing both sides of the equation by 40 sin. 73° 49′ it becomes

$x+5=9\cdot0125$ sin. 86° 56′ cosec. 73° 49′, the same as before, having x in place of p.

46. COROL. In practical cases, the angles of direction of the power and resistance will generally be known, it is therefore unne-

cessary to illustrate the method of finding these two elements by numerical examples, for it will seldom be requisite to compute them; we shall therefore content ourselves by exhibiting the expressions that indicate their values, as derived from the general equation of equilibrium in which they are involved.

If both sides of the general equation *(f)*, be divided by $(2rd+d^2\varphi)$, we get

$$\sin. a = \left(\frac{2p\text{D}+\text{D}^2\varphi}{2rd+d^2\varphi}\right)\sin. b;$$

and again, if we divide both sides by $(2p\text{D}+\text{D}^2\varphi)$, we get

$$\sin. b = \left(\frac{2rd+d^2\varphi}{2p\text{D}+\text{D}^2\varphi}\right)\sin. a.$$

These are the expressions by which the angles of direction can be determined, when they are not given *a priori* from the circumstances of construction, and it is presumed, that from the simple nature of the equations, the intelligent reader will find no difficulty in their application.

47. Problem 30. *Given the magnitude of the power p, its distance from the centre of motion* D; *the magnitude of the resistance r, and its distance from the centre of motion d; with a and b, the angles of direction; to find φ, the weight of an unit in length of the lever.*

The general equation *(f)*, in its expanded form, is as below,

$$2p\text{D} \sin. b + \text{D}^2\varphi \sin. b = 2rd \sin. a + d^2\varphi \sin. a,$$

by transposition we have

$$\text{D}^2\varphi \sin. b - d^2\varphi \sin. a = 2rd \sin. a - 2p\text{D} \sin. b,$$

or, since either of the terms on both sides of the equation may exceed the other, the value of the unknown quantity may be thus expressed,

$$\varphi = \frac{2(rd \sin. a \backsim p\text{D} \sin. b)}{\text{D}^2 \sin. b \backsim d^2 \sin. a}.$$

In this equation $p\text{D}$ is the moment of the power, and rd the moment of the resistance, the same as we denoted them when treating of the straight lever; but since these products are here modified by the angles of direction, the terms $p\text{D} \sin. b$ and $rd \sin. a$ may, in the case of the bent lever, very properly be denominated the moments of the power and resistance; the practical rule will then be expressed in words at length as follows.

Rule. *Multiply the squares of the distances of the power and resistance from the centre of motion by the natural sines of the respective angles of direction, and find the difference of the products according as the one or other is the greater; then, divide twice the difference of the*

moments of the power and resistance, by the difference of the products thus found, and the quotient will give the weight of an unit in length of the lever, expressed in the same terms as p and r.

EXAMPLE 1. In a bended lever of the first order, a power of 560 pounds, acting at the distance of 24 feet from the centre of motion, is found to balance a load of 8556 pounds acting at the distance of one foot; what is the weight of one foot of the lever, supposing the angles of direction of the power and resistance to be respectively 80° and 75°?

$$\text{nat. sin. } 80° = \cdot 98481$$
$$\text{nat. sin. } 75 = \cdot 96593;$$

then, by the rule, we have

$$24 \times 24 \times \cdot 98481 = 567 \cdot 24056,$$
$$1 \times 1 \times \cdot 96593 = \quad \cdot 96593,$$
$$\overline{566 \cdot 27463}$$

difference of the products.

$$\text{moment of the power} \quad 560 \times 24 \times \cdot 98481 = 13235 \cdot 8464,$$
$$\text{moment of the resistance} \quad 8556 \times 1 \times \cdot 96593 = 8764 \cdot 4971,$$
$$\overline{4471 \cdot 3493}$$

difference of the moments;

then $\dfrac{4471 \cdot 3493 \times 2}{566 \cdot 27463} = 15 \cdot 79$ pounds, the weight required; the length of both arms of the lever together is 25 feet; consequently, its whole weight is 394·75 pounds; hence, the aggregate weight on the fulcrum is $9510\frac{3}{4}$ pounds.

Another example resolved analytically will here be sufficient.

EXAMPLE 2. In a bended lever of the first order, a power of 4cwt. acting at the distance of 8 feet from the centre of motion, is found sufficient to balance a load of 380cwt. acting at the distance of three feet; what is the weight of the lever, the angles of direction being equal?

It has already been stated, that when the angles of direction are equal, the conditions of equilibrium are the same, whether the lever is straight or bended; therefore,

let x = the weight of one foot in length of the lever,

then according to the fundamental principle, we have

$$4(1+x) : 380 + 1 \cdot 5x :: 3 : 8; \text{ therefore,}$$

by making the product of the mea. terms equal to the product of the extremes,

$$\text{we have } 32(1+x) = 1140 + 4.5x,$$

and by transposition, we get

$$27.5x = 1108,$$

therefore, by division $x = 40.29$ pounds, the weight of one foot, but the whole length of the lever is $8+3=11$ feet, consequently its whole weight is $40 \cdot 29 \times 11 = 443 \cdot 19$ pounds.

48. If the resistance r be withdrawn, the term in which it occurs will vanish, and the general equation *(f)*, becomes, $(2p\text{D}+\text{D}^2\varphi)$ sin. $b=d^2\varphi$ sin, a, the equilibrium being maintained by the effort of the extreme arm of the lever alone; consequently, the value of φ as deduced from this equation is

$$\phi=\frac{2p\text{D sin. } b}{d^2 \text{ sin. } a-\text{D}^2 \text{ sin } b},$$

and when the angles of direction are equal to one another, this expression becomes

$$\varphi=\frac{2p\text{D}}{d^2-\text{D}^2};$$

but $d+\text{D}=$the whole length of the lever, and if w represent its whole weight, we obtain

$$w=\frac{2p\text{D}\,(d+\text{D})}{d^2-\text{D}^2}$$

and because, the product of the sum and difference of any two quantities is equal to the difference of their squares, we have

$$w=\frac{2p\text{D}(d+\text{D})}{(d+\text{D})\,(d-\text{D})};$$

therefore, by reducing the fraction, we finally get

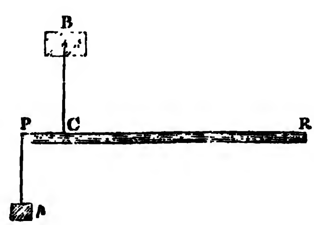
$$w=\frac{2p\text{D}}{d-\text{D}}.$$

The practical rule afforded by this equation may be expressed as follows,

RULE. *Divide twice the moment of the power, by the difference between the arms of the lever, and the quotient will give the whole weight of the beam, in the same name as the power applied.*

This rule was given in another form for the straight lever, (problem 9), and it was there remarked, that it might be applied to the determination of the weight of large uniform bodies, when their weight and magnitude are such, as to render it difficult to weigh them by any other method; since that remark was written, an experiment has been tried, the result of which renders the theory perfectly conclusive.

The experiment was conducted in the following manner; a straight uniform piece of wood exactly $41\frac{1}{8}$ inches in length, was suspended by a small cord to the nail n fixed purposely into the end of a projecting beam B; the point C at which the bar was suspended was exactly 10.9375 inches from P, and consequently, 30.1875 inches from R; the power p, required to bring the rod into the horizontal position was exactly $16\frac{1}{4}$ ounces, and when the rod



was weighed after the experiment it was found to weigh $18\frac{1}{2}$ ounces.

By our rule, we have $w=\dfrac{10\cdot9375\times2\times16\cdot25}{30\cdot1875-10\cdot9375}=18\cdot466$ ounces, a very satisfactory result.

49. This experiment, it will be observed, was made on a straight beam, but the same result would have obtained if the beam had been bent at the point C, provided that the power p and the effort of the arm CR were directed in the same angles; in a bent lever however this may often happen not to be the case, it is therefore necessary to retain the angles of direction, in order that our theorem may apply to all positions of the bended lever, whether the forces act in lines parallel or oblique to each other.

The value of φ deduced from the general equation *(f)*, is, as we have shewn above,

$$\varphi=\frac{2p\text{D}\sin.b}{d^2\sin.a-\text{D}^2\sin.b},$$

and this multiplied by the whole length of the lever, must give its whole weight; but the length is $(d+\text{D})$ and the whole weight is represented by w; consequently, by multiplication we have

$$w=\frac{2p\text{D}\sin.b\,(d+\text{D})}{d^2\sin.a-\text{D}^2\sin.b}.$$

The practical rule afforded by this expression may be given in words as follows.

RULE. *Multiply twice the moment of the power by the sum of the arms, or the whole length of the lever; then, divide the result by the difference of the products that arise, by multiplying the square of the length of each arm of the lever by the natural sine of the adjacent angle of direction, and the quotient will give the whole weight of the lever.*

EXAMPLE 1. In a bended lever of the first order, a power of 224 pounds, or 2cwt. acting at the distance of 3 feet from the centre of motion, and directed in an angle of 70° 30′, is found to sustain the extreme arm of the lever horizontally; what is the whole weight of the lever, supposing the length of the horizontal arm to be 28 feet?

Here, since the extreme arm of the lever is sustained in a horizontal position, the line of its effort must coincide with that of gravity; consequently, the angle of its direction is 90 degrees.

Then, nat. sin. 70° 30′ = .94264,
nat. sin. 90 0 = 1.00000,

whole length of the beam or lever, $28+3=31$ feet;

then, by our rule, we have

$$w=\frac{2\times2\times3\times.94264\times31}{28\times28\times1-3\times3\times.94264}=.452\text{cwt. the}$$

whole weight required.

Extension of the foregoing principle to the determination of the cubical contents of bodies.

50. We shall propose another example of a kind likely to occur very frequently in practice, and since the magnitude of a body can always be determined by having its weight given, it is probable that the same principle may be extended to the determination of the cubical centents of bodies when they are uniform in figure and density.

EXAMPLE 2. A log of American pine 64 feet long, and of uniform section throughout the whole of its length, is suspended from a crane at the distance of 12 feet from one of its ends; what is the weight of the log, and how many cubic feet does it contain, supposing that 8cwt. suspended from the shorter end brings it to an equipoise, its specific gravity being 0·46 when that of water is unity?

In this example, the angles of direction are each equal to 90 degrees, therefore the rule which we have laid down for that case must be applied; hence we get

$$w=\frac{2\times8\times12}{52-12}=4.8\text{cwt. the whole weight of the log.}$$

Now we have 1 : 0.46 :: 62.5 : 28.75 pounds, in a cubic foot of American pine; consequently,

$$\frac{4.8\times112}{28.75}=18.7 \text{ cubic feet.}$$

54. If instead of the resistance, we suppose the power to be withdrawn, the term in which it occurs will vanish, and the general equation (*f*), becomes

$$D^2\varphi \sin. b=(2rd+d^2\varphi) \sin. a,$$

from which, by transposition and division, we find

$$\varphi=\frac{2rd \sin. a}{D^2 \sin. b-d^2 \sin. a},$$

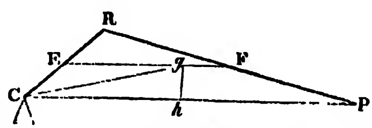
an equation perfectly symmetrical with that which obtained on the supposition that the resistance was withdrawn, therefore, the rules and operations, are on both suppositions the very same.

We shall now proceed to inquire, what are the conditions of equilibrium for a bended lever of the second order, taking into account the effort of its own weight and the effect produced by the action of oblique forces.

SECTION FIFTH.

OF BENDED LEVERS OF THE SECOND ORDER, OR THOSE WHICH HAVE THE RESISTANCE BETWEEN THE CENTRE OF MOTION AND THE POWER.

52. In a bended lever of the second order, the centre of gravity does not exist in any part of the figure itself, but is situated somewhere in the straight line joining the centres of gravity of the two arms, and is consequently either raised or depressed according as the lever is bent downwards or upwards; it therefore becomes necessary to calculate the place of this centre, and reduce it to the proper point in the straight line connecting the power with the centre of motion, for it is at this point that a new force equal to the weight of the lever must be conceived to exert itself, and the moment of this force, added to that of the resistance, must in the case of equilibrium be equal to the moment of the power. Now, since the lever is considered uniform in section and density, the centre of gravity of each arm exists at its middle point, and the common centre of gravity, or that of the whole lever, must therefore be determined by the following construction.



Let CR and PR represent the arms or parts of the lever; then, since the whole is considered uniform in section and density, bisect CR, PR in the points E and F, join EF; then the common or principal centre of gravity exists in EF; *but the common centre of gravity of two bodies divides the straight line joining their respective centres in the inverse ratio of their masses*, and since the parts of the lever CR and PR are considered uniform, their masses are directly as their lengths; therefore, if g represent the position of the common centre, we have

$$\overline{\text{CR}+\text{PR}} : \text{PR} :: \text{EF} : \text{E}g;$$

Join CP and Cg, then EF equal one half of CP; that is, one half of D, and if l be taken to represent the whole length of the lever, or the sum of the parts CR and PR, we have

$$l : l-d :: \frac{\text{D}}{2} : \frac{\text{D}(l-d)}{2l} = \text{E}g.$$

Put $\delta = \text{C}g$, which has to be found by calculation, and $c =$ the angle Cgh, the angle of direction of the new force; then is C$h = \delta$ sin. c. h being the point in CP where the weight of the lever acts in opposition to the power; now, if $l\varphi = w$ be the whole weight of the lever, then its moment or effort at h is δw sin. c; consequently, by employing the moments of the power and resistance as formerly assigned, the equation of equilibrium becomes

$$p\text{D} \sin. b = rd \sin. a + \delta w \sin. c. \qquad (g)$$

This equation supposes that CR, PR, and PC, or CR, PR, and the angle CRP are known, which in practical constructions will always be the case; consequently, the chief difficulty consists in finding *a priori* the values of the distance δ, and the angle of direction c; but supposing these elements to be already determined, the analysis of the equation will be similar to that of equation *(f)*, for which reason, and because our article on the lever has already extended beyond the limits prescribed to it, we shall be very brief in the developement of that analysis.

53. PROBLEM 31. *Let all the factors which constitute the equation of equilibrium (g) be given excepting one, and let it be required to express the value of that factor in terms of the rest.*

Here, by recurring to the original equation, and separately disentangling the quantities, we find the following class of formulæ to fulfil the conditions of the problem.

1. $d=\dfrac{p\text{D} \sin. b-\delta w \sin. c}{r \sin. a}$, 5. $p=\dfrac{rd \sin. a+\delta w \sin. c}{\text{D} \sin. b}$,

2. $\text{D}=\dfrac{rd \sin. a+\delta w \sin. c}{p \sin. b}$, 6. $w=\dfrac{p\text{D} \sin. b-rd \sin. a}{\delta \sin. c}$,

3. $\delta=\dfrac{p\text{D} \sin. b-rd \sin. a}{w \sin. c}$, 7. $\sin. a=\dfrac{p\text{D} \sin. b-\delta w \sin. c}{rd}$

4. $r=\dfrac{p\text{D} \sin. b-\delta w \sin. c}{d \sin. a}$, 8 $\sin. b=\dfrac{rd \sin. a+\delta w \sin. c}{p\text{D}}$,

9. $\sin. c=\dfrac{p\text{D} \sin. b-rd \sin. a}{\delta w}$.

Each equation of the above class affords a rule peculiar to itself, and adapted to the determination of one particular object, and it is presumed, that by turning to the corresponding problems under equation *(f)*, where the rules are drawn out in words at length, the intelligent reader will find no difficulty in supplying them here whenever the circumstances of the case may render it necessary; we therefore omit them, but the subject must be further expounded; for since the elements c and δ are assignable only in terms of CR, PR and CP; or in terms of their representatives d, $(l-d)$ and D; or in terms of some of these in connexion with the angle of deflexion, or the angle CRP contained between the arms of the lever CR and PR; it becomes desirable to know in what manner the values of c and δ are to be determined.

First, then, in the triangle ERF, we have given the three sides $\text{ER}=\frac{1}{2}d$; $\text{FR}=\frac{1}{2}(l-d)$, and $\text{EF}=\frac{1}{2}\text{D}$; consequently, the perpendicular RS, which by the nature of the construction is equal to gh, is expressed as under, viz.

$$\text{RS}=\frac{1}{4\text{D}}\sqrt{4d^2\text{D}^2-\{\text{D}^2-l(l-2d)\}^2}; \qquad (1)$$

and if we put x to represent the angle RES, then we have

$$\sec. x = \frac{2d\text{D}}{\text{D}^2 - l(l-2d)}; \tag{2}$$

now, by the analogy preceding the general equation of equilibrium (g), it has been shown that

$$\text{E}g = \frac{\text{D}(l-d)}{2l}; \tag{3}$$

therefore, in the triangle CEg, we have given the two sides CE and Eg, with the contained angle CEg; to find the side C$g = \delta$; consequently, by Plane Trigonometry, we get

$$\delta = \frac{1}{2l}\sqrt{\text{D}^2(l-d)^2 + dl\{dl + 2\text{D}(l-d)\cos. x\}}; \tag{4}$$

again, from the right angled triangle chg, by having the hypothenuse cg and the perpendicular gh given, we obtain

$$\sin. c = \sqrt{1 - \frac{l^2}{4\text{D}^2}\left\{\frac{(2d\text{D})^2 - \{\text{D}^2 - l(l-2d)\}^2}{\text{D}^2(l-d)^2 + dl\{dl + 2\text{D}(l-d)\cos. x\}}\right\}}. \tag{5}$$

This expression for the value of sin. c is of a very complex character, and when the centre of gravity is reduced to the point h, we have

$$\text{C}h = \frac{1}{2l}\sqrt{\text{D}^2(l-d)^2 + dl\{dl + 2\text{D}(l-d)\cos. x\} - \frac{l^2}{4\text{D}^2}\{(2d\text{D})^2 - \overline{\text{D}^2 - l(l-2d)}^2\}}; \tag{6}$$

the equation just found expresses the distance from the centre of motion at which the lever is supposed to act, and since the whole weight of the lever is represented by w or its equivalent $l\varphi$, its effort or moment when applied at the point h, is expressed by

$$\frac{\varphi}{2}\sqrt{\text{D}^2(l-d)^2 + dl\{dl + 2\text{D}(l-d)\cos. x\} - \frac{l^2}{4\text{D}^2}\{(2d\text{D})^2 - \overline{\text{D}^2 - l(l-2d)}^2\}};$$

Now, if this expression be substituted for δw sin. c in equation (g), the conditions of equilibrium will then be represented in known terms.

If the lever be straight, and consequently $x = 0$; then will cos. $x = 1$ and $l = \text{D}$; therefore, the whole of the second compound member under the vinculum will vanish, and the first becomes simply D^4; from which, by extracting the square root, we have $\frac{1}{2}\text{D}^2\varphi$ for the effort or moment of the lever, the same as it was found to be for the straight lever of the second order.

54. By reason of the very prolix form that the equation of equilibrium would assume, if the preceding expression for the moment of the lever were introduced, it would be almost impossible to render a verbal rule intelligible, and besides the operation would be neither more nor less than determining separately, each of the six preceding quantities, in the same order as they have

occurred in the investigation: we shall therefore propose a numerical example, and calculate successively the several parts in order to show by what means the required object is to be attained, by so blending together the results as to lead us directly to the very point proposed.

EXAMPLE. The whole length of a bended lever of the second order is 25 feet, the shorter arm, or distance between the resistance and centre of motion is 8 feet, the straight line connecting the fulcrum with the remote extremity of the lever being 24 feet; and moreover, the angle of direction of the power is 80° 36′, that of the resistance being 85° 12′; what must be the magnitude of the power to sustain in equilibrio a resistance of 846 pounds, the weight of a foot in length of the lever being 9 pounds?

First then, by substituting the dimensions of the lever in the equation expressing the value of RS or gh, it becomes

$$gh=\frac{1}{4\times 24}\sqrt{(2\times 8\times 24)^2-\{24^2-25\,(25-2\times 8)\}^2}=1\cdot 6223 \text{ feet},$$

and to find the secant of x, we have, by substituting the dimensions of the lever in the second step,

$$\text{sec. } x=\frac{2\times 8\times 24}{24^2-25\,(25-2\times 8)}=1\cdot 094=\text{nat. secant. of } 23^\circ\ 56';$$

then, to find $\text{E}g$, according as its value is exhibited in the third step, we have

$$\text{E}g=\frac{24\,(25-8)}{2\times 25}=8\cdot 16 \text{ feet};$$

and again to find δ, or the distance of the centre of gravity g from the centre of motion c, we have by substitution in the fourth step

$$\delta=\frac{1}{2\times 25}\sqrt{24^2\,(25-8)^2+8\times 25\{8\times 25+2\times 24\,(25-8)\times \cdot 914\}}=$$
11·92 feet.

Here, then, we have determined both gh and cg, or δ; that is, the perpendicular and hypothenuse of the right angled triangle chg, therefore, we have

$$\text{sin. } c=\sqrt{1-\frac{1.6223^2}{11.92^2}}=.99068=\text{nat. sine of } 82^\circ\ 10';$$

consequently ch becomes $11.92\times .99068=11.81$ feet, the distance from the centre of motion to the point where a new power equivalent to the weight of the lever, is supposed to act; therefore, the moment of the lever is $11\cdot 81\times 225=2657\cdot 25$ pounds.

Hence, if 2657·25 be substituted for dw sin. c in the equation of equilibrium (g), it becomes

$$p\text{D sin. } b=rd \text{ sin. } a+2657\cdot 25;$$

or by referring to equation 5, problem 31, we have the value of p, the required quantity expressed as follows, viz.

$$p=\frac{rd \text{ sin. } a+2657\cdot 25}{\text{D sin. } b}.$$

cocsequently, if the given numerical values of r, d and D, with the natural sines of the angles of direction a and b be substituted as above, we shall have

$$p=\frac{846\times 8\times \cdot 99649+2657\cdot 25}{24\times \cdot 98657}=397 \text{ pounds, the power}$$

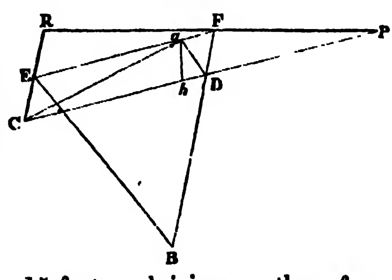
required.

Such then, is the operation necessary to be performed in a bended lever of the second order, to determine the magnitude of the power, when the place of the centre of gravity is not given; and a similar process for any other quantity would be required; but it may here be remarked, that for any practical purpose however delicate, a graphical determination of the centre of gravity will be found sufficiently accurate, and since it is much more easily performed than the numerical process, it may perhaps be found advantageous to adopt it, we shall therefore, take a separate problem for the purpose of showing in what manner the position of the centre of gravity is to be found, and also the point where the new power equivalent to the weight of the lever is supposed to act.

PROBLEM 32. *Of the method of finding the centre of gravity of a bended lever of the second order, when its arms, angles of direction, and resistance with the distance between the fulcrum and the remote extremity of the lever are given.*

56. EXAMPLE. The whole length of a bended lever of the second order is 30 feet, the shorter arm, or distance between the resistance and centre of motion is 6 feet, the distance between the fulcrum and the remote extremity of the lever 26 feet, the angle of direction of the power 88° 30′, that of the resistance 90° or a right angle; what weight will just be balanced by a power of 400 pounds, supposing the whole weight of the lever to be 360 pounds?

In this example it is proposed to determine the centre of gravity by construction as follows: since the whole length of the lever is 30 feet, and that of the shorter arm 6 feet, the longer arm must be 30−6=24 feet. Construct the triangle CRP, having its three sides respectively equal to the given numbers; viz. CR = 6, PR = 24 and CP = 26: bisect CR and PR in E and F, join EF and through F draw FB parallel to CR; make FB equal to half the length of the lever or 15 feet, and join BE; then from B cut off BD equal to FR or FP, and through D draw Dg parallel to BE; then is g the place of the centre of gravity. From g, the



centre of gravity thus determined, demit gh perpendicularly to CD, then will h be the point at which the weight of the lever opposes the power, and ch is the distance from the centre of motion at which it acts; therefore, if ch be measured from the same scale as was used in constructing the triangle, it will be found equal to 11·9 feet very nearly.

We shall now determine the same thing from the sixth step of the investigation, and that the reader may the more clearly perceive the process of substitution, it is deemed expedient to arrange the several terms regularly under each other, in the same order as they occur in the expression, proceeding from right to left, this being the method usually pursued in deducing rules from algebraical formulæ.

$$2d = 6\times 2 = 12,$$
$$l-2d = 30-12 = 18,$$
$$l(l-2d) = 30\times 18 = 540,$$
$$D^2-l(l-2d) = 676-540 = 136,$$
$$\{D^2-l(l-2d)\}^2 = 136\times 136 = 18496,$$
$$(2dD)^2-\{D^2-l(l-2d)\}^2 = 97344-18496 = 78848,$$
$$\frac{l^2}{4D^2}\left\{(2dD)^2-\overline{D^2-l(l-2d)}^2\right\} = \frac{78848}{4\times 26^2}\times 30^2 = 26243.787.$$

This completes the formation of the second member under the vinculum, and the formation of the first is accomplished in the following manner.

From the second step of the preceding investigation, the value of the angle x is found as under,

$$2dD = 2\times 6\times 26 = 312 \quad \log.\ 2\cdot 494155$$

and it has been shown, that $D^2-l(l-2d) = 136 \quad \log.\ 2\cdot 133539$

$$\text{angle REF} = x = 64^\circ\ 9'\ 27'' \quad \log.\ \sec.\ 10.360616$$

$$\text{nat. cos. } 64^\circ\ 9'\ 27'' = \cdot 4359,$$
$$(l-d)\cos. x = 24\times\cdot 4359 = 10\cdot 4616$$
$$2D\,(l-d)\cos. x = 52\times 10\cdot 4616 = 544\cdot 0032,$$
$$\{dl+2D\,(l-d)\cos. x\} = 180+544\cdot 0032 = 724\cdot 0032,$$
$$dl\{dl+2D\,(l-d)\cos. x\} = 180\times 724\cdot 0032 = 130320\cdot 576;$$
$$(l-d)^2 = 24\times 24 = 576,$$
$$D^2\,(l-d)^2 = 676\times 576 = 389376,$$
$$\left.\begin{array}{l} D^2(l-d)^2+dl\{dl+2D\,(l-d)\cos. x\} = \\ 389376+130320\cdot 576 = 519696\cdot 576. \end{array}\right\}$$

This completes the formation of the first member under the vinculum; therefore

$$ch = \sqrt{519696.576-26243.787}\div 60 = 11.71 \text{ feet};$$

hence, the difference in the results as obtained by the two methods of construction and calculation, is about two-tenths of a foot, but the coincidence would be much nearer if the operation were performed with very fine instruments and on a larger scale.

Now, the whole weight of the lever is 360 pounds; therefore, its moment or effort is equivalent to $11.71 \times 360 = 4215 \cdot 6$ pounds; consequently, the equation of equilibrium (g), becomes

$$p\text{D} \sin. b = rd \sin. a + 4215.6,$$

or, by referring to equation 4, problem 31, we have

$$r = \frac{p\text{D} \sin. b - 4215.6}{d \sin. a};$$

therefore, by substituting the numerical values of p, D and d, with the natural sines of the angles of direction a and b, we obtain

$$r = \frac{400 \times 26 \times \cdot 99966 - 4215 \cdot 6}{6} = 1732 \cdot 744 \text{ pounds},$$

for the weight or resistance required.

The investigation for the place of the centre of gravity in a bended lever of the third order, would be precisely similar to that which we have given for the second, and the steps of investigation would be perfectly symmetrical; the only difference being that in this case, the magnitude of the power must always exceed that of resistance, unless when the lever is so much bent, and the magnitude of its arms such, as to bring the resistance nearer to the centre of motion than the power; when this is the case, it is not necessary that the power should exceed the resistance, for the lever again would coincide with that of the second order.

57. Conclusion. In closing the discussion upon the lever, it may not be improper to remark, that our plan originally was, to express algebraically the value of each quantity deduced from the general equations of equilibrium, and from such expression, to draw up a rule in words at length, which again should be illustrated by appropriate practical examples. This plan has been continued throughout for the straight lever, both in case of perfect levity and when it is considered as having weight; and for the bended lever, it has been continued as far as that of the second order, having respect to the effect produced by gravity: at this stage of the subject, we have been compelled to deviate from our plan, on account of the very complicated form which the general equation would assume, if the expanded expression for the effort of the lever were introduced: the values of the several factors derived from the equation so constituted, would unavoidably partake of its prolixity, and as we have already observed, a rule in words could scarcely be rendered intelligible.

It is customary for writers on the lever to enumerate the several simple engines or tools, that are actuated by its principle, thus,

To levers of the first order belong, the balance, the steelyard, scissars, pokers, pincers, snuffers, &c.

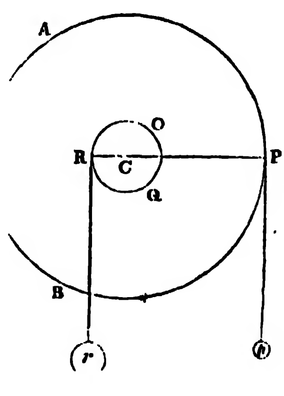
To levers of the second order belong, cutting knives fixed at one end, doors moving on hinges, oars of boats, rudders, wheel-barrows, nut-crackers, &c.

To levers of the third order belong, sheers for sheep, compasses, a ladder while men are raising it up with one end resting on the ground, tongs, the bones of animals, &c.

There are moreover universal and compound levers which we have not considered, but as these are either combinations or multiplications of those above named, the theory would present nothing novel or interesting, we will however have occasion to advert to them, when considering other departments of this work; with this notice, therefore, we quit the lever, and proceed to examine the principles and conditions of equilibrium in the wheel and axle, or as it is commonly called "the *Axis in Peritrochio.*"

2. OF THE WHEEL AND AXLE.

1. THE WHEEL AND AXLE is a machine so named, by reason of its consisting of a wheel and cylinder, having a common axis with pivots fixed in its extremities, on which the whole may revolve. This very simple and useful contrivance, although usually designated a second mechanical power, requires the consideration of no other principles than those already adduced for the lever; it is in fact nothing but a lever, having the radius of the wheel for one arm, and that of the cylinder or axle for the other, the fulcrum being the common centre of both. This machine is also, and not improperly, termed the *Perpetual Lever;* for since the power and the resistance operate respectively at the circumference of a circle revolving about an axis, it is obvious that the rotation must maintain the continuity.



Let PAB represent a section of the wheel, and QOR a section of the axle at right angles to the common axis or line of rotation passing through the centre C; then, since the effort of the resistance to turn the axle is the same at whatever point of its length it may be applied, it will cause no error to suppose the power and the resistance as acting perpendicularly at the points P and R in the same plane, the directions in which they act coinciding with the direction of gravity:—thus circumstanced, the points of application are always as if they were connected by the straight horizontal lever PCR, of which the arms are respectively PC, the radius of the wheel, and RC, the radius of the cylinder or axle, C being the fulcrum or centre of motion; now, while the simple lever could operate only through a very limited space, it is evident that by means of the rotation the power may be made to descend, and the resistance or weight to ascend through any space whatever; hence the name *Perpetual Lever.**

* See art. 2. p. 1. Lever.

The principle of equilibrium for the wheel and axle is as follows.

If the power and the resistance act at right angles to the extremities of their respective radii, an equilibrium will obtain when they are to each other inversely as the radii on which they act.

2. Now this principle, although a little differently expressed, is the same in substance as that on which the theory of the lever was constructed; therefore

Put $D = PC$, the longer arm of the lever, or radius of the wheel,
$d = RC$, the shorter arm of the lever, or radius of the axle,
$p =$ the measure, or magnitude of the power,
and $r =$ the measure, or magnitude of the resistance.

Then, by the principle of equilibrium above enumerated, we have

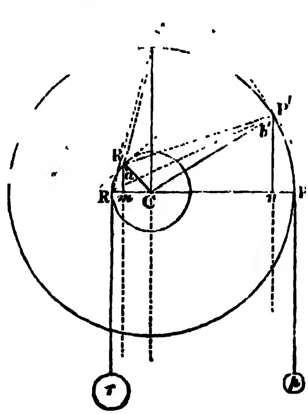
$$p : r :: d : D,$$

and this, by equating the products of the mean and the extreme terms, becomes

$$pD = rd.$$

This is the identical expression which we obtained for the conditions of equilibrium in the straight lever, when it was considered inflexible and void of gravity or weight; consequently, the equations derived from it must be of the same form, and the rules and examples employed in the case of the lever, are equally applicable to the wheel and axle.

3. That the wheel and axle in all its generality coincides in principle with the different orders of the lever, both straight and bended, will become manifest from the following illustration.



The right line PCR is a straight lever of the first order, of which PC and RC are the two arms, and C the fulcrum or centre of motion, the power p acting at right angles to one extremity, and the resistance r acting at right angles to the other, the directions in which they act coinciding with the direction of gravity.

If the power p, instead of acting at the extremity of the horizontal radius CP, be supposed to act perpendicularly to the extremity of the deflected radius CP′, while the resistance r continues to act at right angles to the radius CR, still in the horizontal position, the equation of equilibrium will be the same, and P′CR becomes a bended lever of the first order, whose arms are P′C and RC: but if the power p, instead of acting at right angles to the deflected radius CP′, be supposed to act parallel to the resistance, or in the direction of

gravity, the foregoing equation does no longer hold, for then the arm of the lever or radius CP becomes shortened into Cn; consequently, in order to maintain the equilibrium, all other things remaining, the magnitude of the power must be increased.

Let b represent the angle of direction CP$'n$; then by Plane Trigonometry, Cn becomes D sin. b, and the effort or moment of the power acting at the point P$'$ is expressed by pD sin. b; but this must be equal to the effort or moment of the resistance acting at the point R; that is,

$$p\text{D} \sin. b = rd.$$

Again, if the resistance r, instead of acting at the extremity of the horizontal radius CR, be supposed to act perpendicularly to the extremity of the deflected radius CR$'$, while the power p acts at right angles to CP$'$; then the conditions of equilibrium will still be indicated by the equation

$$p\text{D} = rd,$$

and P$'$CR$'$ continues to be a bended lever of the first order, whose arms are respectively P$'$C and R$'$C; but if the resistance r, instead of acting at right angles to the deflected radius CR$'$ be supposed to act parallel to the power p, while both act perpendicularly to the horizon, or in the direction of gravity; then the equation

$$p\text{D} \sin. b = rd$$

does not express the conditions of equilibrium, because the arm of the lever, or radius CR$'$ becomes shortened into Cm; therefore, the resistance must be increased in the same ratio, all other things remaining as in the equation above.

Let a represent the angle of direction CR$'m$; then by Plane Trigonometry Cm becomes d sin. a, and the effort or moment of the resistance acting at the point R$'$ is expressed by rd sin. a; but this must be equal to the moment of the power acting at the point P$'$, which we have already shown to be expressed by pD sin. b; hence, when the power and resistance act obliquely to the radii of their respective circles, the conditions of equilibrium for the wheel and axle are expressed generally by the following equation, viz.

$$p\text{D} \sin. b = rd \sin. a.$$

This again, is the identical equation which indicated the conditions of equilibrium for the bended lever;* consequently, the values of the several quantities which it involves will be determined after the same manner, and the rules and examples there employed, are equally applicable to the wheel and axle now in progress of discussion.

4. Suppose the radius CP$'$ to revolve about the point C, till it comes into the position CP$''$ at right angles to CP; then, if the

* No. 2.

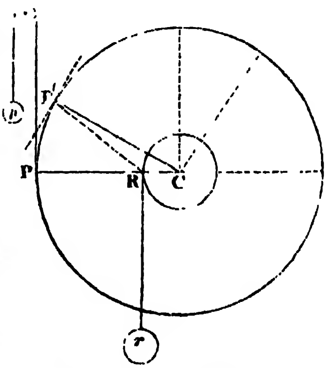
power p be applied perpendicularly to the extremity of the radius CP″, while the resistance r acts at right angles to CR or CR′, the conditions of equilibrium will still be indicated by the equation

$$p\text{D}=rd;$$

but, if the directions of the power and resistance coincide with the direction of gravity, then does $c n$ as well as the angle b become equal to nothing, and the equation

$$p\text{D}\sin. b=rd\sin. a$$

vanishes altogether; consequently, the machine is sustained at rest, either by means of its own weight, or by the power acting singly at P″, or some other point in the vertical line passing through the centre of motion; in this state the lever is about to change its order, for let the radius CP″ continue to revolve about the point C till it assumes the position CP′, then is P′CR a bended lever of the second order, of which the arms are respectively P′C and RC.



If the power p act at right angles to the extremity of the radius CP′, while the resistance r is applied perpendicularly to CR, the equation

$$p\text{D}=rd$$

continues to indicate the conditions of equilibrium; but if the power and the resistance be supposed to exert themselves parallel to each other, or in the direction of gravity, the arm CR remaining horizontal; then, the angle of direction of the resistance is 90 degrees, and its sine is equal to unity; consequently, the equation which indicates the state of equilibrium, is, as we have already shown

$$p\text{D}\sin. b=rd,$$

and if the radius or arm RC, instead of remaining horizontal, becomes inclined to the horizon, while the forces continue to act in the direction of gravity; then the general equation resumes its original form, viz.

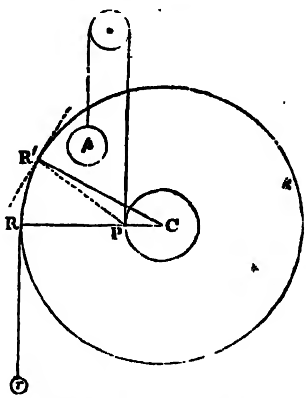
$$p\text{D}\sin. b=rd\sin. a;$$

and moreover, if the radius CP′ continues its rotation till it glides into the position CP, coincident with CR the radius of the axle; then is PRC a straight lever of the second order, whose arms are respectively PC and RC, and the equation of equilibrium is the same as before, viz.

$$p\text{D}=rd.$$

Again, if the two forces be supposed to change places with each other in such a manner, that the power p, instead of being applied

to the circumference of the wheel is applied directly to that of the axle, while the resistance r transfers its point of application to the circumference of the wheel; then is RPC, a straight lever of the third order, whose arms are respectively RC and PC, and when the directions of the power and resistance are coincident with the direction of gravity, if the lever RPC be horizontal, the equation expressing the conditions of equilibrium for this case also, becomes



$$p\text{D}=rd.$$

If the arm CR, revolve about the centre C, till it assumes the position CR′, then is R′CP a bended lever of the third order, whose arms are respectively R′C and PC; now, if the resistance r act at right angles to the arm R′C, while the power p acts also at right angles to the arm PC, then, the equation involving the conditions of equilibrium retains its form as before, viz.

$$p\text{D}=rd\,;$$

but if the directions of the power and the resistance, coincide with the directions of gravity, and become oblique to the arms of the lever or radii of the circles on which they act, the general equation expressing the conditions of equilibrium, again becomes

$$p\text{D}\sin. b=rd\sin. a.$$

Thus then, in every case of its application, have we succeeded in assimilating the conditions of equilibrium for the wheel and axle, to those for the lever formerly developed.

5. The several equations that indicate the state of equilibrium for the wheel and axle, when brought into one view are as under, viz.

1. $p\text{D}=rd,$
2. $p\text{D}=rd\sin. a,$
3. $rd=p\text{D}\sin. b,$
4. $p\text{D}\sin. b=rd\sin. a.$

From the first of these equations we infer, that the angles of direction of the power and the resistance are equal to one another, in which case it is manifest, that they may act either at right angles or obliquely to the radii of the circles to which they are applied.

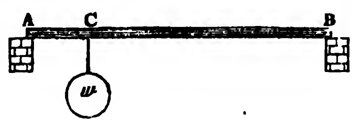
From the second and third we infer a different arrangement, viz. that while one of the forces acts at right angles to the arm where it is applied, the other may act at any degree of obliquity whatever.

The fourth and last equation indicates, that both the power and the resistance act obliquely, and at different angles to their respective radii; hence it appears, that by merely attending to the conditions of equilibrium, the corresponding arrangement of the forces can always be ascertained.

To determine the pressure upon the pivots or gudgeons of the wheel and axle when in equilibrio.

6. In treating of the lever in the foregoing article, it was found necessary, in order to establish generally the conditions of equilibrium, to take into consideration the effect of the lever's weight; but in treating of the wheel and axle there is no necessity for such an enquiry, because, under whatever circumstances the equilibrium may obtain, the portion of matter on one side of the vertical plane passing along the axis of rotation, will always sustain the equal and similar portion on the other side; consequently, the weight of the machine has no tendency either to produce or to derange the state of quiescence, that being maintained solely by the efforts of the opposing forces. But, although the weight of the wheel and axle cannot at all disturb the equilibrium, it may have a very great effect with regard to the pressure on the pivots that support it, and since in many instances, it may be requisite to assign the exact measure of such pressure, we shall here enquire by what means it is to be discovered.

7. Let AB represent a uniform bar or beam of any material whatever resting horizontally on two supports, and let the weight w be suspended from some intermediate point as C; then by the principle of the lever, the load thus suspended will be shared at the supports A and B in the inverse ratio of the distances AC and BC; that is,



the pressure at A, is to the pressure at B, as the distance BC to AC.

Put $m=$ AC, the shortest distance of the support at A from the point of suspension at C,

$n=$ BC, the distance from the support at B, then will

$m+n=$ AB, the whole length of the beam; therefore, we have

$$m+n : m :: w : \frac{mw}{m+n},$$

that portion of the load which is sustained by the support at B; again

$$m+n : n :: w : \frac{nw}{m+n},$$

the other portion of the load, or that which is sustained by the support at A; now, if the weight of the beam be taken into account, it is evident that one half of it will be transferred to each support,

for since it is sustained in a horizontal position, and supposed to be of uniform figure and density throughout its length, there is no reason why it should press more on one support than on the other. Let W represent the whole weight of the beam, then we have

$$\frac{W(m+n)+2nw}{2(m+n)} = \text{the pressure at A, and}$$

$$\frac{W(m+n)+2mw}{2(m+n)} = \text{the pressure at B.}$$

Now, this is exactly the case of the wheel and axle, for the load suspended from the point C, produces the very same effect, as if it were an uniform wheel applied to the axle at that point, and the beam itself can easily be conceived as an axle supported by its pivots at the extremities A and B. The expressions however, can be somewhat simplified, for since $m+n$ denotes the whole length of the beam, if we make $m+n=l$, we have only to substitute l for $m+n$, and we obtain

$$\frac{lW+2nw}{2l} = \text{the pressure at A, and}$$

$$\frac{lW+2mw}{2l} = \text{the pressure at B.}$$

The equations in this form are more convenient for verbal expression, and since it is of great importance in mechanical constructions, to know exactly the quantity of pressure on the pivots or gudgeons, we shall here give a rule by which it can be ascertained; but in the first place we must observe that w in the equation, instead of being a load suspended from C as in the figure, is to be considered as the weight of a wheel fixed there, while l and W are respectively the length and weight of the axle; this premised, the rule may be expressed in words at length as follows.

8. RULE. *Multiply the weight of the wheel by its distance from either pivot, and divide the product by the whole length of the axle; then, to the quotient, add half the weight of the axle for the pressure on the pivot or gudgeon, opposite to the distance by which the weight of the wheel is multiplied.*

EXAMPLE. The weight of a cast iron wheel, fixed on an axle at the distance of 2 feet from one of its ends, is $2\frac{1}{2}$ cwt. or 280 pounds; what is the pressure on either pivot, supposing the weight of the axle is 84 pounds, and its length 16 feet?

By the rule, it is, weight of the wheel 280 pounds
distance from the pivot 2 feet
length of the axle 16)560(
quotient = 35
half the weight of the axle = 42
pressure on the remote pivot = 77 pounds;

consequently the pressure on the other pivot is $280+84-77=287$ pounds.

9. This determination, it will be observed, has respect only to the pressure produced by the wheel and axle, without considering the effect of the forces applied to them, but it is easy to perceive that the complete solution of the question must include the effect of the forces also; that is, the pressure on either pivot of the axle must partake of the three pressures, viz. half the weight of the axle, a proportional part of the weight of the wheel, and a proportional part of the forces. The quantity of pressure produced by the wheel and axle alone has been determined above, it therefore only remains to assign the effect produced by the action of the power and the resistance, when operating in the direction of gravity, and on opposite sides of the axis of rotation.

Now, the power p, and the resistance r, may be conceived to act as a single force equivalent to their sum applied at their common centre; therefore, let δ represent the distance between the wheel and the point where the resistance acts estimated on the axis of rotation;* then by the principles of the lever and centre of gravity we have

$$p+r : r :: \delta : \frac{r\delta}{p+r},$$

the distance betwixt the place of the wheel and that of the common centre of the forces;

$$p+r : p :: \delta : \frac{p\delta}{p+r},$$

the distance betwixt the common centre and the point where the resistance acts, consequently, the distance of one pivot from the common centre of the forces, is

$$\frac{m(p+r)+r\delta}{p+r},$$

and the distance of the other, is

$$\frac{n(p+r)-r\delta}{p+r}.$$

But, as we have already stated, the sum of the forces considered as one force acting at the common centre, must be shared at the pivots or the supports in the inverse ratio of the distances; hence we have

$$l : (p+r) :: \frac{n(p+r)-r\delta}{p+r} : \frac{n(p+r)-r\delta}{l} = \text{proportional pressure}$$

on the pivot at A;

* This may be called the *central* or the *intermediate distance*.

$$l : (p+r) :: \frac{m(p+r)+r\delta}{p+r} : \frac{m(p+r)+r\delta}{l} = \text{proportional pressure}$$
on the pivot at B.

The united pressure of the wheel, the axle, and the forces, is therefore as follows, viz.

$$\frac{n(p+r+w)-r\delta}{l}+\frac{\text{W}}{2}=\text{the whole pressure on the pivot at A;}$$

$$\frac{m(p+r+w)+r\delta}{l}+\frac{\text{W}}{2}=\text{the whole pressure on the pivot at B.}$$

The practical rule afforded by these expressions is as follows.

10. RULE. *Add all three together, the power, the resistance, and the weight of the wheel, and multiply the sum by the distance of the wheel from either pivot; then, to or from the result, according as the lesser or greater distance of the wheel is used, add or subtract the product of the central or intermediate distance by the resistance or force that operates on the axle; then, divide the sum or the remainder by the whole length of the axle, or the distance between the pivots, and to the quotient add half the weight of the axle, for the pressure required.*

EXAMPLE. The weight of a cast iron wheel, fixed on an axle at the distance of 4 feet from one of the pivots, is 336 pounds, the weight of the axle is 112 pounds, and its length 22 feet; and moreover, a power of 56 pounds is applied to the circumference of the wheel, while another power of 265 pounds is applied to the axle, at a point 8 feet distant from the wheel; what is the pressure on either pivot, supposing the forces to act in the direction of gravity on opposite sides of the axis of rotation?

By the rule we have

$$\frac{18\times(336+56+265)-8\times265}{22}+\frac{112}{2}=497\tfrac{2}{11}\text{ pounds, on the}$$

pivot nearest to the wheel, and

$$\frac{4\times(336+56+265)+8\times265}{22}+\frac{112}{2}=271\tfrac{9}{11}\text{ pounds, on the}$$

other pivot.

What we have hitherto done applies only to the case of a single wheel and axle, but the intelligent reader will easily perceive, that the principle is general, and extends to any number of wheels and axles, provided the power be applied to the first wheel, and the resistance to the last axle; whatever may be the mode of connection, whether by bands or by wheel and pinion, the equilibrium obtains when *the power is to the weight as the continual product*

of the radii of all the axles to the continual product of the radii of all the wheels. This is called *Tooth and Pinion Work*, the means and appliances of which may be found in all the treatises on Mechanics since the days of Emerson till the present time; the successive steps of improvement in which may be found particularly detailed in Dr. Olinthus Gregory's Mechanics, and the Treatise on Machinery in the Encyclopædia Metropolitana, compiled by Professor Barlow, of the Royal Military Academy at Woolwich. It formed no part of the plan of our work to copy into it what others have so diligently collected; we therefore the more readily refer the reader to the most established authorities for information on a subject which belongs to the Mechanics of practical men.

3. OF THE PULLEY.

INTRODUCTION.

Our treatise upon the pully presents this mechanical power in a form very different from any that we have seen. The arrangement of the subjects,—its general developement,—and the particular questions introduced under each system of pulleys,—form the assemblage upon which this difference is founded. Out of many combinations that might have been chosen, we have selected eight, the conditions of equilibrium belonging to each of which we have scientifically expounded, by a method similar to that which we have pursued in all our previous enquiries;—and we have commenced our investigation by enunciating a class of equilibrated equations for these eight varieties of the pulley, deducing therefrom three distinct and independent equations, on which the whole theory of the pulley depends. With these three equations, following the order of the arrangement, we are thence enabled to demonstrate at length the eight successive problems which we had selected as sufficient to exhaust the statical properties of the pulley. The examples by which we illustrate our subject, will be found curious; we believe they are novel. To nautical men, the method by which we have uniformly exemplified the pressure or strain on each rope, block or hook, will prove interesting if not instructive. The demonstrations of the pressure on each ring and its continuous pulleys give this value to the solution of the question—taking the sum of these pressures we arrive at the original answer, and thus prove the correctness of our operation. Solutions conducted in this way offer great advantages to the general reader: the man of science sees with pleasure the advances which industry makes to raise to intellectual superiority the ingenious and laborious mechanist. In a word, these solutions unfold the whole doctrine of the pulley, and, by exhausting every principle it involves, they leave nothing unrevealed that should be known.

In the first six problems, the chords are all parallel to one another, in the 7th and 8th they are oblique to one another.

The introduction of White's pulley, in the third problem, assigns to this ingenious contrivance its proper place in the system to

which it belongs, and our illustration of its mechanism, accompanied by two examples, presents an arrangement that should not be altogether overlooked. In the fifth problem, we have investigated the combination which has hitherto been made of one fixed and several moveable pulleys; or of several fixed and several moveable pulleys, and to this system we have added a new combination, by which a still higher mechanical effect will be obtained. This arrangement is exhibited in the sixteenth figure, in which the reader will perceive that each additional cord or moveable block, quadruples the mechanical efficacy of the pulley. The 6th problem is the converse of the 5th; for whereas, that system exhibited the cord passing under the moveable pulley, this (the 6th) treats of the cordage going over the pulley. Like the fifth, it consists of two cases; one, in which the cords are attached to hooks in the load to be raised; the other, in which the cord passes under a pulley that is attached to the load.

Though such a combination may never be met with in practice, its utility and efficiency are demonstrated as clearly as the other, and its introduction, accompanied by examples, will furnish the reader with exercises that are rarely to be met with in the writings of authors who have treated of the mechanical powers. We are perfectly aware, that diagrams similar to figures 14th, 15th, 17th and 18th, may be found in a contemporary publication, but we have not seen in that work any illustrations resembling our own. We may, therefore, claim originality in the whole of these, excepting always the principle of equilibrium, which must be common to all who undertake this branch of mechanical investigation.

In the 7th and 8th problems, the directions of the cords being oblique, the investigation required the introduction of Plane Trigonometry, and the solutions of the equations of equilibrium are necessarily performed by means of logarithms.

With these eight problems we have exhausted every principle of the pulley, and it were to no purpose to have introduced other arrangements and combinations, since every particular respecting them is exemplified in what we have done.

A very curious and useful combination of pulleys is described by Mr. Smeaton, in the 47th volume of the Philosophical Transactions. The model which he showed to the society, consisted of twenty shieves, five on each pin. With this model, which could be carried in the pocket, he raised six hundred weight; and if a larger tackle of the same kind was properly executed, a ton might easily be raised by one man.

Among various recent attempts at improvement in the construction of the pulley, we may be allowed to notice a species of differential pulley, invented by Saxton, for augmenting the speed of carriages on railways and other roads, and of vessels for inland navigation: this contrivance consists in a double pulley, one

side of which is of less diameter than the other, and it may be affixed to the carriage or vessel to be propelled. A rope passing round each grove of the pulley, must also pass along the road or canal in the direction of which the carriage or vessel is to move. If the difference of diameter in the double pulley be as six to seven, the one pulley being one-seventh less than the other, the speed of the carriage or vessel will be multiplied thirteen times; that is to say, the carriage or vessel will move thirteen times faster than the rope which propels it, so that if the rope be made to move 1 mile an hour, the carriage will move 13 miles an hour; or if the rope move 4 mile an hour, the carriage will move 52 miles an hour; or if the rope move 10 miles an hour, the carriage ought to move 130 miles an hour; and the rates of speed may be increased or diminished according to the proportionate diameter of the two sides of the double pulley.

The proportion used in the pulley of a model carriage, which was exhibited in 1834, at the Gallery of Practical Science, Adelaide Street, Strand, London, was as eight to nine, shewing a difference of one-ninth in the diameter of the two faces of the pulley; accordingly, the carriage moved seventeen times faster than the rope which propelled it. The inventor has introduced ingenious and efficient contrivances for stopping and starting the vessel at pleasure, and for keeping it constantly under safe control; also, a compensation for the expansion or contraction to which ropes of great length will be liable from variations of temperature. Carriages on this plan may be moved by the POWER of steam engines, fixed at proper distances; or of horses moving in a straight line, or in the circle of a mill-wheel, or by the power of streams of water, but in whichever way moved, this pulley will always be of very partial use, and upon a very limited scale.

A rope of twisted grass thrown over the branch of a tree, and grasped firmly by a man, who might thereby sustain a load upon it equal to his strength, presents the earliest mode of bodies placed in equilibrium by means of cords. The man's power will in this case sustain the same weight as it would do were a pulley fixed to the bough of a tree, which we may thence very safely consider as the axle of a pulley. But with a pulley, the man would raise a weight much more easily, because, in addition to the purchase he exerted in hauling the rope over the pulley, the pulley itself would afforded very great assistance by revolving in its axis, and thereby removing a vast portion of the friction which would be produced did the rope slide over a fixed in place of a moveable body. Perhaps an idea of this kind suggested to Dr. Hamilton and others the notion, that the pulley "cannot properly be considered a lever of any kind." But there is nothing more obvious, however, than that the fixed pulley is a lever of the first order, and the moveable pulley a lever of the second order; and we behold both these

systems of pulleys in constant action in the daily avocations of mechanics, as well as on the deck of a ship, where the sailor, without moving from the stand to which his duty fixes him, may by means of a rope passing over a pulley, change the direction of a force, and hoist a sail or signal-flag to the top of the loftiest mast. Nay, on ship-board the pulley is the great engine employed, not only for changing the direction of force, and enabling a few men to overcome great resistances, which would require vast numbers to effect the same purpose, but it is in fact the only mechanical power that we can employ so generally. See, for example, with what ease a vast anchor and its ponderous iron cable are weighed by means of blocks and ropes, how masts are raised and placed in their berths, how the whole apparatus of sail-yards can be adjusted with the most perfect order and nicety, how easily vast guns are worked in the hour of battle; and, in short, how every thing aloft or below has its direction changed by a block and tackle, which thereby gives the sailor an entire command of his craft.

We must not, however, confound the block with the pulley; the block is the wooden mass or iron frame in which the pulley revolves on a fixed pin or axle. With this explanation we shall use the words block or pulley indiscriminately.

In demolishing and in rearing buildings, we see labourers perform very curious operations by means of a fixed pulley. Sometimes manual labour is, in this case, supplanted by a horse trotting round a large wheel, attached by a mechanical contrivance to his collar, and by means of pulleys all the materials of stones, bricks, and mortar are thereby effectually carried up or lowered down.

Colliers and well-sinkers will step into a bucket attached to a rope that passes over a pulley, and then, seizing the other end of the rope in their hands, will descend to their work with perfect safety, and remount again to the summit by stepping into the bucket and pulling themselves up by their hands. In descending they employ much less force than is necessary to preserve their equilibrium, but in raising themselves they augment the force, and if they pull with a force equal to half their weight they may easily regain the pulley. Daring adventurers sometimes lower themselves over the brow of a perpendicular rock by fastening a rope round their body, passing the other end over a fixed pulley, and then taking hold of it in their hands, to lengthen it as they descend. By the same means they may raise themselves to the brow of the cliff, but for safety both ends of the rope should be attached to the body. But if the weight of the man be the limit of his strength, such feats appear impossible.

Fire-escapes, attached to a fixed pulley, may thus be worked by individuals when no other means are at hand to render assistance. A pulley attached to a strong girder at the top of a window, with a cradle to which both ends of a rope might be

attached, presents all the apparatus necessary for enabling the most timid to escape from a conflagration.

Before we conclude these prefatory remarks on the pulley we must repeat an observation made, elsewhere, for the purpose of disabusing a common notion, that the pulley, of all the mechanical powers, enables us to save labour. No such thing: and in order to be convinced that even more labour or bodily exertion is expended than would suffice to do the work without pulleys, let us take the first example under figure 14. Here is a system of ten pulleys, upon which a power of 120 pounds weight will sustain in equilibrio a load of 122880 pounds. Now, if one man can lift but 120 lbs., it is very plain that with this tackle of pulleys he is able to balance as much as 1024 men; in fact, as much as a whole regiment of soldiers, or the crew of a ninety-gun ship.

To put this case in another point of view, suppose one man with a tackle of pulleys having ten plies of rope, can raise a weight which it would require ten men to raise at once without pulleys: if the weight is to be raised a fathom, the ten men will raise it by pulling at a single rope and walking one fathom; whereas, the one man at his tackle must walk until he has shortened all the ten plies of rope one fathom each; that is, he must walk ten fathoms, or ten times as far as the ten men walked. In both cases, the same quantity of man's work is employed to accomplish the same end. In the first case, the work is performed by ten men in one minute. In the second, one man consumes ten minutes. And to set this matter in another point of view, (for men work throughout the day,) let us suppose there is a week's work to be done by the ten men; it is plain, there would be ten week's work of the one man; and when the payment came to be made for their labour, we should have to pay ten men one week's work each, and one man ten week's work at the end of the job. If their weekly wages are alike, there is nothing saved in point of expense; and we have shown there was no saving of time. There is, therefore, as Dr. Arnot justly observes, no direct saving of human effort from using pulleys; indeed, there is a loss from the great friction which has to be overcome. But, then, they allow a small force, at its leisure, to produce any requisite magnitude of effect at the expense of additionally overcoming a certain amount of friction; or, of performing, by two or three pairs of hands, operations which hundreds, whom it might be inconvenient to employ, would be required to achieve.

We see with what great ease vast loads are raised by means of pulleys, when the rope is fastened to a wheel and axle, which lift as long as there is rope to be wound up, whereas the pull of a man's strength represents a simple lever that acts only through a small space. In all cranes and pile engines we find pulleys employed, and we see how much may be accomplished by the patient labour of a solitary individual.

Although, therefore, there is no actual saving of labour in the use of the pulley, it is very plain we could not substitute any other contrivance that would so readily do the same service, especially in the rigging of a ship. We could not, for example, place aloft forty or fifty men, to hoist a mast, a topmast, or a top-gallantmast into its berth, or employ a fewer number to raise a vast sailyard to which many hundred yards of canvas were attached; nor could we bring a company of men to the bow of a ship to weigh her anchor and heavy iron cable. In a word, if the reader will attentively look at the method we have employed in estimating the comparative merits of the several systems of pulleys which we have introduced, he will be convinced of the vast importance attached to this mechanical power above any other by maritime people in all ages of the world.

OF THE PULLEY.

Definitions and equations of equilibrium on which the whole theory of the pully depends.

In the two articles immediately preceding, we have treated of the *Lever* and of the *Wheel* and *Axle*, those mechanical powers to which writers in general give the precedence in their discussions, and which they consider as in some measure unfolding the principles of, and constituting a foundation for all the others.

We follow the general system of arrangement, but in our mode of treating the subjects we differ widely from other writers, and we cherish the hope that the plan which we pursue in the illustrations, will, to most of our readers prove both entertaining and instructive.

The next mechanical power which presents itself for our consideration in the order of arrangement is

The Pulley, a small grooved wheel, commonly made of wood, iron, or brass, having a pivot passing through its centre, and fixed in a frame or block, about which it revolves as an axis, by means of a cord passing round the circumference that serves to draw up or sustain the weight.

The pulley is either single or combined; that is, it either acts singly by itself, or there are several pulleys combined together for the purpose of augmenting the energy of the power. It is also either fixed or moveable, that is, it is either confined to one place, or it shifts its position and moves upwards or downwards in obedience to the power and the load.

There are various modes of combining pulleys together, for the purpose of either changing the direction of the power, or deriving some mechanical advantage, some of which are vastly superior to others, but the most interesting and important combinations that have hitherto been applied to practical purposes, are the following, viz.

1. *The single fixed pulley*, over which a cord passes, having the power attached to one end, and the weight or resistance to the other; in this arrangement no mechanical advantage is gained, its principal use being to change the direction of the power. By a machine of this sort, a given power acting in any direction,* can be opposed to an equal resistance acting in any other direction.

The conditions of equilibrium in this system, will be exemplified in the solution of the first problem.

2. *The single moveable pulley*, under which a cord passes, having one end sustained by the power, and the other fixed to some immoveable object, the weight being attached to the frame or block in which the pulley plays; in this arrangement the weight is equal to twice the sustaining power, and by passing the cord over a fixed pulley, the direction of the power can be accommodated to that of the weight.

If the cord, instead of having one of its ends attached to an immoveable object, should be passed over a pulley fixed there, and finally joined to the frame or block which contains the moveable pulley and supports the weight; then, in this arrangement, the weight is equal to three times the power, and if that end of the cord on which the power acts be passed over another fixed pulley, the direction of the power will be changed from an upward to a downward position.

The conditions of equilibrium for this system, will be exemplified in the solution of the second problem.

3. *Several fixed and several moveable pulleys*, in this arangement, the weight is equal to as many times the power, as is denoted by thrice the number of moveable pulleys, and furthermore, every additional moveable pully increases by 2 the efficacy of the sustaining power.

The conditions of equilibrium for this system, will be exemplified in the solution of the third problem.

Note. In the foregoing combinations, with all their varieties of arrangement, it must be observed, that the same cord is supposed to extend from the power to a certain fixed point, and in its progress, to come in contact with, and influence all the pulleys of the system; in this case, the mechanical efficacy can only be augmented by increasing the number of pulleys, but it is manifest, that a far greater mechanical advantage may be obtained by increasing the number of cords, after the manner described in the following arrangements.

4. *One fixed and two moveable pulleys, with two separate cords;* this combination admits of two varieties; in the one, the weight is equal to four times the power, and in the other, it is five times; the whole arrangement is called a *Spanish Burton.*

The conditions of equilibrium for this system will be exemplified in the solution of the fourth problem.

5. *A single fixed and several moveable pulleys*, where each moveable pulley has its own cord passing under it, and fastened to a separate hook; in this arrangement, the power is to the weight, as unity is to that power of 2, in which the exponent is denoted by the number of cords or moveable pulleys. The mechanical efficacy of this system may be greatly augmented, without increasing the number of either the cords or moveable pulleys, by merely passing each cord over a fixed pulley, instead of fastening it to a

hook, and finally attaching the cord to the moveable pulley which it sustains; in the arrangement thus modified, the power is to the weight or resistance, as unity is to that power of 3, in which the exponent is denoted by the number of cords or moveable pulleys.

If another fixed, and another moveable pulley be introduced for each cord, without altering the number of the cords, a still higher mechanical effect will be obtained, for in this case, the power is to the weight or resistance, as unity is to that power of 4, in which the exponent is denoted by the number of cords, or half the number of moveable pulleys.

The conditions of equilibrium for this system, will be exemplified in the solution of the fifth problem.

6. *A single fixed and several moveable pulleys*, where every pulley in the system, has a separate cord passing over it and attached to the load; in this arrangement, the power is to the weight, as unity is to that power of 2, diminished by one, in which the exponent is denoted by the number of pulleys, or by the number of cords attached to the load which is required to be raised, or sustained in equilibrio.

The mechanical effect of this system may be greatly augmented, by passing each cord under a pulley fixed to the load, and returning it to the lower part of the block containing the moveable pulley over which it originally passes. In the arrangement thus modified, the power is to the weight or resistance, as unity is to that power of 3 diminished by one, in which the exponent is denoted by the number of cords, or by the number of pulleys fixed to the load.

The conditions of equilibrium for this system, will be exemplified in the solution of the sixth problem.

Note. In these several combinations, we have considered the pulleys and their containing blocks to be so adjusted, that all the cords may act in parallel directions; but in those that follow, we shall conceive the cords to be strained in directions that are oblique, or inclined to one another.

7. *A single fixed and a single moveable pulley*, where the directions of the power and the weight are inclined to one another; in this arrangement, the power is to the weight, as radius is to twice the cosine of the angle of inclination, or that angle which the direction of the weight makes with the direction of the power.

The conditions of equilibrium for this system, will be exemplified in the solution of the seventh problem.

8. *A single fixed and several moveable pulleys*, where each moveable pulley has its own sustaining cord passing under it, and fastened to a separate hook; in this arrangement, the power is to the weight, as radius is to the continued product of the cosines of half the angles which are made by the cords sustaining the pulleys, drawn into that power of 2, in which the exponent is denoted by the

number of sustaining cords, or by the number of pulleys that are moveable.

The conditions of equilibrium for this system, will be exemplified in the solution of the eighth problem.

9. These are the several varieties of combinations, or systems, which we propose to discuss more at large in their order, for which purpose.

Put p= the power or agent which sustains the weight in equilibrio,

w= the weight or resistance which counterbalances the power,

N= the number of parts of the cord, engaged in sustaining the pulley or block to which the weight is attached;

n= the number of distinct cords in the system:

then, if a, b, c, &c., denote half the angles made by the cords supporting the respective moveable pulleys, we shall have the conditions of equilibrium for each of the foregoing arrangements, expounded by the following equations, viz.

1. $w = \text{N}np$,
2. $w = \text{N}np$,
3. $w = \text{N}np$,
4. $\begin{cases} w = (\text{N}+1)p \\ w = (\text{N}+2)p \end{cases}$,*
5. $w = \text{N}^n p$,
6. $w = (\text{N}^n - 1)p$.
7. $w = \text{N}np \cos. a$,
8. $w = \text{N}^n p, (\cos. a \cos. b \cos. c \cos. n)$.

Of the foregoing class of equilibrated equations, Nos. 1, 2 and 3, have precisely the same form; hence we infer, that however different from one another the particular systems to which they refer may be, yet the principle of their combination is the same, and for this reason, disregarding all extraneous conditions, the mechanical effect must be increased exactly in proportion to the extent of the system; for it is manifest, that if the power p be constant, the value of w varies as the coefficient Nn, and if the factor n be constant in addition to the power, then w varies simply as N; consequently, the greater the value of N, the greater is the effect obtained by the system, the friction and stiffness of the cords not being considered.

The equation No. 7, is of the same import as Nos. 1, 2 and 3, being simply modified for the purpose of estimating the equilibrium in the case of oblique action; if the angle a should vanish, or become equal to *zero*, the arrangement would correspond to that referred to in No. 2; consequently, the mode of estimating the mechanical efficacy would also be the same.

In like manner it may be shown, that No. 8, corresponds to No. 5; for if the angles a, b, c, &c. should vanish, the whole paren-

* These two expressions for the value of the weight, limit the application of this system; in both cases the value of N is 3; consequently,

In the first arrangement, $w = 4p$.
and the second arrangement, $w = 5p$.

thetical expression would become equal to unity, and the value of the weight w, as indicated in Nos. 5 and 8 would be the same.

10. Hence, the distinct and independent equations, on which the whole theory of the pulley depends, are simply as follows

$$1.\ w = \text{N}np,$$
$$2.\ w = \text{N}^{n}p,$$
$$3.\ w = (\text{N}^{n} - 1)p.$$

The method of deducing and applying these equations, with their subordinate varieties, will become manifest, from the solution of the following problems and their appropriate examples.

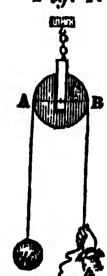
Section First.

WHEN THE CORDS ARE PARALLEL TO ONE ANOTHER.

11. Problem 1. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of a single fixed pulley.*

Let a horizontal line AB, (fig. 1), be drawn through C, the centre of the single fixed pulley; then shall the straight line AB represent a lever of the first order, having the point C for its fulcrum or centre of motion; the weight w acting at the extremity of the arm AC, while the power p acts as a counterpoise at the extremity of the arm BC.

Fig. 1.



Then, because the pivot or fulcrum passing through C is fixed, and the arm AC equal to the arm BC, it is manifest, that in the case of an equilibrium, the following proportion must obtain, viz.

$$p : w :: \text{AC} : \text{BC};$$

consequently, by making the product of the mean terms equal to the product of the extremes, the resulting equation will indicate the conditions of equilibrium; that is,

$$\text{AC}.w = \text{BC}.p;$$

but because the arm AC is equal to the arm BC, we obtain by expunging the common factor,

$$w = p. \qquad (1)$$

This result corresponds to the first of the preceding general independent equations; for there is only one portion of rope or cord engaged in sustaining the weight w, the other being acted on by the power p; and there is only one rope or cord in the system; hence, by our notation, N and n are each equal to unity, which being substituted in No. 1 of the general equations, gives

$$w = p,$$

the identical expression which we obtained from the preceding investigation.

12. The above equation implies, that

If a power sustain a weight in equilibrio by means of a cord passing over a fixed pulley, the power and the weight are equal to each other.

Hence, all fixed pulleys are levers of the first order, in which the arms are equal to one another; consequently, a fixed pulley adds nothing to the efficacy of the power, but only serves to change its direction and to facilitate the motion of the cords.

Let the capital letter P represent generally the pressure or strain on any particular part of any system, whether it be a fixed point, an axis, or portion of the sustaining cords; and let the same letter represent the accumulated pressure of the whole system, viz. the aggregate of the power and the weight taken conjointly; the weight of the apparatus, including pulleys, blocks, and cordage, being, in so far as relates to pure theory, entirely disregarded; then we have

$$P = p + w. \qquad (a)$$

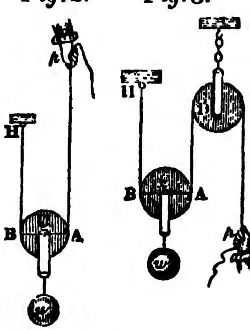
Example. If the weight w be equal to 28 pounds, a force or power of 28 pounds will sustain it in equilibrio; consequently, by equation (a), the accumulated pressure on the pivot or axis passing through C, is

$$P = 28 + 28 = 56 \text{ pounds.}$$

13. Problem 2. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of a single moveable pulley.*

Let a horizontal line AB, (figs. 2 & 3), be drawn through C, the centre of the moveable pulley ACB; then shall AB be a lever of the second order, having the fulcrum or centre of motion at B, the power acting at A, and the weight suspended from the pivot or axis of the pulley passing through C.

Fig. 2. *Fig.* 3.



Then because the arm BA is equal to twice the arm BC, it follows, that in the case of an equilibrium, the power is to the weight, as the length of the arm BC, is to the length of the arm BA; that is,

$$p : w :: BC : BA;$$

consequently, by making the product of the mean terms equal to the product of the extremes, the equation involving the conditions of equilibrium, or the relation subsisting between the power and the weight, becomes

$$BC \cdot w = BA \cdot p;$$

but the length of the arm BA is twice the length of the arm BC,

the one being the diameter and the other the radius of the pulley; therefore, if BC be taken equal to unity, the above equation becomes

$$w=2p. \qquad (2)$$

This result also corresponds to the first of the general independent equations; for there are obviously two portions of the rope or cord employed in sustaining the weight w, viz. the portions Ap and BH, each of them sustaining one half, but the system consists of only one cord; consequently, by our notation, $N=2$, and $n=1$; let these values of N and n be substituted in No. 1 of the general equations, and we get

$$w=2p,$$

the identical expression which we obtained from the preceding investigation.

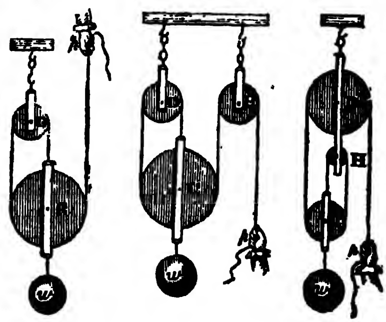
14. The equation thus deduced implies, that

If a power sustain a weight in equilibrio, by means of a cord fixed at one end, and acting on a single moveable pulley, the weight is equal to twice the power.

Hence all moveable pulleys are levers of the second order, having the length of one arm double the length of the other; consequently, a single moveable pulley whose parts are arranged as exhibited in (fig. 2), has the effect of doubling the energy of the power. If the power p, instead of acting in the direction Ap, should have the cord to which it is attached passed over the fixed pulley D, as represented in the third diagram, the same relations would still obtain; for as we have already seen, the fixed pulley D, adds nothing to the efficacy of the power, but only serves to change its direction from an upward to a downward position.

15. The preceding investigation and the results that arise from it, apply only to the case of a single moveable pulley supported by two portions of the sustaining cord; but if one or two fixed pulleys, be brought into combination with the single moveable one, after the manner exhibited in the annexed diagrams, it is manifest, that the weight will then be equal to three times the sustaining power, for since the lower block C, to which the weight w is attached, is supported by three portions of the running cord, each portion must support a third part of the weight; and because the tension of the cord is equal in every part, while its quantity is measured by the magnitude of the power p, it is evident, that the accumu-

Fig. 4. *Fig. 5.* *Fig. 6.*



lated tension of the three portions which sustain the lower block, must be equal to three times the straining power; that is,

$$w=3p. \qquad (3)$$

Here again, the result which we have obtained is identical with that which flows from the general equation No. 1; for in this case, $N=3$, and $n=1$; let these numerical values of N and n be substituted for them in the equation just referred to, and we obtain

$$w=3p,$$

the very same expression which we deduced from the foregoing investigation.

16. The above equation implies, that,

If a power sustain a weight in equilibrio, by means of a cord passing over one or two fixed pulleys, and under a moveable one, the weight is equal to three times the sustaining power.

EXAMPLE 1. What weight will a power equivalent to 28 pounds, be able to sustain in equilibrio, by means of a cord fastened to an immoveable object at the remoter end, and passing under a single moveable pulley, to whose frame or containing block, the weight is attached?

This question is obviously resolvable by means of equation (2), where we have

$$w=2p=28\times2=56 \text{ pounds.}$$

EXAMPLE 2. What weight will a power equivalent to 28 pounds, be able to sustain in equilibrio, when the moveable pulley to which the load is attached, is supported by three portions of the running cord, after the manner exhibited in the fourth, fifth, and sixth diagrams?

This example is resolvable by means of equation (3), where we have

$$w=3p=28\times3=84 \text{ pounds.}$$

EXAMPLE 3. What power will be able to suspend a weight of 224 pounds, by means of a cord having one end fastened to an immoveable object and passing under a single moveable pulley, to whose block the weight is attached?

The conditions of this example are obviously the reverse of the preceding ones, and consequently, the solution requires a reverse operation; that is, the required quantity in this case must be determined by division, or by the reduction of equation (2), as follows

$$p=\frac{w}{2}=\frac{224}{2}=112 \text{ pounds.}$$

EXAMPLE 4. The load remaining, as in the third example; what must be the sustaining power under the conditions indicated

by equation (3), and exemplified in the fourth, fifth, and sixth diagrams?

The mode of solution for this example is obviously similar to that for example 3, and is as follows.

$$p=\frac{w}{3}=\frac{224}{3}=74\tfrac{2}{3} \text{ pounds.}$$

Of the distribution of pressure for the several varieties of pulleys in our arrangement of the foregoing examples.

17. With respect to the distribution of pressure for the several varieties of arrangement, exhibited in the diagrams to which the above examples refer, we have as follows.

For the arrangement represented in the second figure.

The pressure on the hook H, is $P=\frac{1}{2}w=p$,
———————— pivot C, is $P=\ w=2p$:

For the arrangement represented in the third figure.

The pressure on the hook H, is $P=\ \frac{1}{2}w=p$,
———————— pivot C, is $P=\ \ w=2p$,
———————— pivot D, is $P=+\frac{1}{2}w=2p$.

By the two last of these equations, it appears, that the pressures on the axis of the pulleys C and D are equal, but this can only be the case, when the weight of the apparatus, including blocks, pulleys, and cordage, is disregarded; it would obviously be otherwise, if the effect of that weight were taken into the account; for then, the axis passing through D, has to sustain in addition to $2p$, not only the whole weight of the pulley D and cord ADp, but also half the weight of the pulley C; whereas, the axis passing through C has nothing to sustain besides the weight w, and the block to which it is attached. The weight of the other half of the pulley C, is obviously transferred to the hook at H; consequently, the total pressure at H, including the apparatus, is equal to half the weight w, together with half the pulley C and the cord BH.

The power, as determined from the third of the foregoing examples, is 112 pounds; consequently, the respective numerical values of the several pressures, are as below, viz.

Pressure on the hook H, is $P=\ p=112$ pounds,
———————— pivot C, is $P=2p=224$ ——,
———————— pivot D, is $P=2p=224$ ——.

And we have shown in equation (a), that the accumulated pressure of the whole system, is equivalent to the aggregate of the power and the weight taken conjointly; consequently, in the present instance, the total pressure is

$$P=p+w=112+224=336 \text{ pounds,}$$

being equivalent to one and a half times the weight, or three times the power. With respect to the pressure on the axis of the

pulleys C, D, or on C, D and H, in the diagrams illustrative of equation (3), it is manifest, that each of the cords attached to, or engaged in supporting the moveable block C, sustains a third part of the whole weight w; consequently, the pressure on the axis of each pulley, is equal to two thirds of the weight, or twice the power; for the tension of the sustaining cord is the same at every point of it, and since each pulley is acted on by the cord in a similar manner, it is manifest that the axis of each, sustains twice the measure of tension; that is, twice the straining power, or two thirds of the weight; hence we have

For the arrangement represented in the fourth figure,

The pressure on the axis C, is $P=2p=\frac{2}{3}w$,

———————————— D, – $P=2p=\frac{2}{3}w$.

For the arrangements represented in the fifth and sixth figures,

The pressure on the axis C, is $P=2p=\frac{2}{3}w$,

———————————— D, – $P=2p=\frac{2}{3}w$,

———————————— H, – $P=2p=\frac{2}{3}w$.

Now, in the fourth example preceding, the weight is given equal to 224 pounds, and the calculated power is $74\frac{2}{3}$ pounds; consequently, the pressure on the axis of each pulley, is

$$P=\frac{224\times 2}{3}=74\tfrac{2}{3}\times 2=149\tfrac{1}{3} \text{ pounds};$$

and the aggregate pressure of the system according to equation (a), is

$$P=p+w=74\tfrac{2}{3}+224=298\tfrac{2}{3} \text{ pounds}.$$

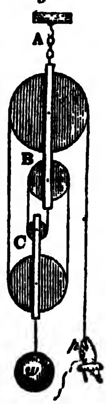
18. PROBLEM 3. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of several fixed and several moveable pulleys, when the same running cord acts upon them all, the number of fixed and moveable pulleys being equal.*

The solution of this problem, depends entirely on the principles developed in the solution of problem 2, but the arrangement in the present instance is somewhat different, and by reason of a more extensive range of combination, a greater variety of curious and important systems may be employed; but whatever variety of systems may be called into action under this principle, there is one important feature characteristic of them all, viz. that whatever number of pulleys may be arranged in the fixed block, the same number must be similarly arranged in the moveable one; especially when the power acts in the most advantageous direction.

The annexed diagram represents a system of this description, where one and the same cord acts on all the pulleys, both in the upper block B, which is fixed and immoveable, and in the lower block C, which moves upwards and downwards with the weight and the power.

It is manifest from the inspection of the figure, that the weight w is sustained by all the cords attached to the lower block, each cord supporting an equal portion of the whole.

Fig. 7.



Now, since the cord which goes round all the pulleys, commences at the lower extremity of the fixed block B, it is evident, that there must be twice as many portions of the cord engaged about each block as there are pulleys in it; but N by our notation, denotes the number of portions of the cord engaged in supporting the weight; consequently, in the case of an equilibrium, whatever may be the degree of tension on one part of the cord, the same degree of tension must exist in them all, and since the tension is universally measured by the magnitude of the power, it is manifest, that the tension of all the cords that support the weight, must be equal to as many times the power as there are cords so engaged; but we have just stated that N denotes the number of cords; therefore, we have

$$w = \text{N}p. \qquad (4)$$

Now in the diagram, fig. 7, there are only two pulleys in each block, consequently, the number of cords supporting the weight is 4; therefore, if 4 be substituted for N, in the above equation, we obtain,

$$w = 4p.$$

In this case also, the result which we have deduced from the foregoing reasoning, is identical with that which flows from the general equation No. 1; for here we have $\text{N}=4$, and $n=1$; consequently, by substitution, the general equation becomes

$$w = 4p,$$

the very same expression afforded by the investigation, as it ought to be.

19. The above equation (4) implies, that

If a power sustain a weight in equilibrio, by means of a running cord acting on all the pulleys of the system, the weight is equal to as many times the power, as there are portions of the cord engaged in supporting the moveable block to which the weight is attached.

Hence, it is manifest, that every additional moveable pulley, adds twice the magnitude of the power to the effect.

Example 1. In a system of pulleys of which two are fixed and two moveable, having the same cord acting on them all; what weight will a power of $9\frac{1}{2}$ pounds sustain, the parts of the cords being supposed paralled to one another?

In this example we have given $\text{N}=4$, and $p=9\frac{1}{2}$; let these numbers be substituted for N and p in equation (4), and it becomes

$$w = 4 \times 9\tfrac{1}{2} = 38 \text{ pounds}.$$

Example 2. By means of a system of pulleys, of which 6 are moveable, the same cord going round them all; what power will it require to sustain a weight of 336 pounds, the portions of the cord being supposed parallel to one another?

Here we have given $N=12$, and $w=336$ pounds; let these numbers be substituted for N and w in equation (4), and it becomes

$$12p=336;$$

therefore, dividing by 12, we get

$$p=\frac{336}{12}=28 \text{ pounds.}$$

It is manifest that the same equation (4), is applicable to the determination of the number of moveable pulleys; for if both sides of the equation be divided by p, we have

$$N=\frac{W}{p};$$

consequently, when the weight and its sustaining power are given, the number of parts of the cord engaged in supporting the weight, is found by simply dividing the weight by the power, and the number of moveable pulleys in the system, is obviously equal to half the number of sustaining cords: thus, suppose the weight to be 336 pounds, and the power 28; then, substituting these numbers for w and p in equation (4), we get

$$28N=336;$$

and dividing by 28, it is

$$N=\frac{336}{28}=12, \text{ the number of parts of the}$$

sustaining cord, one half of which, viz. 6, is the number of moveable pulleys, corresponding to the number proposed, in the second example above, as it ought to do.

Of the distribution of pressure in this system.

20. With respect to the distribution of pressure in a system of this sort, it is obvious, that the weight w is supported by all the cords attached to the lower block C, and since the tension is the same on every part of the cord throughout the system, each cord must support an equal portion of the load; but the number of cords engaged in supporting the load is, as we have stated above, equal to twice the number of moveable pulleys; consequently, the pressure on each cord is represented by the following equation, viz.

$$P=p=\frac{w}{N};$$

therefore, taking the load to be 336 pounds, as in the foregoing example, and the number of sustaining cords 12; then, according to the equation immediately preceding, we get

$$P=p=\frac{336}{12}=28 \text{ pounds.}$$

This degree of tension is the same as the power applied; for, in our description of the diagram which exhibits the system, we have shown that the tension on each portion of the cord engaged in supporting the weight, is equal to the tension on the line of traction, or that portion of the cord on which the power acts.

The whole pressure on the chain at A, setting aside the weight of the apparatus, is, according to equation (*a*), expressed as follows, viz.

$$P = p + w;$$

but by equation (4) we have

$$w = Np;$$

consequently, substituting this value of w in the above expression for the aggregate pressure on the chain at A, we obtain

$$P = (N+1)p. \qquad (b)$$

This equation in its present form, may perhaps require an example to illustrate its application; let us, for instance, take the following.

EXAMPLE 3. Suppose it were required to determine the pressure of the chain at A, in a system of 12 fixed and 12 moveable pulleys, the sustaining force being equivalent to 256 pounds.

Here we have given $N=24$ and $p=256$; let these numbers be substituted for N and p in equation (*b*), and it gives

$$P = (24+1) \times 256 = 6400 \text{ pounds.}$$

The same result, however, may be obtained otherwise, in the following manner, viz. by the equation marked (4) we have

$$w = 24 \times 256 = 6144 \text{ pounds.}$$

and by the equation marked (*a*) it is

$$P = 6144 + 256 = 6400 \text{ pounds, the same as before.}$$

This pressure, it must be observed, is wholly independent of the effect produced by the weight of the apparatus, including the blocks, pulleys, and cordage; but in cases where very great precision is required, it will be found necessary to take these into the account also; for in estimating the strength of the support to which the system is attached, the weight of the apparatus, if neglected, may be such as to produce rupture, even in cases where with the weight and the power only, there would have been no danger.

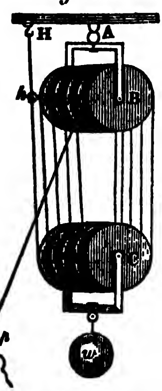
21. Systems of fixed and moveable pulleys, having the same cord acting on them all, are frequently combined otherwise than that which is exhibited in the foregoing diagram; but provided that the portions of the sustaining cord retain their parallelism, the conditions of equilibrium, and the relation that subsists between the weight to be supported and the power applied, must be the same in all; consequently, the equations (4 and *b*), will still obtain.

One of the most common arrangements is, when the wheels are placed, one by the side of another, on the same pin or axis, and are separated from one another by thin plates of iron or other materials,

each pulley occupying, as it were, a separate compartment in the block or frame that holds them together.

The subjoined diagram represents a system of this sort, where B is the upper block or frame fixed, to the beam by means of the sling and staple as at the point A, and C is the lower or moveable block, ascending or descending as influenced by the weight and power; w is the load to be raised or sustained in equilibrio, and p is the power acting at the extremity of the running cord, whose other extremity is fastened to the hook at H, or to some point h in the fixed block B. The method of estimating the conditions of equilibrium, is here the very same as in the preceding combination, but the pressure on the hook or support at A is not precisely the same, by reason of the running cord being fixed into the hook at H, which transfers a portion of the pressure to that point equivalent to the magnitude of the power applied; the pressure at the point A, is therefore only equal to the load, the weight of the apparatus being disregarded, but the aggregate pressure on the beam at the points A and H must, in all such arrangements, be the same; that is

Fig. 8.



$$P=(N+1)p.$$

If the end of the running cord, instead of being fastened to the supporting beam by a hook at H, should be attached to the immoveable block B, by a hook or bracket as at h; the whole pressure would be transferred to the beam at the point A, in which case, both the power and the pressure of this arrangement, would be determined by the same equations, as the power and the pressure of the arrangement immediately preceding; that is, with the same data, and under similar circumstances, the same results would ensue.

EXAMPLE 1. What weight will a power of 450 pounds be able to sustain in equilibrio, in the arrangement represented at fig. 8; the number of moveable pulleys being 5, and the portions of the sustaining cords that act upon the pulleys being parallel to one another?

Here we have given, $p=450$, and $N=10$; let these numbers be substituted for p and N in equation (4), and it becomes

$$w=450\times 10=4500 \text{ pounds.}$$

EXAMPLE 2. What power will be able to sustain a block of marble containing 24 cubic feet, by means of a system of 4 moveable and 4 fixed pulleys, arranged in the manner represented in fig. 8, the portions of the sustaining cord being parallel to one another?

It has been found by experiment that a cubic foot of common white marble weighs very nearly 170 pounds avoirdupois; conse-

quently, 24 cubic feet must weigh $170 \times 24 = 4080$ pounds; hence, we have given, $N = 8$, and $w = 4080$ pounds; let these numbers be substituted for N and w in equation (4), and it becomes

$$8p = 4080;$$

or, dividing both sides of the equation by 8, we get

$$p = 510 \text{ pounds.}$$

Of the distribution of pressure in the several points of this system, together with the accumulated strain, which is shown to be equal to the weight and power taken conjointly.

The 510 pounds determined above is also the pressure on each portion of the cord engaged in supporting the lower block; consequently, the pressure on the supporting beam and the hooks at the points A and H, are respectively as follows, viz.

The pressure at A is, $P = 510 \times 8 = 4080$ pounds,

——————— H —, $P = 510 \times 1 = 510$ ———,

and the accumulated pressure at the points A and H, calculated by equation (b), is,

$$P = (8 + 1) \times 510 = 4590 \text{ pounds.}$$

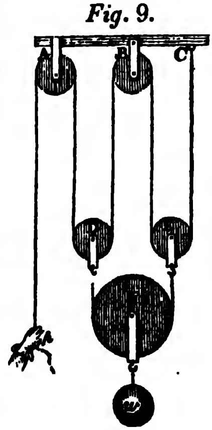
If the end of the cord, instead of being carried up to the hook at H, should be fastened to the fixed block as at h; then, the staple and ring at A will have to sustain the whole pressure of the weight and the power, which is, as we have just shown, equivalent to 4590 pounds.

22. Another curious and important combination of this kind is that which is implied in the following problem, in which the parts of the running cord supporting the several moveable pulleys in any system are parallel to one another.

PROBLEM 4. *To determine the conditions of equilibrium, and the relation that subsists between the power and the weight under the following circumstances.*

A power p, is applied to the extremity of a running cord which passes over a fixed pulley A, under a moveable pulley D, over a fixed pulley B, under a moveable pulley E, and is finally fixed into a hook at C. To the centre of the pulley D, is attached a cord which passes under a moveable pulley F and is fixed to the centre of the pulley E, and at the centre of the pulley F, the weight w is applied.

Fig. 9.



Find the conditions of equilibrium, and the pressures at the points A, B *and* C.

Since the parts* of the running cord are all parallel to one another, it is manifest, that the tension is uniform throughout the system, and equivalent to the sustaining power p; and because the tensions of the cords about the

pulley F are equal, and also those about the pulleys D and E, it is manifest, that the weight w rises vertically, and therefore the pulleys D and E are constantly in the same horizontal line; hence it follows, that the efficacy of the power p, is nothing augmented by means of the pulley F or the cord by which it is supported, its use being to maintain the parallelism of the parts of the running cord, and ensure the vertical movement of the weight;

consequently, we have

The pressure on $A = 2p$,
———————— $B = 2p$,
———————— $C = p$;

but the accumulated pressure on the supporting beam, disregarding the weight of the apparatus, is obviously equal to the sum of the weight and the power taken conjointly, hence we have

$$p + w = 5p; \text{ that is}$$
$$w = 4p.$$

Now, the number of effective pulleys in the system under consideration is only two, each of which is supported by two portions of the running cord; consequently, we have $N = 4$, which being substituted for N in the general equation No. 1, gives

$$w = 4p,$$

the same expression that we derived for the value of w, from the foregoing distribution of the pressure on the several points of support.

The general expression for the aggregate pressure at the different points of support, is obviously expressed by the following equation, viz.

$$P = (N + 1)p;$$

for we have shown above, that

$$p + w = 5p,$$

but N in the present system is equal to 4; consequently $(N + 1) = 5$, the coefficient of p in the above expression for the total pressure; hence it is manifest, that the equations which we have here obtained for the values of w and P, are precisely the same as those for the systems described under problem 3; and generally

If the parts of the running cord supporting several moveable pulleys in any system, be parallel to one another, the equations (4 and b) determine the conditions of equilibrium and pressure.

EXAMPLE 1. What weight will be sustained by a power of 386 pounds, under the circumstances represented in figure 9 immediately preceding?

Here we have given $p=386$, and $N=4$; let these numbers be substituted for p and N in equation (4) and it becomes

$$w=4\times 386=1544 \text{ pounds.}$$

With respect to the pressure at the points of support A, B and C, we have already shown, that,

$$\begin{aligned} \text{The pressure on } A &= 2p = 2\times 386 = 772, \\ \text{———————— } B &= 2p = 2\times 386 = 772, \\ \text{———————— } C &= p = 1\times 386 = 386; \end{aligned}$$

and the accumulated pressure on the beam at the several points of support, as determined from equation (b), is

$$P=(N+1)p=(4+1)\times 386=1930 \text{ pounds.}$$

EXAMPLE 2. What power will be sufficient to suspend a ton weight, by means of a system of pulleys arranged as in fig. 9, all the parts of the cord in contact with the moveable pulleys being parallel to one another, and what is the pressure at each point of support?

In this example we have given, $w=2240$, and $N=4$; let these numbers be substituted for w and N in equation (4), and it becomes

$$4p=2240,$$

and dividing both sides by 4, we get

$$p=560 \text{ pounds};$$

hence, the pressure at each point of support is,

$$\begin{aligned} \text{The pressure on } A &= 2p = 2\times 560 = 1120 \text{ pounds}, \\ \text{———————— } B &= 2p = 2\times 560 = 1120 \text{ ——}, \\ \text{———————— } C &= p = 1\times 560 = 560 \text{ ——}. \end{aligned}$$

COROL. There are various other combinations of pulley besides these which we have illustrated, wherein the same cord runs throughout the system from the point of application of the power, to some other point in opposition to it, where it is finally fixed; but every arrangement of this description is expounded by the same theory, and they are all of them liable, in a greater or lesser degree to the following objections, viz.

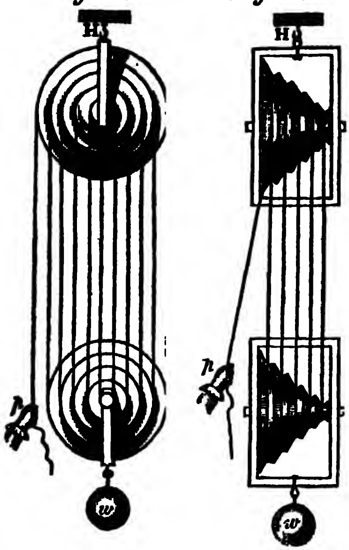
1. *The great quantity of friction generated on the pivots or axis, and against the sides of the blocks in which the wheels revolve.*
2. *The great irregularity of wear, occasioned by the different velocities with which the pulleys revolve on their axis.*

Of White's pulley, the wheels of which are a series of unequal concentric grooves.

22. But these and several other minor defects, are in a great measure obviated by the contrivance represented in figs. 10 and 11,

where the pulleys, instead of being a series of different wheels of equal diameters, revolving on the same or different axles, are represented by a series of unequal concentric grooves in the same wheel, whose diameters correspond to the portion of the cord that moves over their circumferences in a given time.

Fig. 10. Fig. 11.



By this arrangement, the velocity of the circumference of each groove, is adjusted to that of the cord passing over it, which evidently equalizes the wear of the machine, and the friction is reduced to that of the pivots, and the rubbing of one wheel on the sides of the block in which it revolves.

In order that the velocity of the several circumferences may be accommodated to that of the cord passing over them, it is necessary that the diameters of the several grooves, should follow the proportion of the series of consecutive numbers 1, 2, 3, 4, &c. so far as the number of grooves in both blocks, which in the present instance is ten, viz. five in each block; but for the actual construction of the respective grooves, it is requisite that the numbers of the series should be alternated, in the following manner, viz. For the lower or moveable block, the diameters are respectively 1, 2, 3, 4 and 5; and for the upper or fixed block, the diameters are respectively 2, 4, 6, 8 and 10. In practice however, the diameters of the grooves are not proportioned to the above numbers, nor can they be so if the diameter of the cord be taken into account, the proportion only holds when the cord is considered inextensible and of no sensible magnitude, but this in actual practice can never be the case; consequently, in the mechanical construction of the pulley, and its application to actual practice, the diameter of the cord must be deducted from each term of the foregoing series, otherwise the small grooves will have a tendency to rise and fall more rapidly than the larger ones, and thus render the contrivance more defective than the common systems which it is intended to supersede.

In this, as in every other arrangement, in which there is but one cord running throughout the system, the weight is equal to as many times the power, as there are parts of the cord engaged in supporting the weight, or the lower block, to which it is attached; now, in the present instance, there are ten portions of the cord thus employed; consequently, we have

$$w = 10p.$$

The very same result would flow from equation (4), or from the first of the general equations formerly referred to; hence it is manifest, that whatever may be the nature of the arrangement, provided that the same cord traverses the system, the conditions of equilibrium are universally expressed by the same equation.

EXAMPLE 1. What weight will be sustained in equilibrio, by a power equivalent to 600lbs. acting on a system like that represented in figs. 10 and 11, supposing that each block contains five grooves, and what will be the pressure on the hook at H, the weight of the apparatus not being considered?

Here we have given, $p=600$, and $N=10$; let these numbers be substituted for p and N in equation (4), and we obtain

$$w=600\times 10=6000 \text{ pounds.}$$

For the pressure on the hook H, equation (b), gives

$$P=(N+1)p=(10+1)\times 600=6600 \text{ pounds.}$$

EXAMPLE 2. The arrangement and the number of grooves remaining as in the last question; what power will be able to suspend a weight of 12564lbs, the weight of the apparatus not being considered, and what is the pressure on the hook at H?

In this example we have given, $w=12564$, and $N=10$; let these numbers be substituted for w and N in equation (4), and we have

$$10p=12564,$$

and dividing both sides of the equation by 10, we get

$$p=1256\cdot 4 \text{ pounds.}$$

For the pressure on the hook H, equation (b), gives

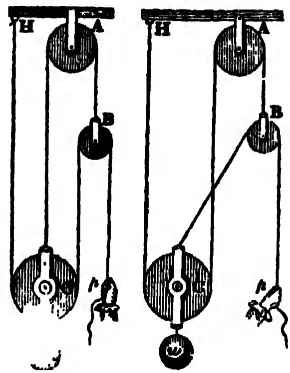
$$P=(N+1)p=11\times 1256.4=13820\cdot 4 \text{ pounds.}$$

We have now illustrated the principal combinations of pulleys in which there is but one cord running throughout the system, and we have also remarked in passing, that the same formula expresses the conditions of equilibrium for them all; but there are other arrangements frequently employed in practice, in which a greater number of cords is introduced, and by which a greater mechanical efficacy is obtained, and moreover, such as require different formulæ to expound their theoretical relations; a few of the most important of these, it is now our intention to investigate.

24. PROBLEM 4. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of one fixed and two moveable pulleys, acted on by two separate cords.*

The conditions specified in this problem are represented by the arrangements in figs. 12 and 13, in each of which we observe one fixed, and two moveable pulleys, with two separate cords, of which there are three portions communicating with, or engaged in sustaining the lower block to which the weight is attached.

Fig. 12. Fig. 13.



In fig. 12, the tension of the cord, extending from the power p, over the pulley B and under C to the fixed point H, is obviously equal to the power applied, while the tension of the cord extending from the pulley B and over A to the lower block, is equal to twice the power; consequently, the sum of the tensions of the three cords engaged in supporting the weight, is equal to four times the power; that is

$$w = 4p.$$

This equation is the same as that given in the note to No. 4 of the table of formulæ, immediately under our notation, and is also derivable from the first form of the general equation there given; for N in the present instance is equal to 3, to which if unity be added, gives 4 for the coefficient of p, the same as above. In fig. 13, the tension of the cord, extending from the power p, to the pulley B, is equal to the power, and this cord being finally attached to the pulley which sustains the weight, supports a portion of the weight, which is equal in magnitude to the power.

The cord from B to A, balances the united tensions of the parts from p to B, and from B to C; consequently, its tension is equal to twice the power, and being brought under the moveable pulley C, which sustains the weight, and finally fixed to the hook at H, it sustains a part of the weight equal to four times the power; therefore, the aggregate tensions of the three cords BC, AC and CH, which are engaged in supporting the weight, must be equal to five times the power applied; that is,

$$w = 5p.$$

This equation also, corresponds to that given in the note above alluded to, and is derivable from the second form of the general equation; for here N is equal to 3, to which if 2 be added, gives 5 for the coefficient of p, the same as has been determined from the figure.

In both of these combinations, it is manifest, that while the weight of the pulley B assists the power in sustaining the weight, that of C opposes it, and therefore, to have their effects exactly to

balance one another, it is necessary, that in the combination, figure 12, they should be equal to one another, but as represented in figure 13, if the weight of the pulley B be equal to half the weight of C, they will destroy each others effects, for in that case, the one assists the power just as much as the other opposes it.

Example 1. What weight will be sustained in equilibrio, by a power equivalent to 360 pounds, in the arrangements represented in figs. 12 and 13, the weight of the apparatus not being considered, and what is the pressure at the points A and H in both cases?

Here, in the case of fig. 12, we have given $p=360$, and $(N+1)=4$; but in the case of fig. 13, it is $p=360$, and $(N+2)=5$; consequently, the value of w for both cases, is as follows, viz.

For the arrangement in fig. 12, $w=360\times 4=1440$ lbs. and
For the arrangement in fig. 13, $w=360\times 5=1800$ lbs.
The pressure on the point A in fig. 12, is, $P=4p=1440$ lbs. and
The pressure on the point H ——— is, $P=p=360$ lbs.
And for fig. 13, the pressure on the point A ——— is, $P=4p=1440$ lbs. and
The pressure on the point H ——— is, $P=2p=730$ lbs.

Example 2. What power will be required to equipoise a weight of 864 pounds, by means of a system of pulleys arranged as represented in figs. 12 and 13, the weight of the apparatus not being taken into the account?

Here, in the case of fig. 12, we have

$$4p=864,$$

or, dividing both sides by 4, it is

$$p=216 \text{ pounds.}$$

In the case of fig. 13, we have

$$5p=864; \text{ that is,}$$

$$p=172\cdot 8 \text{ pounds.}$$

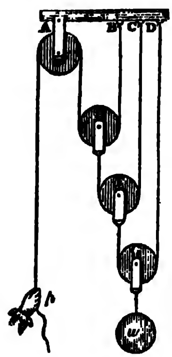
Section Second.

WHEN EACH MOVEABLE PULLEY HAS ITS OWN CORD PASSING OVER IT, AND IS ATTACHED TO A SEPARATE HOOK.

25. Problem 5. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of one fixed and several moveable pulleys, each moveable pulley having its own cord passing under it, and attached to a separate hook.*

When each moveable pulley in the system, has a separate cord passing under it and fastened to the separate hooks B, C and D as in the diagram; then, the determination of the conditions of equilibrium, or the relation that subsists between the power and the weight, requires the consideration of other principles; for by the second problem foregoing, in the case of a single moveable pulley, such as the pulley E in the diagram, it has been shewn, that

Fig. 14.



$$p : w :: 1 : 2,$$

and for the same reason, the weight sustained by the pulley E, is to the weight sustained by the pulley F in the same ratio; that is,

$$p : w :: 1 : 2,$$

and after the same manner may it be shown, that the weight sustained by the pulley F, is to the weight sustained by the pulley G in the same ratio; that is,

$$p : w :: 1 : 2,$$

and so on, as far as the number of moveable pulleys, or separate cords in the system; or generally,

$$p : w :: 1 : N^n,$$

where N according to our notation is the number of cords sustaining the pulley to which the weight is attached, and n, the number of distinct cords in the system; consequently, by equating the products of the extreme and mean terms, we get

$$w = N^n p. \qquad (5)$$

Now, in the arrangement which we are just considering, the number of cords engaged in supporting the pulley to which the weight is attached, is only 2; therefore, if 2 be substituted for N in equation (5), we shall obtain

$$w = 2^n p. \qquad (6)$$

This equation implies, that,

If a power sustain a weight in equilibrio, by means of a system of pulleys arranged as in Fig. 14, where each moveable pulley has its own cord passing under it, and finally attached to a separate hook; the weight is equal to as many times the power, as there are units in that power of 2, whose exponent is denoted by the number of moveable pulleys, or by the number of separate cords in the system.

From which it is manifest, that each additional cord or moveable pulley doubles the efficacy of the power.

EXAMPLE 1. What weight will be sustained by a power of 120 pounds acting by means of a system of 10 moveable pulleys, arranged as in fig. 14, each pulley having a separate cord passing under it, and finally attached to a separate hook; all the parts of the suspending cords being supposed parallel, and the weight of the apparatus not considered?

In this example there are given, $p=120$, and $n=10$; let these numbers be substituted for p and n in equation (6), and it becomes

$$w=2^{10}\times 120,$$

but the tenth power of 2, as indicated by the principles of involution, is

$$2^{3+2}\times 2^{3+2}=32\times 32=1024;$$

consequently, we obtain

$$w=1024\times 120=122880\text{ lbs.}$$

EXAMPLE 2. What power will suspend a weight of 4 tons by means of a system of 4 moveable pulleys, arranged as in fig. 14, all the parts of the suspending cords being supposed parallel, and the weight of the apparatus not considered?

In order to resolve this example, it is necessary to disengage the unknown or required quantity p, from the factor with which it is combined in equation (6), and by so disengaging it, we shall obtain

$$p=w\div 2^n;$$

now, in the question, $n=4$, and $w=4$ tons, or 8960 pounds; let these numbers be substituted for n and w in the foregoing expression for the value of p, and it becomes

$$p=\frac{8960}{16}=560\text{ pounds.}$$

EXAMPLE 3. A power of 5 pounds is found to suspend a weight of 640 pounds, by means of a system of pulleys arranged as in fig. 14, where the moveable pulleys are all suspended by separate cords, and the blocks and pulleys so adjusted, that all the parts of the suspending cords are parallel to one another; required the number of separate cords, or the number of moveable pulleys in the system?

Let both sides of equation (6) be divided by p, and we get

$$2^n=\frac{w}{p},$$

but by the nature of logarithms, it is

$$n.\log. 2=\log. w-\log. p,$$

or by division, it is

$$n=\frac{\log. w-\log. p}{\log. 2};$$

now, $w=640$, and $p=5$ by the question; consequently, by substitution, the above expression for the value of n becomes

$$n=\frac{\log. 640-\log. 5}{\log. 2}$$

or by actually employing the logarithms, we obtain

$$n=\frac{2\cdot806180-0\cdot699970}{0\cdot301030}=7.$$

Of the distribution of the weight on the hooks and frame.

26. With respect to the distribution of the weight on the hooks D, C, B and the frame or bolt at A, it is obvious, that

$$\begin{aligned}\text{The pressure on } D &= \tfrac{1}{2}\, w=(2^{n-1})\cdot p,\\ \text{———————— } C &= \tfrac{1}{4}\, w=(2^{n-2})\cdot p,\\ \text{———————— } B &= \tfrac{1}{8}\, w=(2^{n-3})\cdot p,\\ \text{———————— } A &= \tfrac{1}{2n}\, w=(2^{n-n})\cdot p;\end{aligned}$$

but the aggregate pressure on the supporting beam, setting aside the weight of the blocks, pulleys, and cordage, is manifestly equal to the power and the weight considered conjointly; hence we have

$$P=p+w=(1+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-n})\cdot p, \qquad (c)$$

now, we have already shown in equation (6), that

$$w=2^n p;$$

consequently, by substitution, we get

$$w+p=2^n p+p; \text{ that is}$$

$$P=(2^n+1).p. \qquad (d)$$

In the second example preceding, the weight is 8960 lbs, and the power is 560 lbs.; consequently, we have for the accumulated pressure at the several points of support, according to equation (a),

$$P=8960+560=9520 \text{ pounds,}$$

or according to equation (d), we have

$$P=(2^4+1)\times560=9520 \text{ pounds, the same as before.}$$

Of the pressure sustained by each point of support.

27. But to assign the portion of this pressure which is sustained by each point of support, we must have recourse to equation (c), where the law of the series is developed by which the pressure on each point is to be ascertained.

Now, in the present instance, we have given $n=4$, and $p=560$; consequently, by separating the terms of the series, as far as is indicated by the number of pulleys in the system, we shall have for the pressure on the several points of support, as follows, viz.

$$\begin{aligned}\text{For the pressure on } E \text{ it is } P &= (2^{n-1})\times560=4480,\\ \text{———————— } D \text{ —— } P &= (2^{n-2})\times560=2240,\\ \text{———————— } C \text{ —— } P &= (2^{n-3})\times560=1120,\\ \text{———————— } B \text{ —— } P &= (2^{n-4})\times560=\ \ 560,\\ \text{———————— } A \text{ —— } P &= (1+1)\times560=\underline{1120,}\end{aligned}$$

Then, by taking the sum of these several portions, we get . . . . . } 9520 pounds

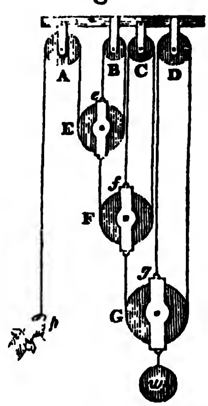
for the aggregate pressure on the supporting beam, the same as before.

The mechanical efficacy is augmented by passing the cord over a fixed pulley in the load.

28. We have elsewhere stated, that the mechanical efficacy of this system may be greatly augmented, without increasing the number either of the cords or the moveable pulleys, by merely passing each cord over a small fixed pulley instead of fastening it to a hook, and finally attaching the cords to the moveable pulleys which they respectively sustain.

An arrangement of this kind is represented in fig. 15, where it is manifest, that the tension of the cord extending from the power p to the pulley A, is measured by the magnitude of the power, and since the same cord passes over the pulley A, under the pulley E, over the pulley B and is finally fixed to the hook at e, it is evident, that the three parts of the cord engaged in sustaining the pulley E, support a portion of the weight w, which is equal to three times the power.

Fig. 15.



But the tension of the cord EF, is equal to the united tensions of the three cords sustaining the pulley E; that is, equal to three times the power; consequently, the three parts of the cord EFCf, which are engaged in sustaining the pulley F, support a portion of the weight w, which is equal to nine times the power.

Again, the tension on the cord FG, is equal to the united tensions of the three cords sustaining the pulley F; that is, equal to nine times the power; consequently, the three parts of the cord FGBg, which are engaged in sustaining the pulley G, support a portion of the weight equal to twenty seven times the power; and so on in the same proportion for any number of cords whatever, or as far as is indicated by the number of moveable pulleys. Hence generally,

$$p : w :: 1 : N^n,$$

where N is the numbers of cords engaged in sustaining the pulley or block to which the weight is attached, and n, the number of distinct cords in the system; consequently, by multiplying extremes and means, we obtain

$$w = N^n p.$$

This expression corresponds in form with equation (5), but in this case, the value of N, or the number of cords engaged in supporting each moveable pulley is 3; therefore, if the number 3 be substituted for N in the above equation, we obtain

$$w = 3^n p. \qquad (7)$$

This equation implies, that,

If a power sustain a weight in equilibrio, by means of a system of pulleys arranged as in fig. 15, where each

distinct cord is passed over a fixed pulley, and finally attached to the moveable one which it sustains; the weight is equal to as many times the power, as there are units in that power of 3, whose exponent is denoted by the number of moveable pulleys, or by the number of separate cords in the system.

From which it is manifest, that each additional cord or moveable pulley, triples the efficacy of the power.

EXAMPLE 1. What weight will be sustained by a power equivalent to 256 pounds, acting by means of a system of 8 moveable pulleys, arranged as in fig. 15, where each cord is passed over a fixed pulley, and finally attached to the moveable one which it sustains; all the parts of the suspending cords being supposed parallel, and the weight of the apparatus not being considered?

In this example there are given, $p=256$, and $n=8$; let these numbers be substituted for p and n in equation (7), and it becomes

$$w=3^8\times 256,$$

but the 8th power of 3, as indicated by the principles of involution, is

$$3^{2+2}\times 3^{2+2}=81\times 81=6561;$$

consequently, we obtain

$$w=6561\times 256=1679616 \text{ lbs.}$$

The same example performed according to equation (6), would give a weight of 65536 pounds, to be sustained in equilibrio, by a power equivalent to 256 pounds; consequently, the system represented in fig. 15, is, with the same number of pulleys and the same applied power, nearly $25\frac{3}{4}$ times more efficacious than the system represented in fig. 14.

EXAMPLE 2. What power will suspend a weight of 6746 pounds, by means of a system of 5 moveable pulleys, arranged as in fig. 15, all the parts of the suspending cords being supposed parallel, and the weight of the apparatus not considered?

Here we must disengage the unknown or required quantity p, from the factor with which it is combined in equation (7), and by so disengaging it, we shall get

$$p=w\div 3^n;$$

now, in the example, $n=5$, and $w=6746$; let these numbers be substituted for n and w in the above expression for the value of p, and it becomes

$$p=\frac{6746}{243}=27\tfrac{185}{243} \text{ lbs.}$$

EXAMPLE 3. A power of 5 pounds is found to suspend a weight of 405 pounds, by means of a system of pulleys, arranged as in fig. 15, where the blocks and pulleys are so adjusted, that all the parts of the suspending cords are parallel to one another; required the number of separate cords, or the number of moveable pulleys in the system?

Let both sides of equation (7) be divided by p, and it is

$$3^n=\frac{w}{p},$$

but by the nature of logarithms, we have

$$n\log.3=\log.w-\log.p;$$

that is, $n=\dfrac{\log.w-\log.p}{\log.3}$;

now, $w=405$, and $p=5$ by the question; consequently, by substitution, the above expression for the value of n becomes

$$n=\frac{\log.405-\log.5}{\log.3},$$

or by actually employing the logarithms, it is

$$n=\frac{2\cdot607455-0\cdot698970}{0\cdot477121}=4.$$

Of the distribution of the weight on the fixed blocks.

29. With respect to the distribution of the weight on the fixed blocks D, C, B and A, it is manifest, that

The pressure on $\text{D}=\frac{2}{3}w=2(3^{n-1})\cdot p$,
———— $\text{C}=\frac{2}{9}w=2(3^{n-2})\cdot p$,
———— $\text{B}=\frac{2}{27}w=2(3^{n-3})\cdot p$,
———— $\text{A}=\frac{1}{3^n}w=(3^{n-n})\cdot p$;

but the aggregate pressure on the supporting beam, setting aside the weight of the blocks, pulleys and cordage, is manifestly equal to the power and the weight taken conjointly; hence we have

$$\text{P}=p+w=2(1+3^{n-1}+3^{n-2}+3^{n-3}+3^{n-n})\cdot p, \qquad (e)$$

now, we have already shown in equation (7), that

$$w=3^np;$$

consequently, by substitution, we get

$$w+p=3^np+p; \text{ that is,}$$

$$\text{P}=(3^n+1)\cdot p. \qquad (f)$$

In the third example preceding, the weight is 405 pounds, the power 5 pounds, and the number of moveable pulleys as found by calculation is 4; consequently, the accumulated pressure at the several points of support is, by equation (a), as under, viz.

$$\text{P}=405+5=410 \text{ pounds,}$$

or, according to equation (f), we have

$$\text{P}=(3^4+1)\times5=410 \text{ pounds, the same as above.}$$

Of the distribution of the pressure sustained by each point of support.

30. But to determine the portion of this pressure which is sustained by each point of support, we must revert to equation (e), where the law of the series is developed by which the respective pressures are to be assigned.

Now, in the present example, we have given, $n=4$, and $p=5$; consequently, by separating the terms of the series, continued as

far as is denoted by the number of moveable pulleys in the system, the respective pressures on the several points of support, will be as follows, viz.

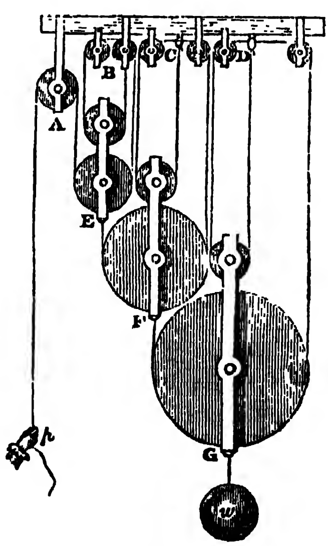
The pressure on E, is $P=2(3^{n-1})\times 5=270$
———— D, – $P=2(3^{n-2})\times 5=\ 90$
———— C, – $P=2(3^{n-3})\times 5=\ 30$
———— B, – $P=2(3^{n-4})\times 5=\ 10$
———— A, – $P=\ \ 2\times 5\ \ =\ 10$

Then, by taking the sum of these several portions, we get 410 pounds, for the aggregate pressure on the supporting beam, the same as before.

31. *Of a system of fixed and moveable pulleys, whose mechanical efficacy ascends according to the successive powers of* 4.

We have stated in another place, that if another fixed and another moveable pulley be introduced for each cord, without altering the number of separate cords in the system, a still higher mechanical effect will be obtained; for in this case there are four portions of each cord engaged in sustaining the moveable pulleys; consequently, the mechanical efficacy of the system ascends, according to the successive powers of the number 4.

Fig. 16



An arrangement of this kind is represented in fig. 16, where it is manifest, that the cord pA, extending from p to the fixed pulley A, is strained by a force equivalent to the power which is applied at the point p, and since every part of the same cord as it proceeds from A to the fixed point under the pulley B, is strained by an equal force, it follows, that the four cords which are engaged in sustaining the moveable block E, must support a portion of the weight equivalent to four times the power.

But the tension of the cord EF, is equal to the united tensions of the four cords which sustain the block E; that is, equal to four times the power, and since the same cord, as it proceeds from E to the fixed point at C is equally strained, the four cords which are engaged in supporting the block F, must sustain a portion of the weight w equivalent to 16 times the power.

And after the same manner it may be shown, that the four cords which are engaged in supporting the block G, must sustain a portion of the weight w equivalent to 64 times the power; and

so on in the same proportion for any number of separate cords, or as far as is indicated by the number of moveable blocks in the system. Hence it is

$$p : w :: 1 : N^n,$$

where as heretofore, N is the number of portions of the cord, engaged in supporting the pulley or block to which the weight is attached, and n, the number of separate cords in the system; consequently, by equating the products of the extreme and mean terms, we get

$$w = N^n p.$$

Now, in this case N, or the number of cords engaged in sustaining each moveable block is 4; therefore, if 4 be substituted for N in the above value of w, we shall get

$$w = 4^n p. \qquad (8)$$

This equation implies, that,

If a power sustain a weight in equilibrio, by means of a system of pulleys arranged as in fig. 16; *the weight is equal to as many times the power, as there are units in that power of* 4, *whose exponent is denoted by the number of moveable blocks, or by the number of separate cords in the system.*

From which it is manifest, that each additional cord or moveable block, quadruples the mechanical efficacy of the machine.

EXAMPLE 1. What weight will be sustained in equilibrio, by a power equivalent to 384 pounds, acting by means of a system of pulleys arranged as in fig. 16; there being three moveable blocks in the system, and all the parts of the suspending cords parallel to one another?

In this example there are given, $p = 384$ lbs. and $n = 3$; let these numbers be substituted for p and n in equation (8), and it becomes

$$w = 4^3 \times 384,$$

but $4^3 = 64$; consequently, we have

$$w = 64 \times 384 = 24576 \text{lbs. nearly 11 tons.}$$

EXAMPLE 2. What power will be sufficient to suspend a weight of 10000 pounds, by means of a system of pulleys arranged as in fig. 16; there being 6 moveable blocks or separate cords in the system, and all the parts of the cords parallel to one another?

Here we have given $w = 10000$ lbs. and $n = 6$; let these numbers be substituted for w and n in equation (8), and it becomes

$$4^6 p = 10000,$$

but the sixth power of 4 as indicated by the principles of involution is

$$4^3 \times 4^3 = 64 \times 64 = 4096;$$

consequently, we have

$$p = \frac{10000}{4096} = 2 \cdot 44 \text{lbs. nearly } 2\tfrac{1}{2} \text{ pounds.}$$

EXAMPLE 3. A power equivalent to 4 pounds, is found to suspend a weight of 262144 pounds, by means of a system of pulleys arranged as in fig. 16, where the blocks and pulleys are so adjusted, that all the parts of the suspending cords are parallel to one another; required the number of separate cords, or moveable blocks in the system?

Let both sides of equation (8) be divided by p, and it is

$$4^n=\frac{w}{p},$$

but by the nature of logarithms, we get

$$n\cdot\log.4=\log.w-\log.p,$$

which, by division, becomes

$$n=\frac{\log.w-\log.p}{\log.4};$$

now, $w=262144$, and $p=4$ by the question; consequently, by substitution, the above expression for the value of n becomes

$$n=\frac{\log.262144-\log.4}{\log.4}$$

or by actually employing the logarithms, it is

$$n=\frac{5.418540-0.602060}{0.602060}=8.$$

Of the distribution of the weight on the pulleys and rings of this system.

32. With respect to the distribution of the weight on the pulleys and rings attached to the supporting beam, it is manifest, that taking them in order, from the right side of the system towards the point where the power acts, we have

For the pressure on the ring D, and its two contiguous pulleys, $P=\frac{3}{4}w=3(4^{n-1})\cdot p.$

For the pressure on the ring C, and its two contiguous pulleys, $P=\frac{3}{16}w=3(4^{n-2})\cdot p,$

For the pressure on the pulley to the right of B, $P=\frac{2}{64}w=2(4^{n-3})\cdot p,$

—— ———— pulley to the left of B, $P=\frac{1}{64}w=\ (4^{n-3})\cdot p,$

———— pulley ———— A, $P=\frac{1}{4^n}w=\ (4^{n-n})\cdot p,$

but the aggregate pressure on the supporting beam, setting aside the weight of the apparatus, is evidently equal to the power and the weight considered conjointly; consequently, we have

$$P=p+w=\{2+3(4^{n-1}+4^{n-2}+4^{n-3}..\&c.)\}\cdot p, \qquad (g)$$

now, we have shown in equation (8), that

$$w=4^np;$$

consequently, by substitution, we get

$$w+p=4^np+p,$$

or, which is the same thing,

$$A=(4^n+1)\cdot p. \qquad (h)$$

Now, in the third of the foregoing examples, the weight is 262144lbs., the power 4lbs. and the number of separate cords, as

found by calculation is 8; consequently, the accumulated pressure at the several points in the supporting beam is, by equation (*a*),

$$P=262144+4=262148 \text{ pounds,}$$

or according to equation (*h*), we have

$$P=(4^8+1)\times 4=262148, \text{ the same as above.}$$

But to determine the portion of this pressure that falls on each ring and its contiguous pulleys, we must have recourse to equation (*g*), which being extended to the specified number of terms, will indicate the pressure for each, as follows

The pressure on ring	I	and pulley is, $P=3(4^{n-1})\times 4=$	196608,
———	H	——— $P=3(4^{n-2})\times 4=$	49152,
———	G	——— $P=3(4^{n-3})\times 4=$	12288,
———	F	——— $P=3(4^{n-4})\times 4=$	3072,
———	E	——— $P=3(4^{n-5})\times 4=$	768,
———	D	——— $P=3(4^{n-6})\times 4=$	192,
———	C	——— $P=3(4^{n-7})\times 4=$	48,
——— pulley	B	——— $P=3(4^{n-8})\times 4=$	12,
——— pulley	A	——— $P=\ 2\times 4\ =$	8.

Then, by taking the sum of these several portions, we have for the aggregate, ... } 262148lbs.,

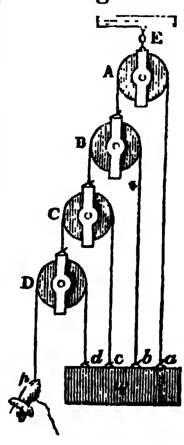
the same as before.

Section Third.

WHEN THE MOVEABLE PULLEY HAS ITS OWN CORD GOING OVER IT, AND IS FINALLY ATTACHED TO THE WEIGHT OR RESISTANCE.

33. Problem 6. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of one fixed and several moveable pulleys, each moveable pulley having its own cord going over it, and finally attached to the resistance or weight to be sustained, all the sustaining cords being parallel to one another.*

Fig. 17.



Here in the case of an equilibrium, if we suppose the power *p*, to act upon a cord passing freely over the pulley D, and to be fixed to the weight *w* in the point *d*; then it is manifest, that the whole pressure on the pulley D, is equivalent to twice the power *p*, because the tension on the part of the cord extending from *p* to D, is equal to the tension on the part extending from D to *d*, and each of them is equal to the power *p*; therefore, the pressure on

$$D=2p.$$

But the cord CD supports the whole force or pressure on the pulley D, and because the cord DC*c*, passes freely over the pulley C, the tension on the part C*c* is also equal to the pressure on the pulley D; consequently, the whole force or pressure on C, is twice the force or pressure on D; that is, the force or pressure on the pulley

$$C=4p.$$

Again, the cord BC sustains the whole force or pressure on the pulley C, and because the cord CBb passes freely over the pulley B, the tension on the part Bb, is equal to the tension on the part BC; consequently, the whole force or pressure on the pulley B, is equal to twice the force or pressure on the pulley C; that is, the pressure on the pulley

$$B = 8p.$$

And after the same manner it may be shown, that the pressure on the pulley A, is equal to twice the pressure on B; that is, the pressure on

$$A = 16p.$$

And thus may the process of induction be pursued as far as the specified number of pulleys in the system, or as far as the number of separate cords attached to the load, which datum in the present arrangement, limits the terms of the series.

Hence, the whole force of the cords Aa, Bb, Cc, Dd &c., that operate to support the weight w, is evidently equivalent to half the accumulated force, or pressure on the several pulleys A, B, C, D, &c., as far as the number of pulleys; that is,

$$w = \tfrac{1}{2}(2^1 + 2^2 + 2^3 + 2^4 \ldots 2^n) \cdot p,$$

or which is the same thing

$$w = (1 + 2 + 2^2 + 2^3 \ldots 2^{n-1}) \cdot p. \qquad (i)$$

Now, the sum of the geometric series within the parenthesis, is manifestly expressed by $2^n - 1$, where n denotes the number of terms in the series, corresponding to the number of separate cords by which the weight is supported; hence we get

$$w = (2^n - 1) \cdot p. \qquad (9)$$

This equation implies, that

If a power sustain a weight in equilibrio, by means of a system of pulleys arranged as in fig. 17, where each pulley has its own cord passing over it and attached to the weight; the weight is equal to as many times the power as there are units, less one, in that power of 2 whose exponent is denoted by the number of cords sustaining the load.

Hence it appears, that whatever may be the intensity of the power at any particular term of the series, that intensity is more than doubled at the next succeeding term, by the magnitude of the single power.

For, let $(2^n - 1)p$ measure the intensity of the power at any term of the series; then will $(2^{n+1} - 1)p$ measure the intensity at the $(n+1)^{\text{th}}$ term; consequently, we have

$$\frac{(2^{n+1} - 1)}{(2^n - 1)} = 2 + \frac{1}{2^n - 1}; \text{ that is,}$$

the numerator of the fraction divided by the denominator gives a quotient of 2 with a remainder p; hence the truth of the foregoing inference is manifest.

EXAMPLE 1. In a system of 4 pulleys, arranged as in fig. 17, where the cord round each pulley is attached to the weight; it is required to ascertain what load will be kept in equilibrio by a force equivalent to 8 pounds, the suspending cords being supposed parallel to one another, and the weight of the blocks, pulleys, and cordage not being considered?

Here we have given, $p=8$, and $n=4$; let these numbers be substituted for p and n in equation (9), and we shall have

$$w=(2^4-1)\times 8;$$

that is, $w=(16-1)\times 8=120$ pounds.

EXAMPLE 2. What power will be required to hold in equilibrio a weight of 2040 pounds, by means of a system of 8 pulleys, arranged as in fig. 17, where the cord round each pulley is attached to the weight, the suspending cords being all parallel to one another, and the weight of the apparatus not considered?

In this example there are given, $n=8$, and $w=2040$ pounds; let these numbers be substituted for n and w in equation (9), and it becomes

$$(2^8-1)p=2040,$$

but the eighth power of 2, as indicated by the principles of involution, is

$$2^{2+2}\times 2^{2+2}=16\times 16=256;$$

consequently, we have

$$(256-1).p=2040;$$

therefore, dividing each side of the equation by $(256-1)=255$, we obtain

$$p=\frac{2040}{255}=8 \text{ pounds.}$$

Hence, in a system of pulleys, such as that described in the question, a power of 8 pounds is sufficient to balance a load of 2040 pounds, or the power is to the weight in the ratio of unity to 255.

EXAMPLE 3. In a system of pulleys arranged as in fig. 17, where the cord passing over each pulley is attached to the weight, a power of 5 pounds is sufficient to hold in equilibrio a weight or resistance of 635 pounds; required the number of pulleys in the system, supposing the suspending cords to be all parallel to one another, and the weight of the apparatus not considered?

Here we have given $p=5$, and $w=635$; let these numbers be substituted for p and w in equation (9), and it becomes

$$5(2^n-1)=635,$$

divide both sides of the equation by 5, and we get

$$2^n-1=127,$$

transpose and we have

$$2^n=128,$$

but by the nature of logarithms

$$n \log. 2=\log. 128;$$

hence, by division, we obtain

$$n=\frac{\log. 128}{\log. 2},$$

or by actually employing the logarithms, we get

$$n=\frac{2\cdot107210}{0\cdot301030}=7 \text{ pulleys.}$$

This is the method of resolving a particular example, but that nothing may be wanting to render the subject complete, we shall here deduce the general formulæ, by which any question similar to the above may be resolved, without going through the several steps exhibited in the particular solution.

34. *General formulæ for the solution of similar examples.*

From equation (9), we get by division

$$2^n-1=\frac{w}{p},$$

which by transposition becomes

$$2^n=\frac{w+p}{p},$$

and by the nature of logarithms, we get

$$n. \log. 2=\log. (w+p)-\log. p,$$

which by division gives

$$n=\frac{\log. (w+p)-\log. p}{\log. 2}.$$

Of the distribution of pressure throughout each of the sustaining cords.

35. With regard to the strain or pressure on the beam E, it is obviously equal to the weight w, and the power p taken conjointly; that is

$$P=w+p,$$

but by equation (9), we have

$$w=(2^n-1).p;$$

consequently, by substitution, we get

$$P=2^n p. \qquad (k)$$

This equation expresses the total strain or pressure on the beam at the point of support, and is, as we have already stated, equivalent to the sum of the weight and the power considered conjointly, omitting the effect produced by the weight of the blocks, pulleys, and cordage.

In the second example foregoing, the weight w is equal to 2040 pounds, the power as found by calculation is 8 pounds, and the number of pulleys or separate cords is 8; consequently, by equation (u), the whole pressure on the supporting beam at E is

$$P=2040+8=2048 \text{ pounds,}$$

or by equation (k) we have

$$P=2^8\times8=256\times8=2048, \text{ the same as above.}$$

But to assign the portion of this strain or pressure, which is communicated to the point E, through each of the sustaining cords, we must recur to the series developed in equation (i), from whence we obtain the following series of numbers, for the pressure or strain on the respective cords, beginning with that which is farthest from the point where the power is applied; thus:

Pressure or strain transmitted by the 1st cord is	$(2^{n-1}).p=$	$128\times 8=$	1024
———————————2nd	$(2^{n-2}).p=$	$64\times 8=$	512
———————————3rd	$(2^{n-3}).p=$	$32\times 8=$	256
———————————4th	$(2^{n-4}).p=$	$16\times 8=$	128
———————————5th	$(2^{n-5}).p=$	$8\times 8=$	64
———————————6th	$(2^{n-6}).p=$	$4\times 8=$	32
———————————7th	$(2^{n-7}).p=$	$2\times 8=$	16
———————————8th	$(2^{n-8}).p=$	$1\times 8=$	8
		sum of the portions $=$	2040

to which add the power, and we shall have $2040+8=2048$ pounds for the aggregate pressure, or strain on the beam, the same as before.

36. The equation marked (k) for the pressure, corresponding to the arrangement of fig. 17, is evidently identical with equation (6) for the equipoising weight, corresponding to the arrangement in fig. 14; and because the aggregate pressure, or strain upon the beam in both cases, is equivalent to the sum of the weight and the sustaining force; it follows, that with the same number of suspending cords, the strain on the beam in the former case, exceeds that in the latter by the magnitude of the power, and this circumstance leads us to a comparison of the different systems, with regard to their mechanical effect.

Let it for instance, be required to exhibit the comparative merits of the systems, whose conditions of equilibrium are respectively expounded by the three following equations, viz.

$$1.\ w=N^{n}p,$$
$$2.\ w=2^{n}p,$$
$$3.\ w=(2^{n}-1).p.$$

Where N, n, p and w must be understood to denote the same quantities specified in our notation, and must be applied accordingly in the comparison.

Now, supposing the power applied in each system to be 28 pounds, and let $N=n=6$; then, we shall have for the respective mechanical effects, as follows, viz.

In the first system, $w=28\times 6 = 168$ pounds,
In the second system, $w=28\times 2^{6} = 1792$ ———,
In the third system, $w=28\times(2^{6}-1)= 1764$ ———,

where it is manifest, that the second system has the advantage, when the comparison proceeds on the supposition of an equal

number of cords being engaged in sustaining the weight; but it is obvious, that this will not hold, in the case of an equal number of moveable pulleys, which is after all the most exact mode of comparison.

37. Let us then suppose the number of moveable pulleys in each system to be 6, the power remaining as above; then the equations of equilibrium for a combination of 6 moveable pulleys are respectively as below, viz.

1. $w=12p$,
2. $w=2^6p$,
3. $w=(2^7-1)p$;

which equations being actually expanded, we obtain

In the first system, $w=28\times 12=336$ pounds,
In the second system, $w=28\times 64=1792$ ———,
In the third system, $w=28\times 127=3556$ ———.

Hence it appears, that on the supposition of the same number of moveable pulleys, the arrangement exhibited in fig. 17 has by far the advantage in point of mechanical energy; and this advantage arises from the circumstance of all its pulleys, both fixed and moveable, proving effective; and what is more, the power gained by the fixed pulley in this system, exceeds the power gained by all the moveable ones be they ever so many; whereas, in the other systems, the fixed pulleys gain no advantage whatever, but only serve to change the direction of the power, and facilitate the motion of the cords.

These remarks, however, it will be seen, apply only to the theory of the several systems according to the tenor of our investigations, and have no reference whatever to their practical applications; the reader will therefore keep it in mind, that the above comparison is merely theoretical, and not at all intended to excite prejudice for or against this or that system, in point of practical utility.

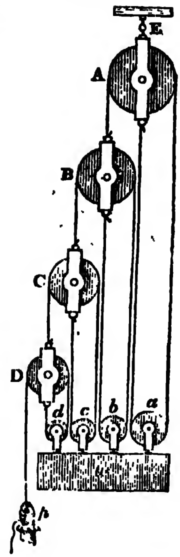
It has been stated in another place, that the mechanical effect of the arrangement represented in fig. 17 may be greatly augmented, by merely passing each cord under a small pulley fixed to the load, and finally attaching it to the moveable pulley over which it originally passes, as in the following diagram.

Of a single fixed and several moveable pulleys, in which every small pulley in the system has a separate cord passing under it, while the pulley itself is fixed into the load or resistance.

38. A combination of this sort will not, it is true, be frequently met with in practice, nevertheless we think proper to give it in this place, not only because it is well adapted for exercise to the reader in the art of arrangement, but because of its effect in impressing on the memory the general principle on which the equilibrium of the pulley depends.

A system such as we refer to is represented in fig. 18, in which it is manifest, that the tension of the cord extending from p to D, is measured by a force equivalent to the power, and since the same cord passes freely over the pulley D and under d, till it is finally attached to the lower part of the block in which the pulley D revolves, it is evident that the whole strain, or pressure on the pulley D, is equivalent to three times the power; for the tension on every part of the cord is the same throughout its length, and since there are three parts of it exerting their influence on the pulley D; it follows, that the pressure on

Fig. 18.



$$D = 3p.$$

But the cord CD supports the whole force or pressure on the pulley D, and because the cord DcC passes freely over C and under c, till it is finally attached to the lower part of the block in which the pulley C revolves; it follows, that the force or pressure on the pulley C, is equal to three times the pressure on the pulley D; that is, the pressure on

$$C = 9p.$$

Again, the cord BC supports the whole strain or pressure on the pulley C, and by reason of the uniform tension, and free passage of the cord over B and under b, till it is finally attached to the lower part of the block in which the pulley B revolves; it follows, that the pressure on

$$B = 27p.$$

And in like manner it may be shown that the whole force, or pressure on the pulley A, is three times the pressure on B; that is, the pressure on

$$A = 81p.$$

And thus may we proceed, as far as the specified number of separate cords in the system may be extended.

But the whole force of the several cords that tend to support the weight w, is obviously equivalent to two thirds of the accumulated force or pressure on the pulleys A, B, C, D, &c., as far as the number of pulleys may be extended: therefore we have

$$w = \tfrac{2}{3}(3+3^2+3^3+3^4 \,..+3^n)\cdot p,$$

or which is the same thing,

$$w = 2(1+3+3^2+3^3 \,..+3^{n-1})\cdot p, \qquad (l)$$

or by summing the terms, we get

$$w = (3^n - 1)p. \qquad (10)$$

This equation implies, that,

If a power sustain a weight in equilibrio, by means of a system of pulleys arranged as in fig. 18; *the weight is equal to as many times the power, as there are units, less one, in that power of* 3, *whose exponent is denoted by the number of separate cords in the system.*

Whence it appears, that whatever may be the intensity, or mechanical effect of the power at any particular term of the series, that intensity will be more than trebled at the next succeeding term, by twice the magnitude of the power applied.

For let 3^n-1 represent any particular term of the series, then will $3^{n+1}-1$ represent the next succeeding term; consequently, we have

$$\frac{3^{n+1}-1}{3^n-1}=3+\frac{2}{3^n-1},$$

which result confirms the above inference.

EXAMPLE 1. In a system of 4 pulleys, arranged as in fig. 18, it is required to ascertain what load will be suspended by a power of 12lbs., the suspending cords being supposed parallel to one another, and the weight of the apparatus not considered?

Here we have given $p=12$, and $n=4$; let these numbers be substituted for p and n in equation (10), and it becomes

$$w=(3^4-1)\times 12,$$

but the fourth power of 3, as indicated by the principles of involution, is

$$3^2\times 3^2=9\times 9=81;$$

consequently, we have

$$w=(81-1)\times 12=960\text{lbs.}$$

EXAMPLE 2. What power will be required to suspend a weight of 4368lbs. by means of a system of 6 pulleys, arranged as in fig. 18, where all the parts of the suspending cords are parallel to one another, and the weight of the blocks, pulleys and cordage not taken into the account?

In this example there are given, $n=6$, and $w=4368$ pounds; let these numbers be substituted for n and w in equation (10), and it becomes

$$(3^6-1)p=4368,$$

but the 6th power of 3, as indicated by the principles of involution, is

$$3^3\times 3^3=27\times 27=729;$$

hence, we obtain

$$p=\frac{4368}{729-1}=6\text{ pounds.}$$

EXAMPLE 3. In a system of pulleys, arranged as in fig. 18, where all the parts of the suspending cords are parallel to one another, a power of 4 pounds is found to balance a weight of

26240 pounds; what is the number of pulleys, or separate cords in the system, the weight of the apparatus not being considered?

Here we have given, $p=4$, and $w=26240$ pounds; let these numbers be substituted for p and w in equation (10), and it becomes

$$4(3^n-1)=26240,$$

divide both sides of the equation by 4, and it is

$$3^n-1=6560.$$

transpose, and we get,

$$3^n=6561,$$

but by the nature of logarithms, we have

$$n.\log.3=\log.6561,$$

hence, by division, we obtain

$$n=\frac{\log.6561}{\log.3},$$

or by actually employing the logarithms, we get

$$n=\frac{3.816970}{0.477121}=8 \text{ pulleys, or separate cords.}$$

Of the accumulated strain on the beam.

39. With respect to the strain or pressure on the beam at E, it is, as has been formerly observed, equivalent to the sum of the weight and power; that is

$$P=w+p,$$

but by equation (10), we have

$$w=(3^n-1)p;$$

consequently, by substitution, we get

$$P=3^n p. \qquad (m)$$

In the first of the examples to equation (10), the calculated weight is 960 lbs., and the given power is 12 pounds; the number of pulleys or separate cords being only 4; consequently by equation (a), the accumulated pressure at E is

$$P=960+12=972 \text{ pounds,}$$

or by equation (m), we have

$$P=3^4\times 12=972, \text{ the same as above.}$$

Of the pressure on each cord in the system.

40. But in order to determine how much of this accumulated strain, is transferred to the point E, through each of the suspending cords, we must return to equation (l), where the law of the series is developed by which the values of the several terms are to be assigned; thus,

The pressure or strain transmitted By the two parts of the first cord is, $2(3^{n-1})\times 12=648,$
———————— second ———, $2(3^{n-2})\times 12=216,$
———————— third ———, $2(3^{n-3})\times 12=\ 72,$
———————— fourth ———, $2(3^{n-4})\times 12=\ 24,$
sum of the portions $=960$, to which add the given power, and we get

$$P=960+12=972\text{lbs. the same as before.}$$

In the whole of the foregoing problems, we have supposed the blocks and pulleys to be so adjusted, as to admit all the parts of the suspending cords to act in parallel directions; it therefore only remains, to consider a case or two, in which the directions of the cords are oblique to one another.

Section Fourth.

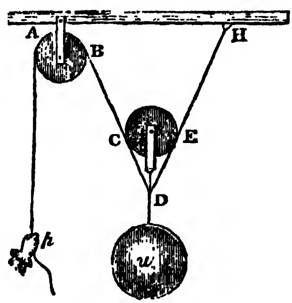
WHEN THE DIRECTION OF THE POWER AND WEIGHT ARE OBLIQUE TO ONE ANOTHER.

41. Problem 7. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of a single fixed and single moveable pulley, when the directions of the power and the weight are not parallel, but oblique to one another.*

Let pBEH be a running cord, attached to the power at p, passing over the fixed pulley B, under the moveable pulley E, and finally fastened to the hook at H.

Produce the cords pB and HE, to meet in the point D; join DC, and draw CE at right angles to DC, the point C being the centre of the moveable pulley E; then, because the cord is equally strained throughout its length, and the magnitude of the strain is equal to the power p, it follows, that the weight w will always assume that position, in which the parts BC and HE of the sustaining cord, will be equally inclined to the vertical line DC; consequently DC bisects the angle BDH.

Fig. 19.



Let ED, the production of the cord HE, represent the magnitude of the force acting in the direction DH, and conceive the force ED to be resolved into the two forces DC and CE perpendicular to one another: then shall DC represent that portion of the power p, which acting in the direction of the cord EH, becomes available in supporting the weight in the direction of the vertical line DC, the other part CE acting at right angles adds nothing to the effect.

Now, because the tension of the cord BC, is equal to the tension of the cord HE, it is manifest, that the whole power expended in supporting the weight w in the direction of gravity, is represented by twice DC, while the power exerted in straining the cord, is represented by twice DE, but DC is obviously the cosine of the angle EDC to the radius DE; consequently, we have by Plane Trigonometry,

$$2\text{DE} : 2\text{DC} :: \text{rad.} : 2\cos.\text{EDC};$$

but 2DE as we have already shown, is equal to the power p, and 2DC equal to the weight w; therefore, it is

$$p : w :: \text{rad.} : 2\cos.a,$$

where the letter a, according to our notation, denotes the angle EDC; that is, half the angle BDH made by the directions of the cords which sustain the weight.

Hence by taking the radius equal to unity, and equating the products of the extreme and mean terms, we get

$$w = 2p \cos. a. \tag{11}$$

This equation implies, that,

If a power sustain a weight in equilibrio, by means of a cord acting on a single fixed and a single moveable pulley, where the parts of the suspending cord are oblique to each other, the weight is equal to twice the power, diminished in the ratio of the cosine of the angle of direction.

42. Hence it appears, that the greater the angle of inclination between the direction of the weight and the power, the less is the effect produced; this is manifest, for if the angle of inclination increases to a quadrant or 90 degrees; then $\cos. a = 0$, and equation (11), becomes

$$w = 0;$$

that is, whatever may be the magnitude of the power applied, it has no tendency to move the weight in the direction of gravity, its whole energy being expended directly on the fixed point or hook at H.

If the angle of direction continues to increase, its consine becomes negative, in which case no equilibrium can obtain, for the force or power has then a tendency to urge the weight w in its own direction, instead of causing it to ascend in an opposite direction.

If the angle of direction increases to a semicircle or 180 degrees; then cos. a is equal to the radius considered negatively, in which case equation (11), becomes

$$w = -2p;$$

that is, the weight and the power act both in the same direction, for the weight instead of ascending in opposition to the power, descends in the same direction with a double intensity.

If the angle of direction vanishes, then the parts of the suspending cord become parallel to one another, in which case, cos. a is equal to the radius taken affirmatively; consequently, equation (11), becomes

$$w = 2p,$$

agreeing with equation (2), as it obviously ought to do.

Example 1. What weight will be held in equilibrio, by a power of 28lbs., attached to a cord passing under a moveable pulley and over a fixed one, supposing the portions of the sustaining cord, to make with each other an angle of 86 degrees?

In this example we have given $2a=86°$, or $a=43°$, and $p=28$lbs.; let these numbers be substituted for a and p in equation (11), and it becomes

$$w=2\times 28\times \cos. 43°,$$

but the natural cosine of 43° is 0·73135; consequently, by substitution, we get

$$w=56\times 0\cdot 73135=41 \text{ lbs., very nearly.}$$

If the cords sustaining the weight had been parallel to one another, as indicated by the foregoing inference, the value of w would have come out 56lbs. instead of 41, making a difference of 15lbs. in the result, produced by reason of the obliquity of the parts of the sustaining cords.

Example 2. A weight of 112lbs. is kept in equilibrio, by a power as in the foregoing example; what is the magnitude of the power, supposing the parts of the sustaining cord to be inclined to each other in an angle of 40 degrees?

Here we have given, $2a=40°$, or $a=20°$, and $w=112$lbs.; let these numbers be substituted for a and w in equation (11), and it becomes

$$2p\cos. 20°=112,$$

or by halving the equation, we get

$$p\cos. 20°=56,$$

but the natural cosine of 20° is, 0.93969; consequently, by substitution and division, we get

$$p=\frac{56}{0\cdot 93969}=59\tfrac{1}{2}\text{ lbs.}$$

If the portions of the sustaining cord were parallel to one another, a power of $59\frac{1}{2}$ pounds would balance a weight of 119lbs. being 7lbs. more than the balanced weight; the difference in this case being small, on account of the acuteness of the angle of direction, which in the present instance is only 20 degrees.

Example 3. A power of 256lbs. is found to balance a load of 486lbs. in a system arranged as in fig. 19; what is the magnitude of the angle, formed by the parts of the suspending cord?

In this example we have given, $p=256$, and $w=486$lbs.; let these numbers be substituted for p and w in equation (11), and it becomes

$$2\times 256\times \cos. a=486,$$

let both sides of the equation be divided by 2×256, and we get

$$\cos. a=\frac{486}{512}=0.94922=\text{nat. cos. } 18° 20' 15'';$$

consequently, the angle made by the portions of the sustaining cord at the point of convergence below the moveable pulley, is

$$18° 20' 15''\times 2=36° 40' 30''.$$

Of the stress on the supporting beam and also upon the several portions of the system.

43. With regard to the stress or pull on the supporting beam at the points A and H, it is manifestly equivalent to the sum of the weight and power, considered conjointly; that is,

$$P = w + p,$$

but the expression for the value of w, as found from equation (11), is

$$w = 2p \cos. a;$$

hence, by substitution, we get

$$P = (2 \cos. a + 1)p. \qquad (n)$$

Now, in the second example preceding, we have $w = 112$lbs.; $p = 59\frac{1}{2}$lbs., and $a = 20$ degrees; consequently, by equation (a), we have

$$P = 112 + 59\tfrac{1}{2} = 171\tfrac{1}{2}\text{lbs.},$$

or, by equation (n), we obtain

$$P = (2 \times 0.93969 + 1) \times 59\tfrac{1}{2} = 171\tfrac{1}{2}, \text{ the same as above.}$$

But the respective portions of the strain or pressure, communicated at the points A and H, are as below, viz.

Pressure or strain on the point, is $P = p + \frac{1}{2}w = (\cos. 20° + 1) \times 59\frac{1}{2} = 115\frac{1}{2}$,

—— on the point H, - $P = \frac{1}{2}w = \cos. 20° \times 59\frac{1}{2} = 56$,

consequently, the total pressure on the beam $= 171\frac{1}{2}$lbs. the same as above.

Section Fifth.

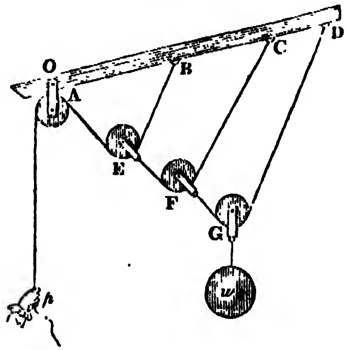
OF A FIXED PULLEY AND SEVERAL MOVEABLE PULLEYS ACTING OBLIQUELY TO ONE ANOTHER.

44. Problem 8. *To determine the conditions of equilibrium, or the relation that subsists between the power and the weight, in the case of one fixed and several moveable pulleys, each moveable pulley having its own cord passing under it and fastened to a separate hook; the portions of the sustaining cords being supposed to act obliquely to each other.*

The solution of this problem manifestly depends upon the principles unfolded in the solution of problems 5 and 7; for with the exception of the oblique cords, it is akin to the one, and by reason of that obliquity it becomes assimilated to the other; it is therefore evident, that its solution must combine the principles peculiar to them both.

Let A, E, F and G, fig. 20, represent the system of pulleys described in the problem, of which A is fixed, and E, F and G moveable, ascending and descending as determined by the weight and power; the moveable pulleys having separate cords passing under them, and finally attached to the hooks B, C and D; w is the weight or resistance connected to the lowest pulley G, and p is the power applied at the extremity of the cord passing over the pulley A.

Fig. 20.



Let $2a$, $2b$ and $2c$ denote the angles made by the cords suspending the moveable pulleys, viz. the angles AEB, EFC and FGD respectively; then, according to our notation, a, b and c represent the halves of those angles; and if w, w' and w'' denote the parts of the weight supported at the pulleys G, F and E; then, by combining the principles of problems 5 and 7, we obtain the following class of analogies, viz.

$$p \quad w'' :: \text{rad.} : 2\cos. a,$$
$$w'' \quad w' :: \text{rad.} : 2\cos. b,$$
$$w' \quad w :: \text{rad.} : 2\cos. c;$$

combining the analogies, we get

$$pw''w' : w''w'w :: \text{rad}^3 : 2^3\cos. a\cos. b\cos. c,$$

which, by casting out the common factor $w''w'$, gives

$$p : w :: \text{rad}^3 : 2^3\cos. a\cos. b\cos. c;$$

consequently, by taking radius equal to unity, and $n=3$, the number of separate cords in the system, we have by substitution

$$p : w :: 1 : 2^n\cos. a\cos. b\cos. c.$$

This analogy corresponds to the system exhibited in the diagram, viz. that in which there are but three separate cords; but by our notation, n represents generally the number of separate cords in any system, and since there must be as many angles as there are separate cords in the arrangement, the general analogy will become

$$p : w :: 1 : 2^n(\cos. a\cos. b\cos. c \ldots \cos. n);$$

which, by equating the products of the extreme and mean terms, gives

$$w = 2^n p(\cos. a\cos. b\cos. c \ldots \cos. n). \qquad (12)$$

This equation implies, that

If a power sustain a weight in equilibrio, by means of a system of pulleys where each moveable pulley has its own cord attached to a separate hook; the power is to the weight, as radius to the continued product of the cosines of half

the angles made by the portions of the sustaining cords, into that power of 2, whose exponent is denoted by the number of moveable pulleys, or by the number of separate cords in the system.

Hence we infer, that whatever may be the intensity of the power at any term of the series, that intensity will be doubled at the next succeeding term, except in so much as it is reduced, by drawing it into the cosine of half the angle, made by the portions of the sustaining cord at that term.

EXAMPLE 1. In a system of pulleys arranged as in fig. 20; what weight will be sustained in equilibrio by a power of 84 pounds, supposing that there are 5 separate cords in the system, and that the angles formed by the portions of each, beginning with that on which the power acts, are respectively 88°, 82°, 76°, 70° and 64°?

In this example we have given, $n=5$; $p=84$; $a=44°$; $b=41°$; $c=38°$; $d=35°$ and $e=32°$; let these numbers be respectively substituted for their representatives in equation (12), and we get

$$w=2^5\times 84\times(\cos.\,44°\ \cos.\,41°\ \cos.\,38°\ \cos.\,35°\ \cos.\,32°);$$

here, however, the operation will be more easily performed by logarithms, on account of the number of factors to be multiplied, and the number of decimal fractions that unavoidably enter the process.

2^5	log.	1·505150
84	log.	1·924279
44°	log. cos.	9·856934
41°	log. cos.	9·877780
38°	log. cos.	9·896532
35°	log. cos.	9·913365
32°	log. cos.	9·928420
natural number = 798·84	log.	2·902460

Hence it appears, that in a system of pulleys, such as the example describes, a force of 84 lbs. will balance a weight or resistance of 798·84 lbs. which is at the rate of $9\frac{1}{2}$ lbs. to 1.

EXAMPLE 2. In a system of pulleys arranged as in fig. 20; what power will be sufficient to balance a weight of 784 lbs. suspended from the centre of the lowest pulley, the number of moveable pulleys being 4, and the angles formed by the parts of their suspending cords being respectively 135, 120, 90 and 60 degrees?

Here we have given, $n=4$; $w=784$; $a=67°\,30'$; $b=60°$; $c=45°$, and $d=30°$; let these numbers be substituted for their representatives in equation (12), and it becomes

$$2^4 p(\cos.\,67°\,30'\ \cos.\,60°\ \cos.\,45°\ \cos.\,30°)=784,$$

or, dividing both sides of the equation by

$$2^4(\cos.\,67°\,30'\ \cos.\,60°\ \cos.\,45°\ \cos.\,30°),$$

we shall obtain

$$p=784\times 0·0625(\sec.\,67°\,30'\ \sec.\,60°\ \sec.\,45°\ \sec.\,30°);$$

here again, and for the same reason as stated in the preceding example, the operation will be more easily performed by logarithms, in the following manner.

0·0625	log.	8·795880
784	log.	2·894316
67° 30′	log. sec.	0·417160
60°	log. sec.	0·301030
45°	log. sec.	0·150515
30°	log. sec.	0·062469
natural number=418·2	log.	2·621370

Whence it appears, that in a system of pulleys like that described in the example, a force of 418·2lbs. will be required to sustain in equilibrio a weight of 784lbs., being something less than 2lbs. of resistance to one pound of power applied.

45. If all the angles of inclination should vanish, the parts of the cords become parallel, in which case equation (12), reduces to

$$w=2^n p,$$

the same as equation (6),

and if all the angles are equal to one another, equation (12), reduces to the following form, viz.

$$w=2^n p \cos^n a. \tag{13}$$

This equation requires to be illustrated by the resolution of two or three numerical examples, as follows.

EXAMPLE 1. In a system of pulleys arranged as in fig. 20, where each pulley hangs by a separate cord, the weight is suspended from the centre of the lowest pulley, and the power applied to a cord passing over the fixed pulley A; now, supposing the number of moveable pulleys to be 4, and the angle formed by the portions of the cords at each equal to 90°; what weight will be sustained in equilibrio by a force equivalent to 32 lbs.?

In this example there are given $a=45°$; $n=4$, and $p=32$lbs.: let these numbers be substituted for a, n and p in equation (13), and it becomes

$$w=2^4\times 32 \cos^4.45°,$$

the operation performed logarithmically, is as follows

2	log.	0.301030	
45°	log. cos.	9.849485	add
	sum=	0.150515	
		4	
$2^4\cos^4.45°$	log.	0.602060	
32	log.	1.505150	
natural number=128	log.	2.107210	

From which it appears, that in a system of pulleys, such as is described in the question, a force equivalent to 32lbs. is sufficient

to counterbalance a weight of 128lbs. being in the proportion of 4 to 1.

Since the natural cosine of 45° is equal to $\frac{1}{2}\sqrt{2}$, this example may be very readily performed without logarithms, thus,

$$w = 2^4 \times 32 \times (\tfrac{1}{2}\sqrt{2})^4,$$

but the fourth power of 2 is 16, and the fourth power of $\frac{1}{2}\sqrt{2}$, is $\frac{1}{4}$; consequently, we have

$$w = 16 \times \tfrac{1}{4} \times 32 = 128, \text{ the same as before.}$$

EXAMPLE 2. Suppose the weight to be 128lbs., and the number of moveable pulleys with the inclinations of their sustaining cords, the same as in the last example; what power will be required to sustain the system at rest?

Here also, we have $a = 45°$; $n = 4$, and $w = 128$; let these numbers be substituted for a, n and w in equation (13), and it becomes

$$2^4 \cos^4 45° p = 128,$$

by division we obtain

$$p = \frac{128}{2^4 \cos^4 45°},$$

but we have shown above that $2^4 = 16$, and $\cos.^4 45° = \frac{1}{4}$; hence we get

$$p = \frac{128}{4} = 32\,\text{lbs.}$$

EXAMPLE 3. The power, weight and inclination of the sustaining cords, being as above; what is the number of separate cords?

By substitution, equation (13) becomes

$$2^n \cos^n 45° = 4,$$

but by the nature of logarithms, we have

$$n \cdot (\log. 2 + \log. \cos. 45°) = \log. 4,$$

which by division gives

$$n = \frac{\log. 4}{\log. 2 + \log. \cos. 45°} = 4.$$

Of the aggregate strain on the several points of the supporting beam, together with the pressure on the several parts of the system.

46. With respect to the aggregate strain, or pull on the several points of the supporting beam, it is obviously as in the other cases, equivalent to the weight and the power taken conjointly; but in the present arrangement, by reason of the obliquity of the cords, the result must be reduced in the ratio of the continued product of the cosines of half the angles, made by the portions of the sustaining cords at each of the moveable pulleys; that is,

$$\text{P} = (2^n + 1)\,(\cos. a \cos. b \cos. c. \cos. n)p. \qquad (o)$$

Now, in the second example to equation (12), the weight is 784lbs. the power 418·2lbs. and $n = 4$, the angles of inclination

being respectively as there specified; consequently, we have by equation (o), the aggregate pressure on the beam as below, viz.

$$P = 418 \cdot 2 \times 17 \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = 833 \text{lbs.}$$

And the respective pressure on each point of support, is as follows, viz.

The pressure on the

$$\begin{array}{llr}
\text{point E is, } P = 2^{n-1}p \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = & 392 & \text{lbs.,} \\
\text{—— D} - P = 2^{n-2}p \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = & 196 & \text{—} \\
\text{—— C} - P = 2^{n-3}p \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = & 98 & \text{—} \\
\text{—— B} - P = 2^{n-}p \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = & 49 & \text{—} \\
\text{—— A} - P = \quad 2p \cos. 67^\circ 30' \cos. 60^\circ \cos. 45^\circ \cos. 30^\circ = & 98 & \text{—}
\end{array}$$

Then, by taking the sum of the several portions, we have P = 833 lbs. for the aggregate, the same as before.

In the second example of equation (13), the weight is 128 lbs. and the reduced power is 8 lbs.; hence the total strain on the oblique beam, is

$$P = 128 + 8 = 136 \text{lbs.}$$

but the general expression for the total pressure or strain on the beam in this case, is

$$P = (2^n + 1) p \cos.^n a; \qquad (p)$$

consequently, by substituting the numerical values of n, p and a, we shall get

$$P = (2^4 + 1) \times 32 \times \tfrac{1}{4} = 136 \text{lbs. the same as before.}$$

And moreover, for the respective portion of the strain transmitted to each point of suspension, we have as follows, viz.

The pressure on the

$$\begin{array}{llr}
\text{point E is, } P = 2^{n-1}p \cos^n 45^\circ = 8 \times 32 \times \tfrac{1}{4} = & 64 & \text{pounds,} \\
\text{—— D} - P = 2^{n-2}p \cos^n 45^\circ = 4 \times 32 \times \tfrac{1}{4} = & 32 & \text{——} \\
\text{—— C} - P = 2^{n-3}p \cos^n 45^\circ = 2 \times 32 \times \tfrac{1}{4} = & 16 & \text{——} \\
\text{—— B} - P = 2^{n-4}p \cos^n 45^\circ = 1 \times 32 \times \tfrac{1}{4} = & 8 & \text{——} \\
\text{—— A} - P = \quad 2p \cos^n 45^\circ = 2 \times 32 \times \tfrac{1}{4} = & 16 & \text{——}
\end{array}$$

Then, by taking the sum of these several portions, we have P . . . . } = 136 pounds for the total pressure, the same as above.

Conclusion.

Such then is the theory of the pulley, and the several systems which we have investigated involve the principles that apply to any other combinations whatever, and the general equations in their present form, extend without exception or modification to every system in common use. Other arrangements and combinations of pulleys might easily have been proposed, but every particular respecting them, could be determined from the theory already exemplified, and consequently, since no new principle would be introduced by an extension of our enquiries, we have thus thought proper to bring them to a close.

4. OF THE INCLINED PLANE.

INTRODUCTION.

There are, properly speaking, three cases of the inclined plane: one in which the incumbent load is supported by a power acting parallel to the plane; another in which the load is supported by the power acting parallel to the base of the plane; and a third in which the angle of traction as well as the angle of the plane's inclination is less than ninety degrees. Each of these cases we shall expound in the sequel. A body to be sustained in equilibrium upon the inclined plane must be acted on by three forces. Having defined these forces, and noticed two conditions which must be fulfilled in order that an equilibrium may obtain between these said forces, we proceed to establish the general principle introduced in the first case, that

> *When a weight is sustained in equilibrium on an inclined plane by a power acting in a direction parallel to the sloping face, the magnitude of the power is to the weight of the body, as the height of the plane is to its length.*

The several cases of analysis implied in this theorem lead to the resolution of four problems, which we have expressed by rules, and illustrated by examples worked at length.

But since it is necessary to find at times the angle of declivity as well as the length or height of the plane, we proceed to show how this may be accomplished; and in the first example of the fourth problem we have given a practical rule for determining the declivity, when the sloping length and height of the plane are given. We have concluded this case by a geometrical construction which illustrates the relation that obtains between the power, the weight, and the reaction of the plane, when the direction in which the power acts is parallel to the face of the plane. This construction, which determines the magnitude of the force of pressure or the resistance of the plane, we have illustrated by examples solved logarithmically.

The next case unfolds the doctrine of equilibrium, when the power acts in a direction parallel to the base of the plane, in which as before,

> *The magnitude of the sustaining power is to the weight of the incumbent load, as the height of the plane is to its base.*

And here also, as in the preceding case, if any three of these terms be given, the fourth is determined by one or other of the problems derived from the several cases of analysis implied in the equation of equilibrium for this case, in which we have investigated the theorem which determines the inclination of the plane from having the magnitude of the sustaining power and the weight of the incumbent body given. Considering that a graphical construction would assist in the illustration of this case, we have introduced the sixth figure and its accompanying details, from which we have drawn another rule for determining the magnitude of the force of pressure, or the resistance of the plane when the magnitude of the sustaining power and the weight of the incumbent body are given. This rule is likewise illustrated by examples, which for their solution require simply a knowledge of the square root.

We come now to the consideration of the general case, in which the angle of traction as well as the inclination of the plane may be any magnitude not exceeding ninety degrees. The demonstration of this case involves several elegant problems, which we have deprived of their analytical dress, and expressed in a practical rule by words at length, accompanied by appropriate examples.

We complete our investigation by a general example, showing how the value of each quantity may be found in terms of the rest, (as many of them as may be found necessary,) directly from the particular equation in which it occurs; or how, by a simple substitution, the value of any term may be obtained from one or another of the equations in which it does not originally appear. The solution of this curious question divides itself into six cases, under the third of which we have verified our results by a geometrical construction exhibited in the seventh figure. In the sixth case it becomes necessary for its resolution to seek the aid of other principles than any of the preceding cases furnish; but these principles are supplied from properties which the figure itself unfolds; and they lead to the same results to which our previous enquiries conducted us.

Having thus established the theory of the inclined plane, considered under all the circumstances of action in which the sustaining power can be applied, we have completed our undertaking by the addition of some select problems, the solution of which we have drawn out at full length for the purpose of enabling the reader to apply their appropriate formulæ to the solution of any particular question that may occur in the course of his enquiries.

These problems are resolved graphically, analytically, and numerically, by a process similar to that which we have pursued in all our previous enquiries upon this branch of mechanical science. We shall therefore content ourselves with believing that these miscellaneous problems give variety enough to the inclined plane to entitle our discussion of it to some credit for originality. We

may, however, notice particularly, the method we have taken in the geometrical construction of verifying our numerical calculations, and also the investigation of the problems assigned to a heavy body supported by two planes, when it is required to determine the relation between the weight of the body and the pressure on each plane, which unfolds some curious and interesting particulars analagous to properties that fell under our consideration in the Composition and Resolution of Forces.

All stairs are inclined planes, so are ladders, by which we ascend to the top of a house or a tree; but both these are very steep planes, the ascent on which is by means of steps or rounds. In a stair the steps are cut into horizontal and perpendicular surfaces, to afford a firm footing; in a ladder, the steps are round pieces of tough wood, as ash or oak, to enable us by their cylindrical figure, to grasp them more firmly with our hands, and disengage our feet the more readily from the step as we climb or come down. The shrouds of a ship are inclined planes, resembling ladders, and the ratlines are the steps by which the seamen mount aloft to their perilous duties in the yard-arm. In short, the spaces between edge and edge of the steps in a stair, between round and round in a ladder, and between ratline and ratline in a ship's shrouds, are so many inclined planes, which we can step over, and place our bodies as the incumbent load upon another plane which is horizontal.

Railroads are generally level, but sometimes inclined, as when heavy loads are passing only in one direction; and in some canals where locks could not be formed we meet with inclined planes, constructed for the purpose of facilitating the passage of boats, moved on these planes by machinery, in which the capstan, the wheel and axle, or pulleys, are used as mechanical agents.

There is an elegance about the inclined plane, as about all the mechanical powers that fascinates by its utility. Witness the brewer's-men in London, discharging the contents of their large drays upon inclined planes, and then lowering the heavy butts into the cellars of the publicans. And see upon an inclined plane how a couple of men will remove from a cart a hogshead of sugar, or other merchandise, which a dozen of labourers could not place perpendicularly upon the ground. Indeed, these are the precise situations in which no other mechanical power could so readily be applied. The beds of rivers present fine specimens of inclined planes; and if any portion of these is subject to the action of the tides, we can fancy the two bodies of water to resemble a couple of wedges laid over each other, so as to produce a dead level at the flood. Roads over hills present by their acclivities inclined planes, which are easy of ascent, if they wind in a zig-zag form, but difficult if the line of direction goes over the summit as the crow flies. Coachmen who have to descend steep roads, take the hill in time, and check the speed of their horses till they have

gone over two thirds of the descent, when, with perfect safety, they increase the velocity to meet the opposing ground at the bottom, and keep the team to their collars till they have gained upon the next ascent. Galileo could do no more.

The observation we made respecting the lever, holds good in the inclined plane, for though it may not be mathematically correct to say, that "the advantage gained by it, is as great as its length exceeds its perpendicular height;" it is, nevertheless, scientifically true that when a plane is four times as long as it is high, a fourth part of the power necessary to lift a load perpendicularly to the summit of the plane, will roll it up the ascent; but then, the one-fourth power in rolling it, is four times as long in performing the work as the quadruple power would be in drawing it up the side of a wall, equal in height to the top of the plane.

Hence the nearer an inclined plane slopes to the horizon, the less will be the strength necessary to roll a cask up its ascent, and the steeper the plane, the quicker will any body descend to its base. Majestic ships launched upon inclined planes, demonstrate the great utility of these planes. Here the plane is slightly inclined, and the power which would sustain the vessel in equilibrio in the plane, (allowing the effects of friction,) is equal to the force with which she descends, which, allowing for the small elevation, is comparatively trifling. We are not at liberty, however, to confound the principles of statics with dynamics, and unite the doctrine of equilibrium with that of motion.

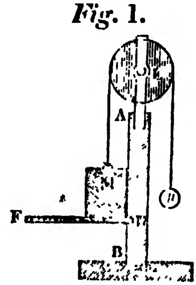
All those devices known as slips by which vessels are brought out of the water for repairs, in place of being floated into a dock that at low tide becomes dry, may be considered as inclined planes—on these slips a vast carriage, constructed upon truck wheels, and moveable as on a railroad, receives the vessel while afloat, and by means of a strong mechanical purchase, the carriage is drawn up, carrying upon its bosom the ponderous load. The chain of the mechanical power is attached solely to the carriage upon which the vessel rests, so that her hull is not exposed to any strain. It is even possible in one and the same tide to haul up, inspect, and lower a large vessel of five hundred tons burden. The cradle, or carriage, may be worked at the rate of two and a half feet per minute, by means of six men upon the purchase to every hundred tons.

OF THE INCLINED PLANE.

Definitions and conditions of equilibrium.

1. All planes are either *perpendicular*, *horizontal*, or *oblique*. The upright walls of a building present us with the idea of planes that are said to be perpendicular to the horizon, and on all such planes, bodies will descend in the direction of gravity, with a force equivalent to their whole weight, unless they are prevented from descending by an opposite force of equal intensity; thus, for instance, the body M fig. 1, which is supposed to be placed in contact with the upright plane AB, will descend in the direction of gravity, with a force or intensity equal to its own weight, if it be not prevented by the power *p* of equal intensity, acting over the pulley E, or otherwise sustaining it in the direction FD.

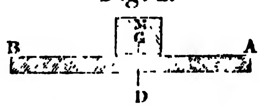
Fig. 1.



In this case, the direction of gravity coincides with the direction of force.

The floor of a room, the surface of a billiard table, a bowling green, or a frozen lake, conveys to us the idea of a horizontal plane; which, if we indicate by the straight line AB fig. 2, we may place upon it the body M without any fear of its further descent.

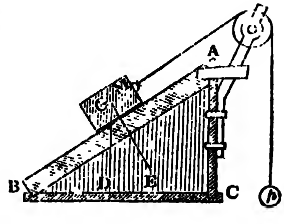
Fig. 2.



Here it is manifest, that the mass or body M presses on the plane AB with a force equivalent to its own weight, and in the direction of gravity GD; consequently, the direction of gravity and the direction of pressure coincide, as in the case of the perpendicular plane, with regard to the direction of force; but in the present instance, the body is prevented from descending by the reaction or resistance of the plane itself, without the intervention of any other force, and is therefore naturally in a state of rest.

The roof of a building, the side of a hill, or a sloping bank, may give us the idea of an oblique plane; or if we elevate one end of the plane AB fig. 3, by putting a prop AC under it, while the other end B is suffered to rest on the horizontal level BC; then, the plane in this position being oblique to the horizon, is called *the Inclined Plane.*

Fig. 3.



2. *The Inclined Plane*, as a mechanical power, performs its office by diminishing the weight of the incumbent load,

which is moved with greater ease in consequence of the inclination. In this plane, the obliquity is measured by the angle ABC, formed by the horizontal or base line BC, and the slope or length AB. Here, the centre of gravity of the body rises with the plane, and in consequence, its direction will become inclined to the direction of pressure in an angle DGE, equal to the plane's elevation.

In this case however, it is manifest, that the plane AB does not support the whole weight of the body, part of it being transferred into the line GD in the direction of gravity perpendicular to the horizon, and part of it into the line GE in the direction of pressure, perpendicular to AB the plane on which the body rests; in which case, the body has a tendency to slide or roll down the plane, if not sustained by another force acting at its centre of gravity, with an intensity equal to the difference between the weight of the body, and the reaction of the plane, or the force soliciting the body in the direction of pressure.

From this it appears, that in order to have a body sustained in equilibrio on an inclined plane, it is necessary to consider it, as being acted on by the three following forces, viz.

1. *The power p, which, acting at its centre of gravity, resists its tendency to slide or roll down the plane.*
2. *The force of gravity, which urges the mass or body* M *in the direction* GD, *perpendicular to the horizon.*
3. *The reaction, or the resistance of the plane in the direction* GE, *opposite to the force of pressure, or perpendicular to the plane on which the body rests.*

3. Before we proceed to develope the principles of the inclined plane, it becomes necessary to premise the following conditions, which must be fulfilled, in order that an equilibrium may obtain between the forces by which the body is solicited.

The first condition is, that

The power p, which sustains the body in equilibrio, or that which prevents it from sliding or rolling down the plane, must have its direction in a vertical plane passing through the centre of gravity of the body.

The second condition is, that

The reaction, or the resistance of the plane on which the body rests, must be wholly destroyed by the resultant of the other two forces.

These conditions being admitted, the mechanical principles of the inclined plane, may be deduced in the following manner.

SECTION FIRST.

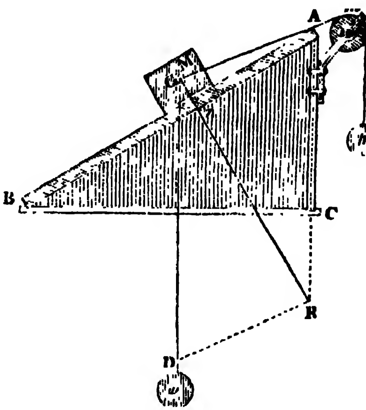
WHEN THE DIRECTION OF THE POWER IS PARALLEL TO THE PLANE.

4. Let ABC fig. 4, represent a vertical section of the inclined plane, passing through G, the centre of gravity of the mass or

body M, which is supposed to be sustained in equilibrio, by means of a power *p* acting at the extremity of the cord *p*AG, passing over the fixed pulley *m*, touching the summit of the plane at A, and finally attached to G, the centre of gravity of the incumbent mass or body M.

From G, the centre of gravity of the body, let fall the perpendicular G*n*, cutting AB the face of the plane in the point *n*; produce G*n* to meet the prolongation of the height AC in the point R; through R draw RD parallel to GA, and through G, the centre of gravity of the body M, draw GD parallel to AR and meeting RD in the point D; then is the figure AGDR a parallelogram, of which GR is the diagonal.

Fig. 4.



Now, if the diagonal RG in that direction, represent the reaction or resistance of the plane AB on the incumbent mass or body M, it is manifest, that the two other forces by which the body is sustained in equilibrio, will be respectively represented in magnitude and direction, by GA and GD, the sides of the parallelogram, of which forces, the diagonal GR is the resultant.

But it is a well known fact in the doctrine of mechanics, that

When one body acts upon another body by any kind of force whatever, that force is exerted in a direction perpendicular to the surface on which it acts.

Consequently, the force of pressure of the mass or body M on the plane AB, is exerted in the direction of the diagonal GR, which by construction is perpendicular to the face of the plane; but the force of pressure and the reaction of the plane are, in the case of an equilibrium, equal to one another; therefore, since the reaction of the plane, is represented in magnitude and direction by the diagonal RG, it follows, that the force of pressure is represented in magnitude and direction by GR, the diagonal of the parallelogram constructed on the lines GA and GD, which represent the magnitude and direction of the forces *p* and *w*, which with the reaction of the plane sustain the body at rest.

Let w = the weight of the body, or its tendency to descend in the direction of gravity GD,

p = the magnitude of the power tending to sustain the body at rest upon the plane, or to urge it in the direction BA,

r = the force of pressure on the plane in the point n, or the reaction of the plane in the direction RG.

Then the three forces p, w and r, are respectively proportional to the three lines GA, GD and GR, by which they are represented; that is, proportional to the two contiguous sides GA, GD and the diagonal GR of the parallelogram AGDR; thus,

$$p : w :: \text{GA} : \text{GD},$$
$$p : r :: \text{GA} : \text{GR},$$
$$w : r :: \text{GD} : \text{GR}.$$

Put a = the angle ABC or DGR, the inclination of the plane to the horizon, or the inclination of the direction of gravity GD, to the direction of pressure GR,

b = the angle AGR, the inclination of the direction of pressure GR, to the direction of the power GA,

c = the angle AGD, the inclination of the direction of the power GA to the direction of gravity GD.

Now, because the figure AGDR is a parallelogram, the angle GRD is equal to the angle AGR, and the angles GDR and DGA are supplements to each other; therefore, since an angle and its supplement have the same sine, and because the side DR is equal to AG, it is manifest, that the calculation may be wholly deduced from the triangle GDR, by considering that

The sides of any plane triangle are to one another as the sines of their opposite angles.

Consequently, if instead of the lines GA, GD and GR, we substitute sin. a, sin. b and sin. c, the above analogies will become

$$p : w :: \sin. a : \sin. b,$$
$$p : r :: \sin. a : \sin. c,$$
$$w : r :: \sin. b : \sin. c;$$

or by making the products of the mean terms equal to the products of the extremes, we shall obtain the following class of analogies, viz.

$$w \sin. a = p \sin. b, \qquad (a)$$
$$r \sin. a = p \sin. c, \qquad (b)$$
$$r \sin. b = w \sin. c, \qquad (c)$$

These three equations involve the principles of equilibrium for the inclined plane, and the developement of those principles, under various conditions, will become manifest from what follows.

5. If both sides of the equation marked (a) be divided by sin. b, we shall obtain

$$p = w \sin. a \operatorname{cosec.} b. \qquad (d)$$

From which we infer, that if the weight w and the inclination of the plane are given, the power p varies as cosec. b, and must therefore be the least when cosec. b is the least; that is, when the angle b is a right angle or 90 degrees, for it is well known from the principles of Trigonometry, that the least value of the cosecant of an arc obtains, when the arc itself is equal to a quadrant.

Now, when the direction of the power, is inclined to the direction of pressure in an angle of 90 degrees, it is obviously parallel to the plane on which the body rests, in which case sin. $b=$ rad. or unity; consequently, by substitution, equation (a), becomes

$$p=w \sin. a. \qquad (e)$$

But by Plane Trigonometry, we have

$$\text{AB} : \text{AC} :: \text{rad.} : \sin. a,$$

which, by equating the products of the extreme and mean terms becomes

$$\text{AB} \sin. a = \text{AC rad.},$$

and this, by putting radius equal to unity, gives

$$\text{AB} \sin. a = \text{AC}; \qquad (f)$$

consequently, if both sides of this equation be divided by AB, we shall obtain

$$\sin. a = \frac{\text{AC}}{\text{AB}}. \qquad (g)$$

Let this value of sin. a be substituted for it in equation (e), and it becomes

$$p=\frac{w \cdot \text{AC}}{\text{AB}},$$

or multiplying both sides by AB, it is

$$p \cdot \text{AB} = w \cdot \text{AC}. \qquad (h)$$

Put $l=$ AB, the length of the declivity, or sloping face of the plane,
$h=$ AC, the rise or perpendicular height,
$\beta=$ BC, the horizontal distance or base.

Then, if l and h be substituted for AB and AC in the equation marked (h), we shall obtain

$$lp=hw. \qquad (i)$$

From this equation we infer, that when a weight is sustained in equilibrio on an inclined plane, by a power acting in a direction parallel to its sloping face

The magnitude of the power, is to the weight of the body, as the height of the plane is to its length.

Hence it is manifest, that of the length of the plane, the height of the plane, the power and the weight, any three being given, the fourth can easily be determined; and the several cases of analysis implied in equation (i), are as below, viz.

1. Given h, l and p, to find w,
2. Given h, l and w, to find p,
3. Given h, p and w, to find l,
4. Given l, p and w, to find h.

And the mode of effecting these analysis, will become manifest from the resolution of the following problems.

6. Problem 1. *In the equation $lp=hw$, there are given, h, l and p; to find the value of w.*

Since the unknown or required quantity w, is combined with the height of the plane by the process of multiplication; it is evident, that in order to expound its value, we must divide both sides of the equation by its coefficient; therefore, by division, we get

$$w=\frac{lp}{h}.$$

And the practical rule afforded by this equation, is expressed in words at length as follows.

Rule. *Multiply the magnitude of the sustaining power, by the length, or sloping face of the plane; then, divide the product by the perpendicular height, for the magnitude or value of the weight required.*

Example 1. If a man can suspend a weight of 150lbs. acting in the direction of gravity; what weight will he be able to sustain on a smooth plank 30 feet long, laid aslope from the top of a perpendicular wall 20 feet in height, the line of traction, or the direction in which the man exerts his strength, being parallel to the face of the plank?

Here by the rule, we have

$$w=\frac{150\times 30}{20}=225\text{lbs.}$$

Example 2. A power equivalent to 738lbs. acting in a direction parallel to the face of a smooth inflexible inclined plane, is found sufficient to prevent the descent of a block of marble which is supported on the plane; what is the weight of the block, supposing its perpendicular height to be 16 feet, and the length of its sloping face 48 feet?

Here by the rule, we have

$$w=\frac{738\times 48}{16}=2214\text{lbs.}$$

Now, it has been found by experiment, that on an average, a cubic foot of sound white marble weighs 169lbs.; therefore, if s represent the solidity of the block in cubic feet; then we have

$$s=\frac{2214}{169}=13\tfrac{17}{169}\text{ cubic feet,}$$

which is equivalent to a cubical block, of something more than 2 feet $4\frac{1}{4}$ inches in the side.

7. Problem 2. *In the equation $lp=hw$, there are given, h, l and w; to find the value of p.*

Here, the required quantity is combined by multiplication, with the length or sloping face of the plane; therefore, if both sides of

the equation be divided by the coefficient of the unknown term, we shall have

$$p = \frac{hw}{l}.$$

And the practical rule which this equation affords, is expressed in words at length, as follows.

RULE. *Multiply the weight of the incumbent body by the perpendicular height of the plane, and divide the product by the length of its sloping face, for the magnitude of the power sought.*

EXAMPLE 1. Suppose the whole weight of a ship on the stocks is estimated at 400 tons; what power will be required to prevent her from gliding off the launch, when its head is elevated 3 feet, its whole length being 105 feet, and the direction of the power parallel to its plane?

Here by the rule, we have

$$p = \frac{400 \times 3}{105} = 11\tfrac{3}{7} \text{ tons},$$

being equivalent to the effective power of about 170 men, exerting themselves at a dead pull.

EXAMPLE 2. A weight equivalent to 20000lbs. is sustained in equilibrio by a certain power, acting in a direction parallel to the sloping face, or length of a smooth inflexible inclined plane; what is the magnitude or intensity of the power, the height of the plane being 8 feet, and its length or sloping face 128 feet?

Here by the rule, we have

$$p = \frac{20000 \times 8}{128} = 1250 \text{lbs.},$$

or the sustaining power, is to the weight sustained in the proportion of 1 to 16.

8. PROBLEM 3. *In the equation $lp = hw$, there are given h, p and w; to find the value of l.*

The unknown or required quantity is here combined by multiplication with the sustaining power; consequently, by division, we have

$$l = \frac{hw}{p}.$$

And the practical rule derived from this equation, may be expressed in words at length as follows.

RULE. *Multiply the weight of the incumbent body by the perpendicular height of the plane, and divide the product by the magnitude of the sustaining power for the length of the sloping face of the plane required.*

Example 1. If a power of 300 lbs. be found to sustain a weight of 9600 lbs. on the surface of a smooth inflexible inclined plane, whose perpendicular height is 12 feet; what is the sloping length of the plane, the line of traction, or the direction in which the power acts being parallel thereto?

Here by the rule, we have

$$l=\frac{9600\times 12}{300}=384 \text{ feet},$$

which gives one foot of perpendicular elevatio for 32 feet of sloping face.

Example 2. On the surface of a smooth inflexible inclined plane, whose perpendicular height is 18 feet, a power equivalent to 960 lbs. is found to support a body whose weight is 15300 lbs.; what is the length or sloping height of the plane?

Here by the rule, we have

$$l=\frac{15300\times 18}{960}=143\tfrac{7}{16} \text{ feet},$$

or very nearly $143\frac{1}{2}$ feet, which is equivalent to 7·97 feet of declivity to one foot of perpendicular height.

9. Problem 4. *In the equation $lp=hw$, there are given, l, p and w; to find the value of h.*

The unknown or required term, is here connected by multiplication with the weight of the incumbent body; therefore, by division, we have

$$h=\frac{lp}{w}.$$

And the practical rule which this equation affords, may be expressed in words at length as follows.

Rule. *Multiply the length, or sloping face of the plane by the magnitude of the sustaining power, and divide the product by the weight of the incumbent body, for the perpendicular height of the plane required.*

Example 1. A steam engine of 3 horses' power, or 675 lbs.,* is found, when acting at full pressure, just sufficient to move a train of 16 waggons on the surface of a smooth inflexible inclined

* In the question, the engine is stated to be equivalent to the power of three horses; but the measure of a horse's power in its general acceptation being merely an arbitrary and conventional standard, derived from the assumption of a certain weight being elevated to a given height in a given time, it cannot be admitted in our present enquiry, for this reason, that no motion or velocity is implied, but simply the commencement of motion. And we here conceive the effort of the engine to produce motion in the train, as being equivalent to the direct energy of three strong horses, when exerted horizontally at a dead pull.

Now, it has been ascertained by experiment, that the tractile power of a

plane of 600 feet in length; what is the perpendicular height of the plane, supposing each waggon with its load to weigh 2½ tons, or 89600 lbs.?

Proceeding as the rule directs, we get

$$h=\frac{675\times 600}{89600}=4\cdot 52 \text{ feet},$$

being at the rate of one foot rise for 132·75 feet of declivity, a deviation from the true level, of very little importance in a distance of 600 feet, giving an inclination of something less than 0° 26′.

To find the angle of declivity.

The angle of declivity is found by dividing the height of the plane by the length of its sloping face, for we have shewn in the equation marked (g), that

$$\sin. a=\frac{\text{AC}}{\text{AB}},$$

which equation for the present example, gives

$$\sin. a=\frac{4\cdot 52}{600}=\cdot 00753=\text{nat. sin. } 0^\circ\ 26' \text{ very nearly.}$$

By the inference to equation (i), we have shown, that when the direction of the power is parallel to the face of the plane,

$$p : w :: h : l,$$

or by converting this analogy into an expression of ratio, we shall obtain

$$\frac{p}{w}=\frac{h}{l}. \qquad (k)$$

Hence it is manifest, that the angle of declivity can also be determined from the power and the weight alone, without having regard to the length and the perpendicular height of the plane; for by equation (g), it is

$$\sin. a=\frac{h}{l}, \qquad (l)$$

and consequently, by equation (k),

$$\sin. a=\frac{p}{w},$$

and this, by adopting the numbers of the question, gives

$$\sin. a=\frac{675}{89600}=\cdot 00753,$$

good horse, when moving at any velocity, is very accurately represented by the equation

$$\phi=(w-v)^2,$$

where w denotes the maximum velocity unloaded, v any other velocity, and ϕ the load at the velocity v; consequently, by taking $w=15$, and $v=o$, we obtain

$$p=3\phi=3(15-0)^2=675 \text{ lbs.}$$

which corresponds to the natural sine of 26 minutes, the same as was obtained from the length and perpendicular height of the plane.

EXAMPLE 2. A power equivalent to a load of 224lbs., acting in a direction parallel to the sloping face of a smooth inflexible inclined plane, is found sufficient to sustain a load of 120 tons or 268800lbs.; what is the perpendicular height of the plane, its sloping length being 1320 feet, what is the proportional elevation, and what is the angle of declivity?

By the rule, the height of the plane is

$$h = \frac{13.20 \times 224}{268800} = 1.1 \text{ feet},$$

consequently, the proportional rise, is

$$\frac{1320}{1.1} = 1200 \text{ feet},$$

that is, an elevation of one foot in 1200 feet, and by equation (g), we have

$$\sin. a = \frac{1.1}{1320} = \cdot 00083,$$

which corresponds to the natural sine of 2′ 53″, or, if we divide the power by the weight, it is

$$\sin. a = \frac{224}{268800} = \cdot 00083.$$

10. This latter method of assigning the angle of declivity, may often be usefully applied in the construction of inclined planes, when neither the length nor the height is given; thus for example.

EXAMPLE 3. A power of 12lbs. only, can be applied for the purpose of sustaining a load equivalent to 108lbs. on the sloping surface of a smooth inflexible inclined plane; what must be the elevation of the plane, that the power may just be sufficient to sustain the load?

It has been shewn in the equation marked (l), that the sine of the angle of declivity, is equivalent to the quotient that arises when the sustaining power, is divided by the weight of the incumbent load; consequently, by division, we have

$$\sin. a = \frac{12}{108} = .11111,$$

which corresponds to the natural sine of 6° 23′.

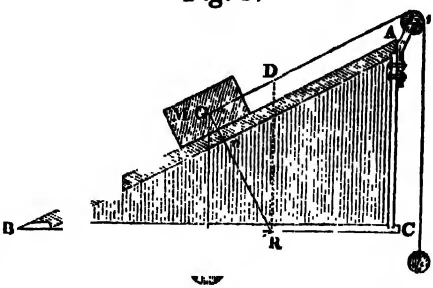
Therefore, if an inclined plane be constructed in such a manner, that its length is to its perpendicular height, as unity to the number .11111; then a power of 12lbs. will be sufficient to sustain a load of 108lbs.

11. *Geometrical and analytical demonstration of the relation subsisting between the power, the weight, and the reaction of the plane, when the power acts parallel to its face.*

The subjoined diagram, will tend to illustrate the relation

that obtains between the power, the weight, and the reaction of the plane, for the case which we have just considered, viz. when the direction in which the power acts is parallel to the face of the plane. Through G, the centre of gravity* of the mass or body M, draw G*n* perpendicular to AB the face of the plane, which produce to meet the base BC in the point R.

Fig. 5.



Then, let GR represent the magnitude of pressure in the direction GR, or the reaction of the plane in the direction RG, and decompose the force GR, into its equivalent component forces GD and G*w*, in the directions of the power and of gravity; that is, in directions respectively parallel to the plane and perpendicular to the horizon; then shall GD represent the magnitude of the power p, and G*w*, that of the mass M or weight w, acting in the direction of gravity.

Now, because GD, the direction of the power, is perpendicular to GR, the direction of pressure, and R*w* parallel to DG; it follows, that *w*GR, the triangle of equilibrium, is similar to the triangle ABC, the vertical section of the plane; therefore, we have

$$w\text{R} : \text{GR} :: \text{AC} : \text{BC},$$
$$w\text{G} : \text{GR} :: \text{AB} : \text{BC},$$

or, by substituting the literal representatives of the respective terms, these analogies become

$$p : r :: h : \beta,$$
$$w : r :: l : \beta,$$

from which, by equating the products of the extreme and mean terms, we obtain

$$\beta p = hr \qquad (m)$$
$$\beta w = lr. \qquad (n)$$

but by Plane Trigonometry, we have

$$\sin. a : h :: \cos. a : \beta,$$
$$\text{rad.} : l :: \cos. a : \beta;$$

from the first of these analogies, we get

$$\beta = h \cot. a,$$

and from the second, it is

$$\beta = l \cos. a.$$

Let these values of β be respectively substituted for it in the equations (m) and (n), and they become

$$r = p \cot. a. \qquad (o)$$
$$r = w \cos. a. \qquad (p)$$

The expressions which we have thus obtained for the value of the pressure, or the reaction of the plane, are also assignable from the general equations marked (b and c), by putting sin. b equal to unity, and substituting cos. a for sin. c, to which in this case it is manifestly equivalent. But another, and an independent expression for the value of r, can easily be found in terms of the power and the weight alone, without having regard to the dimensions and inclination of the plane; for since wGR, the triangle of equilibrium, or its equal DRG is right angled, we have

$$r=\sqrt{w^2-p^2}, \qquad (q)$$

which expression, when modified for logarithmic computation, becomes

$$r=\sqrt{(w+p)(w-p)}.$$

And the practical rule derived from this expression may be enunciated in the following manner.

RULE. *Multiply the sum of the weight and power by their difference, and extract the square root of the product, for the magnitude of the force of pressure or the resistance of the plane.*

EXAMPLE 1. A power equivalent to 28lbs. acting in a direction parallel to the sloping face of an inclined plane, is found to balance or maintain at rest a load of 2240lbs.; with what force does the load press upon the plane?

By the rule we have

$$r=\sqrt{(2240+28)(2240-28)}=2239.82\text{lbs.}$$

In examples of this kind, however, especially when the numbers are large, the operation is somewhat more easily performed by logarithms, in the following manner.

$$w+p=2240+28=2268 \ldots\ldots \text{log. } 3.355643$$
$$w-p=2240-28=2212 \ldots\ldots \text{log. } 3.344785$$

sum of the logs. 6.700428

Natural number or pressure on the plane $=2239\cdot82$ } half sum of the logs. 3.350214.

EXAMPLE 2. A load of 224 tons is sustained at rest upon the surface of a smooth inflexible inclined plane by a power equivalent to 2 tons acting parallel to the sloping face; what is the magnitude of the force counteracted by the plane?

Here, by operating according to the rule, we have

$$(224+2)\times(224-2)=50172;$$

consequently, by extracting the square root, we get

$$r=\sqrt{50172}=223.99 \text{ tons.}$$

The logarithmic performance is as follows.

$w+p=224+2=226$	. . . . . . log.	2·354108
$w-p=224-2=222$	. . . . . . log.	2·346353
	sum of the logs.	4·700461
therefore, the required force $=223.99$ tons	half sum of the logs.	2·350230

SECTION SECOND.

WHEN THE DIRECTION OF THE POWER IS PARALLEL TO THE BASE OF THE PLANE.

12. Returning to the general expressions for the conditions of equilibrium, and dividing both sides of equation (b) by sin. a, we get

$$r=p \text{ cosec. } a \text{ sin. } c. \qquad (r)$$

Therefore, if the magnitude of the power p, and a the inclination of the plane are given, the pressure on the plane varies as sin. c, and must therefore be a maximum when sin. c is a maximum; that is, when c is a right angle or 90 degrees, for then sin. c is equal to the radius, which is manifestly the greatest sine that can be drawn in the circle, as is well known from the principles of Trigonometry.

Now, when the angle which the direction of the power makes with the direction of gravity is a right angle, the direction of the power is parallel to the base of the plane; in which case, the angle made by the direction of the power with the direction of pressure is equal to the complement of the plane's inclination; that is,

$$\text{sin. } b=\text{cos. } a;$$

consequently, by substitution, equation (a) becomes

$$p=w \text{ tan. } a; \qquad (s).$$

but by Plane Trigonometry, we have

$$\beta : h :: \text{rad.} : \text{tan. } a,$$

or, by equating the products of the extreme and mean terms, we shall have

$$\beta \text{ tan. } a=h; \qquad (t)$$

let both sides of this equation be divided by β, and we shall obtain

$$\text{tan. } a=\frac{h}{\beta}, \qquad (u)$$

and if this value of tan. a be substituted instead of it in equation (s), we shall have

$$p=\frac{hw}{\beta}$$

or multiplying both sides by β, the denominator of the fraction, we obtain

$$\beta p=hw. \qquad (v)$$

From this equation we infer, that when a weight is sustained in equilibrio on an inclined plane, by a power acting in a direction parallel to its base,

The magnitude of the sustaining power, is to the weight of the incumbent load, as the height of the plane is to its base.

Hence, it is manifest, that of the height of the plane, the base of the plane, the sustaining power and the incumbent load; any three being given, the fourth can easily be found; and the several cases of analysis implied in equation (v), are as follows.

1. Given β, h and p, to find w,
2. Given β, h and w, to find p,
3. Given β, p and w, to find h,
4. Given h, p and w, to find β.

And the mode of effecting these analyses, will become manifest from the resolution of the following problems.

13. PROBLEM 1. *In the equation $\beta p = hw$, there are given, β, h and p, to find the value of w.*

Here, the unknown or required quantity, is combined with the perpendicular height of the plane by the process of multiplication; consequently, by division, we have

$$w = \frac{\beta p}{h}.$$

And the practical rule which this equation affords, is expressed in words at length in the following manner.

RULE. *Multiply the base of the plane by the magnitude of the sustaining power, and divide the product by the perpendicular height of the plane, for the magnitude of the incumbent load.*

EXAMPLE 1. If a man can suspend 200lbs. by the side of a perpendicular wall 200 feet high; what weight will he be able to sustain on a smooth inflexible plank, laid aslope from the summit of the wall, to a point 45 feet distant from its bottom, supposing the direction in which he exerts his power to be parallel to the ground.

Here by the rule we have

$$w = \frac{45 \times 200}{20} = 450\,\text{lbs.}$$

If 450lbs., the magnitude of the incumbent load, be divided by 200lbs. the magnitude of the sustaining power, we shall have $2\frac{1}{4}$lbs. for the quotient; which implies, that the power and the load in this case, are to one another as the numbers 1 and $2\frac{1}{4}$; and moreover, the number $2\frac{1}{4}$ also indicates the cotangent of the plane's elevation.

EXAMPLE 2. A power equivalent to 112lbs. acting in a direction parallel to the base of a smooth inflexible inclined plane, is found sufficient to sustain a body in a state of rest; what is the weight of the body, the perpendicular height of the plane being 3 feet, and its base being 174 feet?

Here, by operating according to the rule, we have

$$174 \times 112 = 19488,$$

and by division, we obtain

$$w = \frac{19488}{3} = 6496 \text{lbs.}$$

which gives the ratio of the power to the weight as 1 to 58.

14. PROBLEM 2. *In the equation $\beta p = hw$, there are given, β, h and w, to find the value of p.*

The unknown term in this case, being combined by multiplication with the base of the plane, if we divide both sides of the equation by the coefficient of the required power, we shall have

$$p = \frac{hw}{\beta}.$$

And the practical rule derived from this equation, may be expressed in words as follows.

RULE. *Multiply the perpendicular height of the plane by the weight of the incumbent body, and divide the product by the base of the plane for the magnitude or intensity of the sustaining power.*

EXAMPLE 1. What power being attached to the extremity of a rope, passing over a fixed pulley in the king post of a roof, will be able to sustain in equilibrio, a load of 768lbs. upon the rafter, the span of the roof being 48 feet, its rise 14 feet, and that part of the rope between the pulley and the load parallel to the joint?

Here, by operating according to the rule, we have

$$14 \times 768 = 10752,$$

and dividing by 24 feet, half the span, we get

$$p = \frac{10752}{24} = 448 \text{ pounds.}$$

EXAMPLE 2. Suppose the perpendicular height of a smooth inclined plane to be 10 feet, and its base 110 feet; what power will be sufficient to sustain a load of 50000lbs., the direction of the power being parallel to the base of the plane?

Here by the rule we have

$$p = \frac{10 \times 50000}{110} = 4545\tfrac{5}{11} \text{lbs.}$$

15. PROBLEM 3. *In the equation $\beta p = hw$, there are given, β, p and w, to find the value of h.*

Let both sides of the equation be divided by w, the coefficient of the required term, and we shall have

$$h = \frac{\beta p}{w}.$$

And the practical rule afforded by this equation, may be expressed in words at length, as follows.

RULE. *Multiply the magnitude of the sustaining power, by the base of the plane, and divide the product by the weight of the incumbent body, for the perpendicular height of the plane required.*

EXAMPLE 1. A solid block of marble, weighing $3\frac{1}{2}$ tons or 7840lbs. is supported on a slide or inclined plane, whose lower extremity rests upon the ground, at the distance of 36 feet from bottom of an upright wall, on the top of which, the upper extremity of the slide rests. Now, supposing the block to be sustained in equilibrio by means of a power equivalent to 450lbs. acting in a direction parallel to the ground; what is the perpendicular height of the wall on which the plane rests?

Operating according to the rule, we get

$$450 \times 36 = 16200,$$

and dividing by 7840, the weight of the body, it is

$$h = \frac{16200}{7840} = 2.066 \text{ feet}.$$

EXAMPLE 2. A power of 150lbs. is employed to elevate a cask of gun-powder to the top of a fort, from the opposite side of a ditch 42 feet wide; what is the height of the fort, supposing the cask to weigh 270lbs. the power acting in a direction parallel to the horizon?

Here, by proceeding according to the rule, we get

$$42 \times 150 = 6300,$$

and dividing by the weight of the cask, it gives

$$h = \frac{6300}{270} = 23\tfrac{1}{3} \text{ feet}.$$

16. PROBLEM 4. *In the equation $\beta p = hw$, there are given, h, p and w, to find the value of β.*

The unknown or required term in this case, is found connected by multiplication with the magnitude of the sustaining power; therefore, if both sides of the equation be divided by that coefficient, we shall have

$$\beta = \frac{hw}{p}.$$

And the practical rule afforded by this equation, may be expressed in words at length in the following manner.

RULE. *Multiply the magnitude of the incumbent body by the perpendicular height of the plane, and divide the product by the magnitude of the sustaining power, for the base of the plane required.*

EXAMPLE 1. At what distance from the bottom of an upright wall 12 feet in height, must the lower end of an inclined plane or slide be placed, in order that a power equivalent to 368lbs. acting

in a direction parallel to the horizon, may be able to sustain a load of 5 tons, or 11200 lbs, resting on the slide?

Here, by the rule, we have

$$12 \times 11200 = 134400,$$

consequently, dividing by the sustaining power, we get

$$\beta = \frac{134400}{368} = 364 \cdot 78 \text{ feet.}$$

EXAMPLE 2. Suppose that a power, equivalent to 225 lbs. acting in a direction parallel to the horizon, is found to sustain a load of 90000 lbs. on an inclined plane, whose perpendicular height is 125 feet; what is the length of the horizontal level, or base of the plane?

Operating according to the rule, we have

$$125 \times 90000 = 11250000;$$

therefore, dividing by the sustaining power, it is

$$\beta = \frac{11250000}{225} = 5000 \text{ feet, or } \cdot 947 \text{ of a mile.}$$

PROBLEM 5. *To find the angle of declivity of the plane.*

17. The angle of declivity of the plane, is found from the division of the perpendicular height by the length of the horizontal base; for we have shown in equation (u), that

$$\tan. \alpha = \frac{h}{\beta},$$

but by the inference to equation (v), it appears, that when the power acts in a direction parallel to the base of the plane,

$$p : w :: h : \beta;$$

therefore, by converting this analogy into an equation of ratio, we get

$$\frac{p}{w} = \frac{h}{\beta}. \qquad (w)$$

Hence, it is manifest, that the angle of declivity of the plane, can be determined from the magnitude of the sustaining power, and the weight of the incumbent body, without having regard to the dimensions of the plane; for by the equality of ratios, we have from equations (u and w),

$$\tan. \alpha = \frac{p}{w}. \qquad (x)$$

And the practical rule afforded by this equation, may be expressed in words, as follows.

RULE. *Divide the magnitude of the sustaining power, by the weight of the incumbent body, and the quotient will give the tangent of the plane's inclination.*

EXAMPLE 1. It is required to construct an inclined plane such, that a power, whose intensity is equivalent to a pressure of 84 lbs. acting in a direction parallel to the base, shall be sufficient to sustain a load of 798 lbs. upon the sloping surface?

Here, by the rule, we have

$$\tan. a = \frac{84}{798} = 0.10526,$$

which corresponds to the natural tangent of 6° 0′ 32″.

Therefore, if the base and the perpendicular height of the plane be made to one another in the ratio of 1 to 0.10526, the conditions of the question will be found to succeed.

EXAMPLE 2. A load of 78 tons is required to be held in equilibrio on the surface of a smooth inclined plane, by means of a power of 39 tons, acting parallel to its base; what is the inclination of the plane?

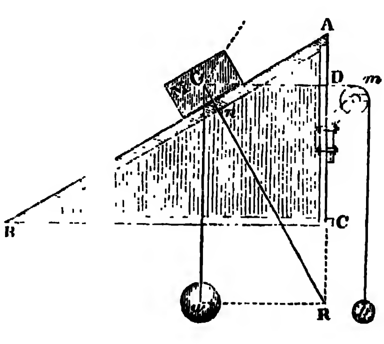
Proceeding according to the rule, we have

$$\tan. a = \frac{39}{78} = 0.5,$$

which corresponds to the natural tangent of 26° 33′ 54″.

18. PROBLEM 6. *To show the relation that obtains between the power, the weight and the pressure, or the reaction of the plane, in that case, where the direction of the power is parallel to the base, or when the base and perpendicular height of the plane are respectively parallel to the base and perpendicular of the triangle of equilibrium.*

Fig. 6.



Through G, the centre of gravity of the mass or body M, draw Gn perpendicular to AB the face of the plane; produce Gn to meet the prolongation of AC, the perpendicular height of the plane in the point R.

Again, through G draw GD parallel to BC, and Gw parallel to AC the height of the plane or perpendicular to BC its base, and through R draw Rw parallel to DG or BC; then is GR the resultant of the two forces p and w, which are respectively represented in magnitude and direction by the straight lines GD and Gw.

Consequently, the diagonal GR represents the force of pressure in magnitude and direction, and RG is the reaction or resistance of the plane; hence it is manifest, that GwR, the triangle of equi-

librium, is similar to ACB, the upright section of the plane; therefore, we have

$$wR : GR :: AC : AB,$$
$$wG : GR :: BC : AB;$$

or by substituting the literal representatives of the respective terms, these analogies become,

$$p : r :: h : l,$$
$$w : r :: \beta : l;$$

which, by equating the products of the extreme and mean terms, become

$$lp = hr, \qquad (y)$$
$$lw = \beta r; \qquad (z)$$

but by Plane Trigonometry, we have

$$\sin. a : h :: \text{rad.} : l,$$
$$\cos. a : \beta :: \text{rad.} : l.$$

Therefore, by putting radius equal to unity, and equating the products of the extreme and mean terms,

$$l \sin. a = h,$$
$$l \cos. a = \beta;$$

from the first of these equations, by reciprocating sin. a, we shall obtain,

$$l = h \operatorname{cosec}. a,$$

and from the second, by reciprocating cos. a, it is

$$l = \beta \sec. a.$$

Let these two values of l, be respectively substituted for it, in the equations (y and z), and we shall obtain

$$r = p \cos. a, \qquad (A)$$
$$r = w \operatorname{cosec}. a. \qquad (B)$$

These expressions for the value of r, are derived at once from the equations (b and c), and those equations flow directly from GwR, the triangle of equilibrium; where c, the angle contained between the direction of the power, and the direction of gravity is a right angle; and b, the angle contained between the direction of the power and the direction of pressure, is equal to the complement of the plane's elevation.

But an independent expression for the value of r, can easily be obtained in terms of the sustaining power and the weight of the incumbent body; for by the property of the right angled triangle, we have

$$GR^2 = Gw^2 + Rw^2;$$

therefore, by substituting the literal values and extracting the square root, we obtain

$$r = \sqrt{w^2 + p^2}. \qquad (C)$$

And the practical rule derived from this equation, may be expressed in words at length, as follows.

RULE. *To the square of the magnitude of the sustaining power, add the square of the weight of the incumbent body, and the square root of the sum will give the magnitude of the force of pressure, or the resistance of the plane.*

EXAMPLE 1. A power equivalent to 128lbs. acting in a direction parallel to the base of an inclined plane, is found to balance a load of 9646lbs. upon its sloping surface; what resistance does the plane oppose to the load thus sustained, or in other words, what pressure does the load exert upon the plane?

Operating as the rule directs, we have

$$p^2 = 128 \times 128 = 16384$$
$$w^2 = 9646 \times 9646 = 93045316$$
$$w^2 + p^2 = 93061700$$

which, by extracting the square root, gives

$$r = \sqrt{93061700} = 9646 \cdot 85 \text{lbs.}$$

being $0 \cdot 85$lbs. more than the actual weight of the body.

This increase in the magnitude of the force of pressure, above the weight of the incumbent body, arises from the tendency of the sustaining power to urge the body down on the plane, in consequence of its oblique direction; but if the direction of the power had been elevated above the plane, as denoted by the dotted line GE, the force of pressure, instead of being increased by the action of the power, would have been diminished; for in that case, the tendency of the power, is to lift the body from the plane, and consequently, to abstract from the pressure, or from the magnitude of the force which urges the body in the direction of the diagonal GR;

Hence it follows, that whether the line of direction of the sustaining power, be elevated above, or depressed below the surface of the plane; if the angles of traction be the same, an equal power will balance an equal weight, but the force of pressure on the plane, is less in the case of an elevated direction, than in the case of a depressed one.*

EXAMPLE 2. A power equivalent to a pressure of 3 tons, acting in a direction parallel to the base of an inclined plane, is capable of sustaining a load of 4 tons on its sloping surface; what is the force of pressure on the plane, or what resistance does it oppose to the incumbent body?

* *The angle of traction*, is the angle which the direction of the power makes with the sloping surface of the pane.

Here, by the rule, we have

$$w^2=4\times4=16$$
$$p^2=3\times3=\underline{\ 9}$$
$$w^2+p^2=25$$

which, by extracting the square root, gives

$$r=\sqrt{25}=5 \text{ tons},$$

exceeding by a whole ton, the weight of the incumbent body.

SECTION THIRD.

WHEN THE DIRECTION OF THE POWER MAKES ANY ANGLE WITH THE FACE OF THE PLANE.

20. We now come to the consideration of the general case, where the angle of traction, as well as the inclination of the plane may be of any magnitude, not exceeding 90°.

The relation that subsists generally, between the magnitude of the sustaining force, the weight of the incumbent body, and the reaction of the plane, or the force of pressure, has already been announced in the equations marked (a, b and c); but to accommodate those expressions to the dimensions of the plane, we must substitute for sin. a, its value as exhibited in equation (g), viz. the quantity $\frac{AC}{AB}$, and for cos. a, the quantity $\frac{BC}{AB}$, in which case, the equations above alluded to become

$$\left.\begin{array}{l} w\cdot AC=AB\cdot p\sin. b,\\ r\cdot AC=p\{AC\cos. b+BC\sin. b\},\\ r\cdot AB=w\{AC\cot. b+BC\};\end{array}\right\}*$$

or, by substituting the literal values of AB, AC and BC, we get

$$hw=lp\sin. b, \qquad (D)$$
$$hr=p\{h\cos, b+\beta\sin. b\}, \qquad (E)$$
$$lr=w\{h\cot. b+\beta\}. \qquad (F)$$

These three equations involve the several particulars respecting the state of equilibrium on the inclined plane, and the method of applying them, will become manifest from the resolution of the following problems.

21. PROBLEM 1. *In the equation $hw=lp$ sin. b, there are given, h, l, p and b, to find the value of w, or the weight of the incumbent body.*

Here, we have the unknown or required term, connected by the process of multiplication with the perpendicular height of the plane; therefore, if both sides of the equation be divided by the perpendicular height, we shall obtain

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* In the triangle of equilibrium GwR, we have the angle RwG$=180°-($RGw$+w$RG$)$; that is, $c=180°-(a+b)$; consequently, sin $c=$sin. $(a+b)=$sin. a cos. $b+$cos. a sin. b, therefore, by substituting this value of sin. c in equations (b and c), we obtain

$$r.AC=p\{AC\cos. b+BC\sin. b\},$$
$$r.AB=w\{AC\cot. b+BC\}.$$

$$w=\frac{lp \sin. b}{h}.$$

And the practical rule which this equation affords, may be expressed in words at length, in the following manner.

RULE. *Multiply together, the magnitude of the suspending power, the sloping length of the plane, and the sine of the angle, which the direction of the power makes with the direction of pressure; then, divide the product by the perpendicular height of the plane, and the quotient will give the weight of the incumbent body.*

EXAMPLE 1. The perpendicular height of an inclined plane is 18 feet, and its length, or sloping height, 32 feet; what weight will be sustained in equilibrio by a power equivalent to 224 lbs., acting in a direction inclined to that of pressure, in an angle of 76° 45′?

Here then, the natural sine of 76° 45′ is ·97338; therefore by proceeding according to the rule, we have

$$224 \times 32 \times .97338 = 6977.18784,$$

which being divided by 18, the perpendicular height of the plane, gives

$$w=\frac{6977.18784}{18}=387.62 \text{ lbs.}$$

EXAMPLE 2. The perpendicular height of an inclined plane is 120 feet, and its sloping length 240 feet; what weight will be balanced on its surface, by a power of 328 lbs. acting in a direction inclined to the direction of pressure, in an angle of 112° 56′?

Here the natural sine of 112° 56′ is ·92096; therefore, by the rule, we have

$$328 \times 240 \times .92096 = 72497.9712;$$

consequently, dividing by the perpendicular height of the plane, we have

$$w=\frac{72497\cdot9712}{120}=604\cdot15 \text{ lbs.}$$

22. PROBLEM 2. *In the equation $hw=lp \sin. b$, there are given h, l, w and b, to find the value of p, or the magnitude of the sustaining power.*

In this case, the unknown or required term, is combined by multiplication, with the product of the length of the plane, into the sine of the angle which the direction of the power makes with the direction of pressure; consequently, if both sides of the equation be divided by that product, we shall have

$$p=\frac{hw}{l \sin. b}.$$

From which expression, the following practical rule is easily derived.

RULE. *Multiply the perpendicular height of the plane, by the weight of the incumbent body; then, divide the product, by the sloping length of the plane, drawn into the sine of the angle which the direction of the power makes with the direction of pressure, and the quotient will give the magnitude of the sustaining power.*

EXAMPLE 1. The perpendicular height of an inclined plane is 28 feet, and its sloping length 168 feet; what power will be required to balance a weight of 6888lbs., supposing the direction of the power, to make with the direction of pressure, an angle of $63^\circ\ 50'$; that is, when the direction of the power is depressed below the plane, in an angle of $26^\circ\ 10'$? Here the natural sine of $63^\circ\ 50'$ is .89752;

therefore, by proceeding according to the rule, we have

$$28\times 6888 \quad = 192864$$
$$168\times \cdot 89752 = 150\cdot 78336,$$

then, dividing the first of these products by the second, we get

$$p=\frac{192864}{150\cdot 78336}=1279\text{ lbs.}$$

EXAMPLE 2. The perpendicular height of an inclined plane is 100 yards, and its sloping length 790 yards; what power will be required to sustain a weight of 42 tons in equilibrio, the direction of the power making with the direction of pressure, an angle of $125^\circ\ 30'$; that is, when the direction of the power is elevated above the plane in an angle of $35^\circ\ 30'$?

Here then, the natural sine of $125^\circ\ 30'$, is .81412; therefore, according to the rule, we have

$$100\times 42=4200,$$
$$790\times .81412=643.1548,$$

then, dividing the first of these products by the second, we get

$$p=\frac{4200}{643\cdot 1548}=6\cdot 53\text{ tons.}$$

In like manner may the other quantities, which are concerned in the composition of the equation marked (*dd*) be determined: but since their determination is of very little practical utility, at any rate, of very rare occurrence, we prefer leaving it for exercise to the reader, and pass on to the resolution of the next problem.

23. PROBLEM 3. *In the equation* $hr = p\{h\cos. b+\beta\sin. b\}$, *there are given,* β, h, p *and* b, *to find the value of* r, *the magnitude of the force of pressure, or the reaction of the plane.*

The unknown term in this instance, is combined with the perpendicular height of the plane by the process of multiplication; consequently, if both sides of the equation be divided by the height of the plane, we shall obtain,

$$r = \frac{p}{h}\{h \cos. b + \beta \sin. b\}$$

And the the practical rule which this equation affords, may be expressed in words at length, as follows.

RULE. *Multiply the perpendicular height and base of the plane, respectively by the cosine and sine of the angle which the direction of the power makes with the direction of pressure; add the products; then, multiply the sum by the magnitude of the sustaining power, and divide by the perpendicular height of the plane, for the magnitude of the force of pressure sought.*

EXAMPLE 1. The perpendicular height of an inclined plane is 56 feet, its base 69 feet, and the angle contained between the direction of the power, and the direction of pressure is 72° 25′; that is, the direction of the power is depressed below the plane in an angle of 17° 35′; what is the magnitude of the force of pressure, or the reaction of the plane, the magnitude of the sustaining power being equivalent to 84 tons?

Here, the natural sine of 72° 25′, is .95328, and the natural cosine, is .30209; therefore, by proceeding according to the rule, we have

$$56 \times .30209 = 16.91704,$$
$$69 \times .95328 = 65.77632,$$

sum of the products $= 82.69336$; consequently, we get

$$r = \frac{82.69336 \times 84}{56} = 124.04 \text{ tons.}$$

EXAMPLE 2. The base of an inclined plane is 124 feet, its perpendicular height 52 feet, and the angle contained between the direction of the power and the direction of pressure is 109° 20′; that is, the direction of the power is elevated above the plane, in an angle of 19° 20′; what force will counterbalance the reaction of the plane, the magnitude of the sustaining power being equivalent to a pressure of 312lbs.?

Here, the natural sine and cosine of 109° 20′, are respectively .94361 and .33106; therefore, by proceeding as the rule directs, we have

$$52 \times \cdot 94361 = 49 \cdot 06772,$$
$$124 \times \cdot 33106 = 41 \cdot 05144,$$

sum of the products $= 90 \cdot 11916$; consequently, we get

$$r = \frac{90 \cdot 11916 \times 312}{52} = 540 \cdot 715 \text{lbs.}$$

24. PROBLEM 4. *In the equation $lr = w\,(h \cot. b + \beta)$, there are given, β, h, w and b, to find the value of r, the magnitude of the force of pressure, or the reaction of the plane.*

The required term in this case, is combined with l the sloping length of the plane; therefore, if both sides of the equation be divided by that length, we shall have

$$r=\frac{w}{l}(h\cot. b+\beta).$$

And the practical rule which this equation affords, may be expressed in words at length, in the following manner.

RULE. *Multiply the perpendicular height of the plane, by the cotangent of the angle which the direction of the power makes with the direction of pressure; to the product add the horizontal base of the plane; then, multiply the sum by the weight of the incumbent body, and divide by the sloping length of the plane for the magnitude of the force of pressure.*

EXAMPLE 1. The perpendicular height of an inclined plane is 30 feet, its horizontal base is 40 feet, and its sloping length 50 feet; what force will counterbalance the reaction of the plane, supposing the weight of the incumbent body to be 120 tons, and the angle which the direction of the power makes with the direction of pressure 74° 12′; that is, having a depression of 15° 48′?

Here, the natural cotangent of 74° 12′, is ·28297; therefore, by operating according to the rule, we have

$$30\times .28297 = 8.4891,$$
the base of the plane $=40.$
$$\overline{48.4891};$$
consequently, by multiplication and division, we get
$$\frac{48\cdot4891\times120}{50}=116\cdot37384 \text{ tons.}$$

EXAMPLE 2. The weight of the incumbent body, and the dimensions of the plane being the same as in the last example: what force will counterbalance the reaction of the plane, when the direction of the power makes with the direction of the pressure an angle of 105° 48′; that is, having an elevation of 15° 48′?

The natural cotangent of 105° 48′, is $-.28297$; therefore, by operating as the rule directs, we have

$$30\times-\cdot28297=-8\cdot4891,$$
the base of the plane $=$ 40
$$\overline{31\cdot5119};$$
therefore, by multiplication and division, we get
$$\frac{31\cdot5119\times120}{50}=75\cdot62856 \text{ tons.}$$

Comparing the results of these two examples with each other, it appears, that by merely changing the direction of the sustaining power, from an angle of 15° 48′ of depression below the plane, to

an angle of $15^\circ\ 48'$ elevation above it; there arises a difference in the force of pressure, or the reaction of the plane, of

$$116\cdot37384-75\cdot62856=40\cdot74528 \text{ tons},$$

being something more than one third of the original weight of the incumbent body.

25. Problem. *To shew in what manner the value of each quantity may be found in terms of the rest, directly from the particular equation in which it occurs.*

The computation of the various examples, having been performed according to the rule derived from the formula appropriated to the determination of each term; we presume, that it will not be considered superfluous in this place, to show, in what manner the value of each quantity, may be found in terms of the rest, directly from the particular equation in which it occurs; or how, by a simple substitution, the value of any term, may be obtained from one or other of the equations in which it does not originally appear. For which purpose, we shall take the following general

Example. Suppose that the horizontal base of a smooth inclined plane is 16 feet, its perpendicular height 12 feet, and its sloping length 20 feet; and let the magnitude or intensity of the sustaining power be equivalent to 72lbs., the inclination of its direction to the direction of pressure $68^\circ\ 57'\ 36''$, or $111^\circ\ 2'\ 24''$; the weight of the incumbent body being 112lbs., and the pressures on the plane corresponding to the acute and obtuse values of the power's direction, 115·45 and 63·75lbs.; find the value of each quantity in terms of the rest, or in terms of as many of them as may be found necessary.

Here then, the numerical values of the several quantities composing the equations in the case of an equilibrium, are $\beta=16$; $h=12$; $l=20$; $p=12$; $w=112$; $u=115.45$ or 63.75, and $b=68^\circ\ 57'\ 36''$ or $111^\circ\ 2'\ 24''$; therefore, we have

1. *To find the value of w in terms of h, l, p and b.*

Let the numerical values of the several given quantities, be substituted for them respectively in equation (D) and it becomes

$$12w=20\times72\times\sin.\ 68^\circ\ 57'\ 36'',$$

but $\sin.\ 68^\circ\ 57'\ 36''=.93$; consequently, by substitution and division, we get

$$w=\frac{20\times72\times.93}{12}=112\text{lbs.}$$

If the length and base of the plane had been given, instead of the length and perpendicular height, all other things remaining; then, by the property of the right angled triangle, we should have

$$h=\sqrt{l^2-\beta^2},$$

which, by substitution gives,

$$w\sqrt{l^2-\beta^2}=lp\ \sin. b;$$

therefore, by restoring the numerical values, we get

$$12\,w=20\times 72\times \cdot 93,$$

and by division, we have

$$w=\frac{20\times 72\times .93}{12}=112\text{lbs.},$$

agreeing with the result obtained by the former process.

2. *To find the value of r, in terms of h, l, p and b.*

Referring to the three equations marked (D, E and F), it soon appears, that none of them in their present form involves the given and required quantities; but the equation (E) can easily be modified, so as to afford the required result; for by the property of the right angled triangle, we have

$$\beta=\sqrt{l^2-h^2}\ ;$$

consequently, by substitution, we get

$$hr=p\{\sin. b\sqrt{l^2-h^2}\pm h\cos. b\},$$

in which equation, by substituting the numerical values of the several quantities, we shall obtain

$$12r=72\{16\sin. 68^\circ\ 57'\ 36''\pm 12\cos. 68^\circ\ 57'\ 36''\};$$

but $\sin. 68^\circ\ 57'\ 36''=.93$, and $\cos. 68^\circ\ 57'\ 36''=.35902$;

therefore, by substitution and division, we get

$$r=\frac{72\{16\times .93\pm 12\times .35902}{12}=115.45\text{ or }63.75\text{lbs.}$$

If the length and base of the plane had been given, instead of the length and perpendicular height, the value of r would then be found by the equation marked (F); for by the property of the right angled triangle, we have

$$h=\sqrt{l^2-\beta^2}\ ;$$

consequently, by substituting, and restoring the numerical values, we get

$$20r=112\{16\pm 12\times \cot. 68^\circ\ 57'\ 36''\},$$

but $\cot. 68^\circ\ 57'\ 36''+\cdot 38466$; therefore, we have

$$r=\frac{112\{16\pm 12\times \cdot 38466}{20}=115\cdot 45\text{ or }63\cdot 75,$$

corresponding to the result obtained above.

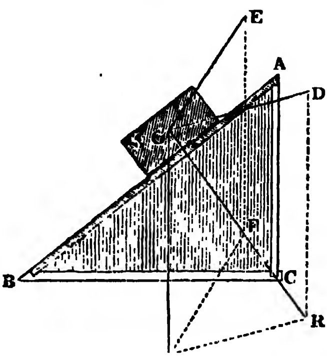
26. The foregoing conclusions verified by a geometrical construction, as follows.

Let ABC, represent a vertical section of the inclined plane, passing through the centre of gravity of the mass or body M, and whose sides AB, AC and BC, are respectively equal to 20, 12 and 16 feet.

Through G, the centre of gravity of the body, draw Gn perpen-

dicular to AB, the sloping face of the plane, and at the point G, make the angle DGn or EGn, equal to the given angle 68° 57′ 36″, or 111° 2′ 24″: from the point G on the line GD or GE, set off GD or GE, equal or proportional to the number 72, the measure of the sustaining force; then, through the points D and E draw DR and EF in the direction of gravity, or parallel to AC, the perpendicular height of the plane, and meeting Gn produced in the points R and F. Again, through G, the centre of gravity of the body M, draw Gw parallel to AC, and through the points R and F, draw Rw, and Fw, respectively parallel to GD and GE, meeting Gw in the same point w; then is Gw the weight of the incumbent body M, and GR and GF the force of pressure, or the reaction of the plane, corresponding to the angles DGR and EGR.

Fig. 7.



Therefore, if Gw, GR and GF be taken in the compasses, and applied to the same scale on which the magnitude of the power GD or GE was measured, they will be found to indicate respectively as follows, viz.

$$Gw = 112,$$
$$GR = 115.45,$$
$$GF = 63.75;$$

corresponding to the numbers previously obtained by the process of calculation.

3. *To find the value of p, in terms of h, l, w and b.*

Let the numerical values of the several given quantities, be substituted instead of them in the equation marked (D), and it becomes

$$20 \times .93 \times p = 12 \times 112;$$

consequently, by division, we obtain

$$p = \frac{1344}{18.6} = 72\text{lbs.}$$

If the length and base of the plane had been given, instead of the length and perpendicular height; then, all other things remaining, we should have

$$20 \times .93 \times p = 112\sqrt{20^2 - 16^2},$$

for by the property of the right angled triangle, it is

$$h = \sqrt{l^2 - \beta^2},$$

therefore, by division, we obtain

$$p = \frac{1344}{18.6} = 72\text{ lbs.}$$

corresponding to the result by the former process.

4. *To find the value of r, in terms of h, l, w and b.*

Referring to the original equations (D, E and F), we find that none of them is composed of the given and required quantities, but at the same time, it is manifest, that the equation (F), can easily be modified so as to express the value of the required term; for by the property of the right angled triangle, we have

$$\beta=\sqrt{l^2-h^2},$$

consequently, by substitution, equation (*ff*), becomes

$$lr=w(\sqrt{l^2-h^2}\pm h\cot. b);$$

wherefore, by restoring the numerical values of the several terms, and calculating both for the acute and obtuse value of the angle b, we obtain

$$20r=112\,(16\pm12\cot.\ 68°\ 57'\ 36''),$$

but cot $68°\ 57'\ 36''=.38466$; therefore, by substitution and division, we get

$$r=\frac{112\,(16\pm12\times.38466}{20}=115.45 \text{ or } 63.75\text{lbs.}$$

5. *To find the value of b in terms of h, w, l and p.*

By substituting the numerical values of the given quantities in equation (*dd*), we get.

$$20\times72\times\sin. b=12\times112;$$

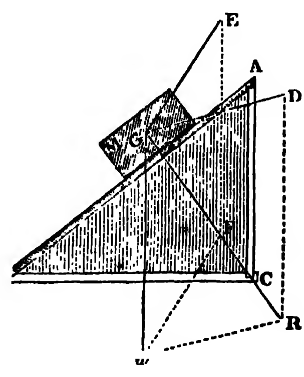
therefore, by division, we obtain

$$\sin. b=\frac{1344}{1440}=.93,$$

which corresponds to the natural sine of $68°\ 57'\ 36''$, or to the natural sine of its supplement $111°\ 2'\ 24''$.

6. *To find the value of r, in terms of p, w and b.*

The resolution of the case now proposed, is obviously more difficult than any of the preceding cases, for none of the original equations involve directly the given and required quantities; it therefore becomes necessary to seek the aid of other principles, and to supply the deficient data from properties which the figure itself unfolds.



Referring therefore, to the preceding diagram it is manifest, that in the triangle of equilibrium G*w*R or G*w*F, there are given the two sides G*w*, *w*R or G*w*, *w*F and the angle GR*w* or GF*w*; therefore, by Plane Trigonometry, we have

$$Gw : \sin. GRw :: Rw : \sin. RGw,$$

or taking the literal representatives, we have

$$w : \sin. b :: p : \sin. a;$$

consequently the angle $GWR = 180° - (a+b)$; hence we have

$$\sin. b : w :: \sin. (a+b) : r,$$

or by equating the products of the extreme and mean terms, we shall obtain

$$r \sin. b = w \sin. (a+b),$$

which, by division becomes

$$r = \frac{w \sin. (a+b)}{\sin. b}.$$

But $\sin. a = \frac{72 \times .93}{112} = .60000$, corresponding to $36° \, 52' \, 12''$;

therefore, by substituting, and restoring the numerical values of b and w, the above expression for the value of r becomes.

$$r = \frac{112 \sin. 105° \, 49' \, 48''}{\sin. 68° \, 57' \, 36''},$$

but $\sin. 105° \, 49' \, 48'' = .96208$, and $\sin. 68° \, 57' \, 36'' = .93$; consequently, we have

$$r = \frac{112 \times \cdot 96208}{\cdot 93} = 115 \cdot 45 \text{ lbs.}$$

The result now determined corresponds to the acute value of the angle b, but for the obtuse value it is

$$r = \frac{112 \times \sin. 147° \, 54' \, 36''}{\sin. 111° \, 2' \, 24''},$$

but $\sin. 147° \, 54' \, 36'' = .53125$, and $\sin. 111° \, 2' \, 24'' = .93$;

consequently, we have

$$r = \frac{112 \times .53125}{.93} = 63.75 \text{ lbs.}$$

27. Such, then, is the theory of the inclined plane, considered under all the circumstances of action in which the sustaining power can be applied; and the several cases, to which our enquiries have been more especially directed, are

1. *When the direction of the power is parallel to the plane.*
2. *When the direction of the power is parallel to the base of the plane.*
3. *When the direction of the power makes any angle with the plane, either ascending or descending.*

And the equations which involve the conditions of equilibrium for each of these cases, are respectively as follows, viz.

1. $hw = lp$,
2. $hw = \beta p$,
3. $hw = lp \sin. b$.

Where it must be kept in mind, that the symbol b in the last equation, universally denotes the angle of inclination between the direction of the sustaining power and the direction of pressure.

SECTION FOURTH.

MISCELLANEOUS PROBLEMS UPON THE INCLINED PLANE, INCLUDING THE CASE OF A HEAVY BODY SUPPORTED BY TWO PLANES.

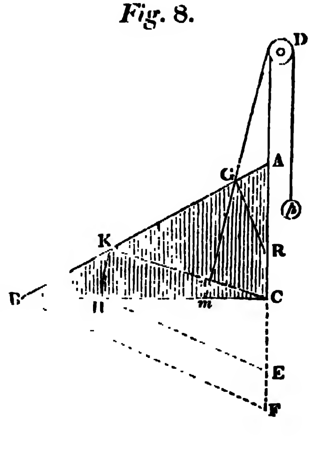
Various important and interesting practical examples may be proposed to illustrate the application of the theory which we have now laid down; but we presume, that from what has already been done, the intelligent and careful reader will find no difficulty in applying the appropriate formula, to the solution of any particular question that may occur to him in the course of his enquiries.

The following select problems with their solutions are all that can be conveniently given in this place, without exceeding the limits which we have assigned to this article; we shall therefore endeavour to sum them up in the shortest and most comprehensive manner possible, as follows.

28. PROBLEM 1. *Upon an inclined plane, whose perpendicular height is h feet, and elevation a degrees, a weight of w lbs. is sustained in equilibrio by a power of p lbs. acting over a fixed pulley, at the distance of d feet from the top of the plane; it is required to find the distance of w from the top of the plane, when the equilibrium takes place?*

When the power acts in a direction parallel to the plane, or parallel to the base of the plane, or in any other direction inclined to the plane, upwards or downwards in a constant angle; it is manifest, that if an equilibrium between the weight and the sustaining power obtain at any one point of the plane, it will also obtain at any other point. But when the power p acts upon a fixed pulley, as in the present instance, it is evident that the angle DGR varies at every point of the plane, and consequently, there can be but one point where the equilibrium will hold, and the situation of that particular point is determined by the following

Fig. 8.



CONSTRUCTION. Draw the straight line BC at pleasure, and at the point C erect the perpendicular AC, making AC equal to h the given height of the plane; at the point A, make the angle BAC equal to the complement of a, the given elevation; then is ABC a vertical section of the inclined plane.

Produce AC to F, making CE and CF respectively equal or proportional to the power p and the weight w; join BF, and through the point E draw EH parallel to BF, meeting BC the base of the plane in the point H; from C as a centre, with the distance CH, describe the arc HK meeting AB the face of the plane in the

point K, and join CK. Produce CA the height of the plane to the point D, making AD equal to d, the distance of the fixed point from the top of the plane; then from the point D let fall the perpendicular Dn, cutting AB the face of the plane in the point G; then is G the position of the point, at which the given power will balance the given weight. Through G draw GR perpendicular to AB, and produce Dn to meet BC the base of the plane in the point m.

Then, in the right angled triangles mDC and mCn, the angle mCn is equal to the angle mDC; and in the triangles ABC and ARG, the angle GRA is equal to the angle CBA; therefore, the triangles CKB and DGR are similar to one another; hence we have

$$\text{KC} : \text{BC} :: \text{DG} : \text{DR},$$

but by the construction we have

$$\text{KC} : \text{BC} :: p : w;$$

consequently, by equality of ratios, it is

$$p : w :: \text{DG} : \text{DR}.$$

Hence we infer, that when the weight is applied at the point G, found as above, the relation between the sides of the triangle DGR, is such, as to produce an equilibrium between the power p and the weight w.

Now, by the principles of Plane Trigonometry, the value of BC, the base of the plane is

$$\beta = h \cot. a:$$

consequently, the value of KC is

$$\text{KC} = \frac{hp \cot. a}{w};$$

therefore, in the triangle BKC, we have given the sides BC, KC, and the angle KBC, to find the other angles BKC and BCK; wherefore, by Plane Trigonometry, we get

$$\text{KC} : \text{BC} :: \sin. \text{KBC} : \sin. \text{BKC},$$

which, by substituting the literal values, becomes

$$\frac{hp \cot. a}{w} : h \cot. a :: \sin. a : \sin. b;$$

consequently, by making the product of the mean terms equal to the product of the extremes, we obtain

$$w \sin. a = p \sin. b,$$

which corresponds with equation (a), hence, by division, we get

$$\sin. b = \frac{w}{p} \sin. a; \qquad \text{(G)}$$

wherefore, the value of the angle b is known.

Now, because of the similarity of the triangles KBC and GHD, the angle DGR is equal to the angle CKB, and GDR to KCB; therefore, because the angle AGR is a right angle, we have

$$\sin. \text{DGA} : \sin. \text{GDA} :: \text{DA} : \text{AG},$$

which, by substituting the literal values, becomes

$$\sin.(b-90) : \sin.(180-\overline{a+b}) :: d : \text{AG};$$

therefore, by equating the products of the extreme and mean terms, we shall obtain

$$\text{AG} \sin.(b-90) = d \sin.(180-\overline{a+b});$$

but $\sin.(b-90) = \cos. b$ taken positively,

and $\sin.(180-\overline{a+b}) = \sin.(a+b)$;

consequently, by substitution and division, we get

$$\text{AG} = \frac{d \sin.(a+b)}{\cos. b}. \qquad \text{(II)}$$

And the practical rule which this equation affords, may be expressed in words at length as follows.

RULE. *To the angle which the direction of the power makes with the direction of pressure, add the angle of the plane's elevation; then, multiply the sine of the sum by the distance of the fixed point from the summit of the plane, and divide the product by the cosine of the angle, which the direction of the power makes with the direction of pressure, then the quotient will express the distance of the point where the weight acts from the summit or top of the plane.*

EXAMPLE 1. Upon an inclined plane, whose perpendicular height is 45 feet, and elevation 16 degrees, a weight of 84 lbs. is sustained by a power of 32 lbs. acting over a fixed pulley at the distance of 26 feet above the summit of the plane; at what distance from the top of the plane on the slant side is the weight applied in the case of an equilibrium?

Find the value of b, the angle contained between the direction of the power and the direction of pressure, as indicated in equation (gg); that is

$$\sin. b = \frac{84 \times \sin. 16^\circ}{32} = .72355,$$

which corresponds to the natural sine of $46^\circ 21'$; consequently, we have

$$\text{the angle } b = 180^\circ - 46^\circ 21' = 133^\circ 39',$$

and by the rule to equation (II), we have

$$133^\circ 39' + 16^\circ = 149^\circ 39';$$

but $\sin. 149^\circ 39' = .50528$, and $\cos. 133^\circ 39' = .69025$,

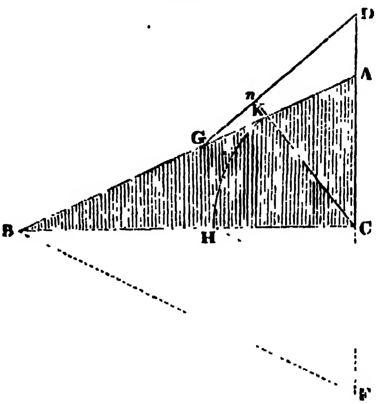
therefore, by substitution, we have

$$AG = \frac{26 \times .50528}{.69025} = 19 \text{ feet nearly.}$$

29. We have already shown the general mode of construction in the preceding figure, but in order to verify the result which we have just obtained, it will be found convenient in this place to repeat the process.

Draw the straight line BC at pleasure, and at the point C anyhow assumed in the line BC, erect the perpendicular AC; then from a scale of equal parts, set off CA equal to 45 feet, the given perpendicular height of the plane; at the point A in the line AC, make the angle BAC equal to 74°, the complement of the plane's elevation; then is ABC a vertical section of the inclined plane.

Fig. 9.



Produce AC both ways to D and F, making AD equal to 26 feet, and CE, CF respectively equal, or proportional to the numbers 32 and 84, which numbers, denote the magnitude of the sustaining power, and the weight of the incumbent body.

Join BF, and through the point E draw EH parallel to BF, meeting BC the base of the plane in the point H; from C as a centre with the distance CH, describe the arc HK cutting AB the face of the plane in the point K; join CK, and from D let fall the perpendicular D*n*, cutting AB the face of the plane in the point G; then shall AG be the required distance, which being taken in the compasses and applied to the proper scale will indicate 19 feet, the same as was found by the process of calculation.

EXAMPLE 2. On an inclined plane whose slant height or length is 58 feet, and elevation 23° 30′, a weight of 53 tons, is sustained in equilibrio by a power of 22 tons acting at a fixed point 19 feet distant from the summit of the plane: at what point on the surface of the plane is the weight applied?

Here by the equation (G), we have

$$\sin. b = \frac{53 \times \sin. 23^\circ 30}{} = \cdot 96062$$

which corresponds to the natural sine of 52′; consequently, we shall have

$$\text{the angle } b = 180^\circ - 73^\circ 52' = 106^\circ 8',$$

and by the rule to equation (H), we get

$$106^\circ 8' + 23^\circ 30' = 129^\circ 38',$$

but sin. $129^\circ 38' = .77014$ and cos. $106^\circ 8' = .27787$,

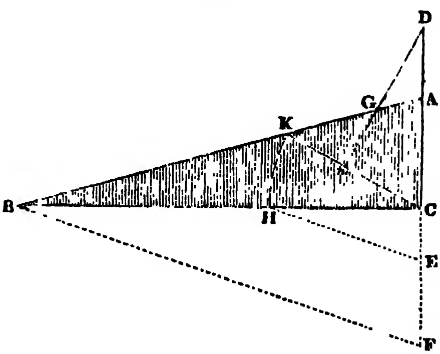
therefore, by substitution, we have

$$\text{AG} = \frac{9 \times \cdot 77014}{\cdot 27787} = 24 \cdot 9 \text{ feet.}$$

30. The truth of the above result may be verified by the following geometrical construction.

Draw the straight line BC of any convenient length at pleasure, to represent the horizontal base of the plane, and at the point B in the line BC, make the angle ABC equal to 23° 30′, the given angle of elevation; from a scale of equal parts of any dimensions whatever, set off BA equal to 58 feet, the given length or sloping height of the plane; from the point A, let fall the perpendicular AC, meeting the base BC in the point C; then is ABC a vertical section of the inclined plane.

Fig. 10.



Produce AC both ways to D and F, making AD equal to 9 feet, and CE and CF respectively equal, or proportional to the numbers 22 and 53, which numbers denote the magnitude of the sustaining power, and the weight of the incumbent body.

Join BF, and through the point E draw EH parallel to BF, meeting BC the base of the plane in the point H; from C as a centre with the distance CH, describe the arc HK cutting AB the face of the plane in the point K; join CK, and from D let fall the perpendicular Dn meeting CK produced in n; produce Dn to meet AB the face of the plane in the point G; then shall AG be the distance from the summit of the plane, of the point at which the weight is applied.

If AG, be taken in the compasses, and applied to the same scale from which the lines AB and AD were taken, it will be found to measure 25 feet very nearly, agreeing with the result obtained by calculation.

OF TWO PLANES SUPPORTING THE SAME LOAD.

31. Problem 2. *A heavy body is supported by two planes of given elevations, it is required to determine the relation between the weight of the body, and the pressure on each plane.*

Before we proceed to the solution or investigation of this problem, it becomes necessary to remark, that, in order that a body solicited by no other force but gravity, may be supported in equilibrio between two inclined planes, it is requisite, that there shall be in the vertical line passing through its centre of gravity, at least one point, from which a perpendicular may be let fall on both the planes, and furthermore, it is absolutely necessary for the condition of an equilibrium, that those perpendiculars shall meet the plane in the points of contact.

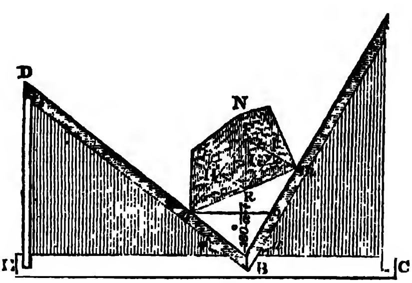
Let *n*N*m* represent a vertical section of the body passing through *n* and *m*, the two points at which it comes in contact with the planes DB and AB; then it is manifest, that the forces by which the body is sustained in equilibrio, are

1. Its own weight urging it in the direction of gravity,
2. The reaction of the plane DB, and
3. The reaction of the plane AB.

And it is evident, that before an equilibrium can obtain, the weight of the body, or the force which solicits it in the direction of gravity, must be wholly destroyed or counteracted by the joint reaction of the planes on which it is sustained.

At the points *n* and *m*, erect the perpendiculars *n*F and *m*F intersecting each other in the point F; then, from F the point of intersection draw FR perpendicular to the horizon and passing through G, the centre of gravity of the incumbent mass. Let the vertical line FR, or any portion of it, represent the weight of the body, and resolve the force FR into its equivalent components FH, HR repectively parallel to the perpendiculars F*n* and F*m*; complete the parallelogram FHRI, then shall its sides FH and FI respectively, represent the force of pressure on the planes DB and AB.

Fig. 11.



Through the point *n*, draw *no* parallel to EC, the horizontal base of the planes; then shall the sides of the triangle *no*B, be respectively perpendicular to the sides of the triangle FRH, and consequently, these two triangles are similar to one another; hence, we have

$$\text{FH} : \text{FR} :: n\text{B} : no,$$
$$\text{FR} : \text{FI} :: no : o\text{B},$$

which, by multiplying the corresponding terms, and casting out the common factors, gives

$$\text{FH} : \text{FI} :: n\text{B} : o\text{B};$$

but the sides nB and oB, are to one another respectively as the sines of their opposite angles; therefore, we have

$$FH : FI :: \sin. noB : \sin. onB,$$

but by reason of the parallels EC and no, the angles noB, onB are equal respectively to the angles ABC and DBE, the elevations of the planes; therefore, by substitution, we get

$$FH : FI :: \sin. ABC : \sin. DBE;$$

from which it appears, that

If a heavy body be sustained in equilibrio between two inclined planes, the pressures on those planes are reciprocally proportional to the sines of their elevation above the horizon.

Let $w=$ FR, the whole weight of the incumbent body,
$r=$ FI, the pressure on the plane AB,
$r'=$ FH, the pressure on the plane DB,
$a=$ ABC, the elevation of the plane AB,
$\varphi=$ DBE, the elevation of the plane DB;

then, we shall have $180^\circ-\overline{a+\varphi}=$ ABD, the angle contained between the two inclined planes, AB and DB. Consequently, by substituting the literal values of the several quantities in the preceding analogy, and equating the products of the extreme and mean terms, we obtain

$$r \sin. a=r' \sin. \varphi. \qquad (\text{I})$$

32. This equation, expresses the relation that subsists between the pressures on the planes and their respective elevations; but to determine the relation that subsists between the weight of the body, and the respective pressures, we have the following analogies, viz.

$$r' : w :: \sin. a : \sin. (180^\circ-\overline{a+\varphi}),$$
$$r : w :: \sin. \varphi : \sin, (180^\circ-\overline{a+\varphi}),$$

or by adding the corresponding terms, and substituting $\sin. (a+\varphi)$ for $\sin. (180^\circ-\overline{a+\varphi})$, we shall have

$$r+r' : w :: \sin. a+\sin. \varphi : \sin. (a+\varphi);$$

from which it appears, that

If a heavy body be sustained in equilibrio between two inclined planes, the sum of the pressures on the planes, is to the whole weight of the body as the sum of the sines of elevation, is to the sine of inclination of the planes.

By equating the product of the extreme and mean terms, the above analogy gives the following equation, viz.

$$(r+r') \sin. (a+\varphi)=w(\sin. a+\sin. \varphi). \qquad (\text{K})$$

Or, because the angle nFm, is equal to the sum of the angles of elevation, we have by equation (a), of the Composition of Forces,

$$w=\sqrt{r^2+r'^2\pm 2rr' \cos. (a+\varphi)}. \qquad (\text{L})$$

And from either of these equations, any one of the quantities, may be determined in terms of the rest, after the manner exhibited in the following problems.

33. PROBLEM 1. *In the equation $(r+r')\sin.(a+\varphi)=w(\sin. a+\sin.\varphi)$, there are given, r, r' a and φ, to find the value of w.*

Let both sides of the equation be divided by $(\sin. a+\sin.\varphi)$, and we shall have

$$w=\frac{(r+r')\sin.(a+\varphi)}{\sin. a+\sin.\varphi}. \qquad \text{(M)}$$

34. PROBLEM 2. *In the equation $(r+r')\sin.(a+\varphi)=w(\sin. a+\sin.\varphi)$ there are given, r, w, a and φ, to find the value of r'.*

Here, if we transpose the term $r\sin.(a+\varphi)$, and divide by the coefficient of r', we shall have

$$r'=\frac{w(\sin. a+\sin.\varphi)}{\sin.(a+\varphi)}-r. \qquad \text{(N)}$$

The expression for the value of the pressure r, determined in a similar manner, is

$$r=\frac{w(\sin. a+\sin.\varphi)}{\sin.(a+\varphi)}-r'. \qquad \text{(O)}$$

35. PROBLEM 3. *In the equation $(r+r')\sin.(a+\varphi)=w(\sin. a+\sin.\varphi)$ there are given, r, r', w and a, to find the value of φ.*

Let the expression $\sin.(a+\varphi)$ be expanded according to the principles of Trigonometry, and our equation becomes

$$(r+r')(\sin. a\cos.\varphi+\cos. a\sin.\varphi)=w(\sin. a+\sin.\varphi),$$

from which, by collecting the terms, we get

$$\{(r+r')\cos. a-w\}\sin.\varphi+(r+r')\sin. a\cos.\varphi=w\sin. a,$$

or dividing both sides of the equation by $\sin. a$, we have

$$\{(r+r')\cos. a-w\}\operatorname{cosec.} a\sin.\varphi+(r+r')\cos.\varphi=w;$$

but $\cos. a\operatorname{cosec.} a=\cot. a$; therefore, by substitution, we obtain

$$\{(r+r')\cot. a-w\operatorname{cosec.} a\}\sin.\varphi+(r+r')\cos.\varphi=w;$$

let all the terms of the equation be divided by $(r+r')$, and we get

$$\cos.\varphi+\left(\cot. a-\left(\frac{w}{r+r'}\right)\operatorname{cosec.} a\right)\sin.\varphi=\left(\frac{w}{r+r'}\right),$$

but $\cos.\varphi=\sqrt{1-\sin.^2\varphi}$; consequently, by substitution, we have

$$\sqrt{1-\sin.^2\varphi}+\left(\cot. a-\left(\frac{w}{r+r'}\right)\operatorname{cosec.} a\right)\sin.\varphi=\left(\frac{w}{r+r'}\right). \qquad \text{(P)}$$

From which equation, by substituting the numerical values of the several given quantities, the value of $\sin.\varphi$ can easily be found.

The expression for the value of the angle a, derived in a similar manner, is as follows, viz.

$$\sqrt{1-\sin.^2 a}+\left(\cot.\varphi-\left(\frac{w}{r+r'}\right)\operatorname{cosec.}\varphi\right)\sin. a=\left(\frac{w}{r+r'}\right). \qquad \text{(Q)}$$

As the form of these equations is a little complicated, it may perhaps be useful to illustrate the method of reduction by the resolution of the following numerical example.

EXAMPLE. A body whose weight is equal to 74 tons, being sustained in equilibrio between the two inclined planes AB and DB, is found to transmit pressures on those planes equivalent to 44 and 68 tons respectively; what is the elevation of the plane DB, that of AB being 64 degrees?

Here we have given $w=74$; $r=44$; $r'=68$, and $a=64°$; let these numbers be respectively substituted for their representatives in equation (P), and it becomes

$$\sqrt{1-\sin.^2\varphi}+(\cot.64-\frac{37}{56}\operatorname{cosec}.64)\sin.\varphi=\frac{37}{56};$$

but $\cot.64°=\cdot48773$, and $\operatorname{cosec}.64°=1\cdot1126$; consequently, by substitution, we get

$$\sqrt{1-\sin.^2\varphi}-\cdot24738\sin.\varphi=\cdot660705.$$

Transposing the term $\cdot24738\sin.\varphi$, and squaring both sides of the equation, we get

$$1-\sin.^2\varphi=\cdot0612\sin.^2\varphi+\cdot32689\sin.\varphi+\cdot43654;$$

therefore, by collecting the terms, we obtain

$$1\cdot0612\sin.^2\varphi+\cdot32689\sin.\varphi=\cdot56346,$$

and dividing all the terms by $1\cdot0612$, we get

$$\sin.^2\varphi+\cdot30804\sin.\varphi=\cdot58157.$$

An adfected quadratic equation, which, being reduced, gives $\sin.\varphi=\cdot59075$, which corresponds to the natural sine of 35° 33′ 40″.

36. It is however manifest, that the same thing may be obtained directly, and much more simply from the triangles of equilibrium RFH and RFI; for we have elsewhere shown, that when a body is at rest upon the surface of an inclined plane, the angle of elevation of the plane is equal to the angle contained between the direction of pressure and the direction of gravity, and the same property holds in the case of two inclined planes; consequently, we have

$$r' : r :: \sin.a : \sin.\varphi.$$

This corresponds to the original analogy, but since it excludes the weight of the incumbent body, it does not illustrate the use of the equations marked (P and Q), in which the weight is an indispensable datum.

The value of the angle φ, or the elevation of the plane DB, determined in this manner, is as follows.

$$\sin.\varphi=\frac{r\sin.a}{r'};$$

therefore, by substituting the numerical values of the several quantities, we shall have

$$\sin.\varphi=\frac{44\times.89879}{68}=.58157,$$

which corresponds to the natural sine of 35° 33′ 40″.

35. The following very curious and interesting property flows immediately from the equation marked (kk), and the same property has already been alluded to in the Composition of Forces, equation (e), thus.

If the angles a and φ, the inclinations of the planes, are each equal to 60 degrees, then equation (kk) becomes

$$(r+r')\sin. 2a=2w\sin. a,$$

but by the Arithmetic of Sines, we have

$$\sin. 2a=2\sin. a\cos. a;$$

therefore, by substitution, we obtain

$$(r+r')\cos. a=w,$$

but since $a=60°$, $\cos. a=\frac{1}{2}$; consequently, we get

$$r+r'=2w.$$

From which we infer, that

If the angles of elevation of the planes be each 60 degrees, the sum of the pressures on the planes will be equal to twice the weight of the body.

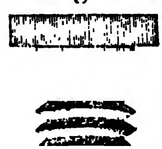
Various other particulars of a kindred nature to the above might have been added in this place; but since they are in general more interestingly curious than really useful, we have thought it better to omit them.

5. OF THE SCREW.

INTRODUCTION.

The screw can hardly be called a simple machine, because it is never used without a lever or winch to move it home, and then it becomes an engine of amazing power and utility, in pressing together substances that have little cohesion, or in raising to short heights ponderous bodies. The smith, the carpenter, the printer, and the packer all use screws in their respective occupations. Bales of wool, cotton, hay, &c. may be compressed by means of a screw into packages, the specific gravity of which shall be much heavier than an equal volume of water. Such packages will then sink in the ocean like a cannon-shot. Moreover, many of our domestic operations are performed by means of presses or screws; as the making of sugar, oil and wine:

Fig. 1.



Cæcubum et prælo domitam Caleno.

Fig. 2.

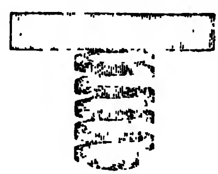
The screw possesses one great and decided advantage over the inclined plane, from which its principle of action may be said to be derived. The great attrition or friction which takes place in the screw, is useful by retaining it in any state to which it has once been brought, and continuing the effect after the power is removed. It is thus the cabinet-maker's cramp, the smith's vice, and all these instruments, made by opticians, in which screws act, can with certainty and advantage be employed.

Screws are made in various ways: some have sharp threads, some square, and others round threads. The first create most friction; and the others, by creating less, are employed in the construction of self-acting hinges, and other mechanical inventions that raise weights with ease. And these threads are differently formed, according to the materials of which they are made, or the use for which they are intended. The threads of wooden screws are generally angular, (as in figures 1 and 2) that they may thereby rest upon a broad base and have their strength increased to the utmost. Small screws, of whatever material they are made, are generally angular also, not only for the same reason as the wooden

ones, but because the angular thread is the most easily made. The metal screws which are used for large presses, vices, cramps, &c. have generally a square thread, (as in figures 3 and 4,) a form which gives great steadiness of motion. A thread of which the sides are parallel, and the top and bottom a little rounded, is perhaps the most perfect of all forms.

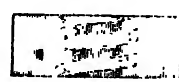
In the common screw, to which the preceding observations are exclusively applicable, the threads are one continued spiral from one end to the other; but where there are two or more separate spirals running up together, as in the worm of a jack, or the principal screw of a common printing press, the descent of the screw in a revolution will be proportionably increased; and, therefore, whatever be the number of spirals, they must, in calculating the power, be measured and reckoned as one thread.

Fig. 3.



We may contemplate the screw with the square thread as a sort of winding wedge which has the same relation to the straight wedge that a road winding up a hill or tower has to a straight road of the same length and acclivity. This thought is fully developed in the subsequent definitions of the screw.

Fig. 4.



To enumerate all the uses of the screw would be impossible; but we cannot pass over its vast importance in measuring or subdividing small spaces: when thus applied it is called a *micrometer*, which may be made to indicate on an index plate, a portion of a turn, advancing the screw less than the fifty-thousandth part of an inch. This enables the mathematical instrument maker and the engraver to mark divisions on their work with a minuteness and accuracy quite extraordinary. If we suppose such a screw to be pulling forward a plate of metal, or pulling round the edge of a circle, over which a sharp pointed steel marker can be let down perpendicularly, always in the same place, the marker if let down once for every turn of a screw having one hundred turns of its thread in the space of an inch, will make just as many lines on the plate as the screw makes turns, but if made to mark at every hundredth or a thousandth of a turn of the screw, which it will do with equal accuracy, it may draw an hundred thousand distinct lines in one inch.*

The use of the screw to raise water in the manner invented by Archimedes, is a very remarkable application of this engine, and appears to have been the earliest pump which mariners employed to bale water from the holds of their ships. Any spar or mast would form a cylinder, round which a tube might be wound in a

* Arnot's Physics, vol. i. p. 179.

spiral form; so as to raise the fluid from the hold of the vessel. But the same machine will raise sand or any body that can pass within the tube. For if a screw thus formed be placed obliquely, so as to make with the vertical, an angle equal to that which the spiral makes with the lines parallel to the axis, there will be in each turn of the spiral a part parallel to the horizon, where, if a body were placed, it would be at rest. If, then, the screw be turned, the body will ascend, because the part of the screw behind it becomes more inclined than the part before it, so that the body is urged forward and consequently ascends. If the screw used in this manner, be turned with great rapidity, the body may acquire a centrifugal force so great as to overcome its gravity, in which case it will descend.*

The coining engine which consists of a screw carrying ponderous arms acts by a kind of percussion: the impulsion accumulated by the swing produces a stroke similar to the concentrator of force; but the violence of the blow is softened, and the shock partly consumed by the prolonged friction of the slanting grooves of the screw, by which the stamper advances to the die.

The principle of the conditions of equilibrium for the screw is precisely the same as that already demonstrated for the inclined plane, when the power is supposed to act in a direction parallel to the base. And in the composition of the general equation which involves these conditions, there are five different quantities engaged, one of which is constant, and four variable, consequently the entire solution required the reduction of four equations, which we have analyzed in as many problems. The practical rule afforded by each problem has been expressed in words, and further illustrated by a couple of examples worked out at length.

Finally we have wound up this theorem by circumscribing its four cases, so as to exhibit all the varieties of questions that can be proposed respecting the screw, on the supposition of its being an instrument combining the principles of the lever and those of the inclined plane: and this is the obvious characteristic of the simple screw and its kindred nut. To effect this we have substituted numerical values for the given quantities in each case of the original equation shewing that, any three of these quantities being given, the remaining one can always be found from the general equation alone, which therefore involves every circumstance respecting the equilibrium when the screw is the instrument through whose means the equilibrium has been produced.

The endlesss crew comes next under our consideration. It is so called because it is cut on an axle which may be turned perpetually without advancing or receding, while as a pinion it works in the

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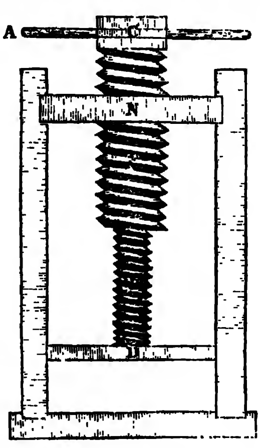
* Playfair's Outlines of Natural Philosophy, vol. i. p. 91. Third Edition. Also Bernouilli Hydro. Sec. 9 §. 26. Hennert sur la vis d'Archimede.

circumference of a wheel which it moves round, and thus accomplishes at once what would otherwise require the intervention of two or three wheels, and though it operates slowly, it has not probably more friction than any of the less simple combinations which might be employed to effect the same object. It possesses the advantage too of moving a wheel with much more steadiness than a pinion, when the workmanship of both is of equal quality. This circumstance is not so much regarded by mechanists as it ought to be, and therefore the endless screw is frequently unemployed in situations where it would be very advantageous.

Having established the equilibrated expression for the perpetual screw, we proceed to the complete analysis of the equation which involves six quantities to be determined, all of which may in their turn be considered variable, consequently six separate equations are required to exhibit the solution in its full extent. Each of these problems affords a practical rule, which we have expressed in common language and illustrated by a couple of examples, whose solutions are drawn out at length, and to accomodate the whole theory of the perpetual screw, considered in combination with the principles of the wheel and the axle to the plan pursued in the simple screw, we have substituted the several given quantities for their representatives in the general equation, by which means a new equation is obtained, involving only one unknown letter combined with the numerical value of all the others. The six equations derived from this process, concentrate the rules in a very neat manner; and give systematic uniformity to investigations, upon which we have spared no pains to render practically useful.

The differential screw, invented by Hunter the surgeon, was first described in the 71st vol. of the Philosophical Transactions, and all writers upon mechanics have introduced it into their lucubrations. It is vastly superior as an invention to the screw of Archimedes. This apparatus as we have defined and explained, consists of two screws, one of which moves within the other, and has one thread fewer an inch than the exterior screw in which it moves. Moreover the exterior screw is a hollow cylinder, the interior of which is a nut adapted to receive the fixed screw, while the exterior surface is an exterior screw working through a fixed nut.



In the diagram before us B is the fixed screw which works in the hollow of S, whose exterior thread moves in the fixed nut N. We shall not anticipate any part of our illustration in the text, if we notice here that if the threads of two screws S and B were equal, their motions would not move the board D; but if S had 20 threads

in an inch while B had 21, in one revolution c would be moved downwards through the 20th of an inch, and it would draw up the screw B $\frac{1}{21} \times \frac{1}{20} = \frac{1}{420}$ of an inch. Thus the differential screw produces an effect much superior to the common screw, for the latter must have 420 threads in an inch before it could produce an effect equal to the former, which would weaken it to such a degree, that it would be unable to resist any considerable force.

The equation of equilibrium which we have established for the differential screw, implies that when a power sustains a weight in equilibrio, the power is to the resistance as the difference between the distances of the threads of the two screws, to the circumference of the circle described by the power. Four problems flowing from the equilibrated expression for the differential screw, exhibit every portion of its analysis, and the examples under each problem sufficiently illustrate the mode of their application. We conclude this subject by shewing in what manner the theory may be abridged, so that if any three of the quantities in the equilibrated equation are given together with the constant number, the value of the fourth quantity can easily be assigned. In conclusion we have set before the reader in three equations of equilibrium, the whole theory of the simple the endless and the differential screw.

In the lever we have seen how a person of very little strength may by means of a simple machine overcome vast and ponderous resistances; in the wheel and axle, when converted into a turning-lathe, we have proof of the workman's foot answering the purpose of a second pair of hands; the pulley has added to our knowledge in mechanics, by unfolding a new power which enables a man, without quitting his stand, to change the direction of a weight that would otherwise have required many men to remove; and in the inclined plane, as a primary mechanical agent, we have witnessed the means of transmitting power and executing work where the rude attempts of unassisted physical force would in vain have sought to triumph. But none of these instruments resembles the *screw*, which is not merely an additional hand that science confers on the workman who uses this appliance, for with a *vice* he is omnipotent over the material he may be fashioning with ease, patience, and skill; but it is a power (ἑκατόγχειςον) superior to an hundred hands—

ὁ γὰρ αὖτε βίῃ οὗ πατρὸς ἀμείνων ——

The cord, itself incapable of generating power, when wound round an axle, may yet become the means of accumulating force in a wondrous degree, as when a heavy weight is lifted and then allowed to fall upon the head of a pile, which is thus driven far into the hard ground; or, employed in the *Concentrator of Force*, the cord firmly elucidates the acquisition and the transfer of impulsive energy; and combined with the wheel and axle in the

turning lathe and its sliding rest, the cord enables us to accomplish the finest specimens of human art in wood, metal, ivory, or in clay, whether we consider the excellence of the workmanship or the unlimited application to which we put the products of a turning-lathe. Moreover it is this same application of the lever and the cord which extends to the construction of a large part of our machinery,—printing-presses by steam, calico-printing from cylinders, embossing calico and leather with which to cover books or furniture, engraving copper-plates by pressure;—in all of which we use rollers which must be turned round by other circular appliances; and without attempting to enumerate the various processes in the useful and the fine arts in which the lever, the wheel and axle, and the pulley are concerned, we may affirm that there are many thousand ways in which the combinations of those three mechanical powers are daily and hourly employed in enabling us to exert to the greatest advantage the forces we employ.

With how much reason may we affirm of the screw, that its marvellous excellence may be best appropriated in the universality of its application as an additional hand to man, a great and powerful engine in numberless occupations in the every day concerns of the busy world, an instrument, when combined with the engraver's art and the delicate mechanisms produced by watchmakers and engineers, steadier far in adjusting mechanism than the nicest eye which ever measured artificial distances, capable of aiding us in paths of art yet unknown, and in operations too delicate for human touch.

OF THE SCREW.

Definitions, Illustrations, and Conditions of Equilibrium.

1. The next mechanical power to which our attention has to be directed for the purpose of establishing the conditions of equilibrium, is

THE SCREW, an instrument of very frequent occurrence, and of very extensive application in the mechanical arts, its vast power rendering it more convenient than most other machines for the generation of intense pressure, while the extreme equability and smoothness of its operation, added to the immense difference between the velocity of the threads and that of the handle or power, adapts it in an eminent degree to the admeasurement and subdivision of minute spaces, "and in *reading off* from them."

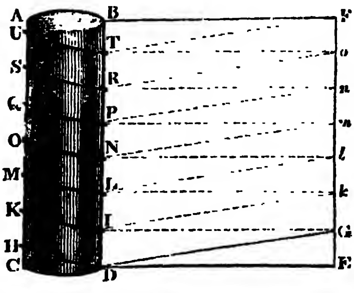
The screw consists of a spiral thread or ridge of uniform thickness, winding obliquely round a cylinder, and every where making the same angle with straight lines drawn upon its surface in a direction parallel to the axis; so that, if the surface of the cylinder were unfolded with this spiral thread upon it, there would be constituted a rectangular parallelogram, on which the development of the spiral thread would trace out an inclined plane,

Whose height would be to its base, as the distance between two contiguous threads in the direction of the axis, to the circumference of the cylinder on which the thread is raised.

Hence it appears, that the screw is simply a modification of the inclined plane, and under this point of view shall we consider it in the following pages.

2. Let ABCD be a right cylinder surrounded by the spiral thread DHIKLM &c., making every where the same angle with the straight lines AC and BD, which are drawn parallel to the axis of the cylinder.

Fig. 5.



Draw BF and DE perpendicular to BD, and conceive the cylinder to roll forward between the parallels BF and DE, until it has made a complete revolution about its axis; then will BD coincide with FE, and BF and DE will each of them be equal to the circumference of the cylinder, of which BD is the height or axis; consequently, the plane BDEF is the developement of the cylindric surface.

Through the points I, L, N, P, R and T, draw the lines IG, L*k*, N*l* &c. respectively parallel to BF or DE, and join DG, I*k*, L*l* &c.; then, because the spiral thread DHIKLM &c. makes every where the same angle with the straight lines AC and BD, which are drawn parallel to the axis of the cylinder, it is manifest, that while the cylinder rolls forward from BD to EF, the several convolutions of the spiral line will trace out the oblique lines DG, I*k*, L*l* &c., while the corresponding circumferences trace out the lines DE, IG, L*k* &c., as far as the number of times that the spiral thread surrounds the cylinder.

Now, because the lines EG, G*k*, *kl* &c. are equal to DI, IL, LN &c., the distances between the convolving portions of the spiral thread, it is obvious, that the evolution of the cylindric surface generates a series of inclined planes, each satisfying the conditions of the preceding proportion; and because the inclination of every part of the spiral line with respect to the base of the cylinder, is the same as the inclination of the several planes; it follows, that the power which is necessary to sustain a weight or resistance on any portion as DHI of the spiral line, will also sustain it on DGE its corresponding plane.

Now, as the cylinder turns round on its axis, each point of the spiral thread DHI acts upon the resistance, with a force analagous to that which sustains a body on an inclined plane when the power acts in a direction parallel to its base; but we have shown, in treating of the inclined plane in the foregoing article, that

> *If a body be held in equilibrio on an inclined plane, by means of a power acting in a direction parallel to the base of the plane, the power is to the weight or resistance of the body, as the height of the plane is to the length of its base.*

3. In the present instance, EG the height of the plane is equal to DI, the distance between two contiguous portions of the spiral line, while DE, the base of the plane, is equal to the circumference of the cylinder ABCD on which the thread is raised; hence we infer, that

> *If a body is sustained on the spiral thread* DHI, *by means of a power acting in a direction parallel to the base of the cylinder, or in a direction perpendicular to the axis, the power is to the weight or resistance of the body, as the distance between two contiguous portions of the spiral line is to the circumference of the cylinder.*

This would manifestly be the case, supposing the force to act close to the surface of the cylinder, but when it acts at any other distance from the axis, its magnitude varies directly as the radius of the cylinder, and inversely as the distance at which it acts.

Put p = the magnitude of the power which sustains the body at rest,
w = the weight of the body,
δ = GE, the height of the plane, or distance between two contiguous portions of the spiral,
β = DE, the base of the plane, or circumference of the cylinder on which the thread is raised,
r = the radius of the cylinder,
and R = the radius of the circle described by the power, or the distance from the axis of the cylinder at which the power is applied.

Then, by the principles of the inclined plane, we have

$$p : w :: \delta : \beta,$$

or by making the product of the mean terms equal to the product of the extremes, we get

$$\delta w = \beta p.$$

This equation has been obtained on the supposition that the power is applied close to the circumference of the cylinder, or, which is the same thing, at the extremity of the radius r; but in order to adapt it to the case of the power acting at any other distance, we must substitute for β considered generally, its value $2\pi R$, obtained in the following manner.

The value of β as referred to the base of the plane DE in the present instance, is obviously

$$\beta = 2\pi r,$$

where $\pi = 3.1416$, the circumference of a circle, whose diameter is unity, and r the radius of the cylinder, or the circle of whose circumference β is the assumed representative; but because the circumferences of circles are to one another as their radii, we have

$$r : R :: 2\pi r : 2\pi R;$$

consequently, by substitution, we get

$$\delta w = 2\pi R p,$$

and finally, by introducing the numerical value of π, it is

$$\delta w = 6.2832 R p. \qquad (a)$$

This equation being independent of r the radius of the cylinder, it is manifest, that at whatever distance from the axis the weight or resistance may be applied, the same power will maintain it at rest on the spiral line, provided that the distance between any two contiguous portions remains unaltered.

The preceding equation, then, involves the conditions of equilibrium precisely as they are unfolded in the operations of the screw, to which instrument our present enquiries are more especially directed, and to which we shall now endeavour to apply them.

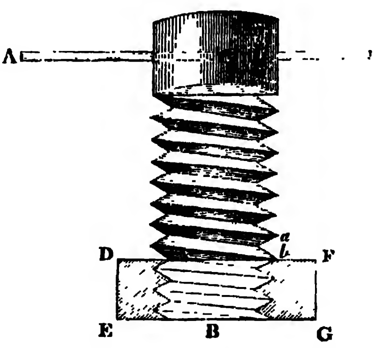
Section First.

OF THE SIMPLE SCREW.

4. For which purpose, let us revert to our definition of the screw, where, in addition to the spiral thread winding about the convex surface of the cylinder in the manner already described, we may conceive a corresponding groove to be cut on the concave surface of a hollow cylinder of the same diameter as the former, into which groove the protuberant spiral on the convex cylinder is admitted, and being turned by a power acting at the extremity of a lever as represented in the subjoined diagram, the operation of the screw becomes manifest.

Let CB represent the convex cylinder whereon the spiral thread is raised which constitutes the screw, and DEFG the block or nut, wherein is the hollow cylinder of the same diameter, having a corresponding groove on its concave surface, in which the spiral thread operates to produce the mechanical effect.

Fig. 6.



The part of the figure DEFG is a section of the nut, showing the manner in which the spiral thread and groove come in contact with one another, and AC is the lever, at whose extremity the power is supposed to be applied, which acting in a direction perpendicular to the axis of the cylinder, opposes the resistance and maintains the equilibrium.

Now, in order to explain the mode of action of the screw fitted as in the annexed diagram, we shall suppose the nut or block DEFG to be fixed, while the cylinder CB revolves about its axis, impelled by a power acting at the extremity of the lever AC, in a direction parallel to the base of the cylinder; then it is evident, from the nature of the contact exhibited in the section DEFG, that when the power acting at A, the extremity of the lever AC, has made a complete revolution, the cylinder will have been elevated or depressed through a space equal to the distance between two contiguous portions of the spiral line, or threads of the screw, according to the direction in which the cylinder is made to revolve; consequently, in the case of an equilibrium,

The power is to the resistance, as the distance between two contiguous threads of the screw is to the circumference of the circle described by the power.

This is the principle of equilibrium in the screw, and is, as we have already shown, the very same as that for the inclined plane when the power is supposed to act in a direction parallel to its base; the same principle is also implied in equation (a), which we proceed forthwith to exemplify.

In the composition of the general equation, which involves the conditions of equilibrium for the screw, it is obvious, that there are five different quantities engaged, one of which is constant and four variable; consequently, the complete solution of the question requires the reduction of four equations, and the manner of performing those reductions will become manifest from the analysis of the following problems.

5. PROBLEM 1. *In the equation* $\delta w = 6.2832 \text{R}p$, *there are given,* δ, R *and* p; *to find the value of* w.

Here it is evident, that the required quantity, combined with one given quantity after the manner of multiplication, occupies one side of the general equation; but it is a principle in the theory of equations, that by whatever process the composition is effected, a process of a nature directly the reverse must be employed to accomplish the analysis; now, the composition indicates a process of multiplication; hence, the analysis must be effected by division.

Therefore, let both sides of the given equation be divided by δ, the coefficient of the required quantity, and we shall obtain

$$w = \frac{6.2832 \text{R}p}{\delta}. \qquad (b)$$

From which it appears, that if R and p are given, the value of w varies inversely as δ, and is therefore the greatest when δ is the least, or infinite when δ is nothing.

The practical rule afforded by equation (b) may be expressed as follows.

RULE. *Multiply the magnitude of the power by the distance from the axis of the cylinder at which it acts, and again by the constant number* 6.2832; *then divide the product by the distance between two contiguous threads of the screw, and the quotient will give the magnitude of the weight or resistance, which can be sustained in equilibrio by the action of the given powe*

EXAMPLE 1. What weight will a power equivalent to 44 lbs. acting at the extremity of a lever 20 inches long, sustain in equilibrio, by means of a screw, in which the pitch,* or distance between the threads is $\frac{1}{8}$ of an inch?

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* The distance between two contiguous threads of the screw is sometimes called the pitch, in order to adapt it to the teeth of wheels.

Here we have given $\delta = \frac{1}{8} = 0\cdot125$; $R = 20$, and $p = 44$; consequently, by equation (b), or the rule derived from it, we have

$$w = \frac{6.2832 \times 20 \times 44}{0.125} = 44233.728\text{lbs.}$$

EXAMPLE 2. What pressure will be produced by means of a power, equivalent to 156lbs. acting at the extremity of a lever 3 feet long, the pitch of the screw, or distance between two contiguous threads being $\frac{1}{2}$ an inch?

By the rule we have

$$w = \frac{6.2832 \times 36 \times 156}{0.5} = 70572.9\text{lbs.}$$

6. PROBLEM 2. *In the equation $\delta w = 6.2832\,Rp$, there are given, δ, w and R; to find the value of p.*

In this case, the required quantity is found in combination with the constant number 6.2832, and also with R, one of the given quantities; consequently, the analysis will be effected, if both sides of the equation be divided by the product 6.2832, thus

$$p = \frac{\delta w}{6.2832R}. \qquad (c)$$

Whence it appears, that if the lever and the weight are given, the value of the power p varies directly as the pitch of the screw.

The practical rule afforded by equation (c), may be expressed as follows.

RULE. *Multiply the given weight by the pitch or distance between two contiguous threads of the screw, and divide the product by 6.2832 times the length of the lever, for the magnitude of the power required.*

EXAMPLE 1. What power acting at the distance of 24 inches from the axis of the cylinder, will be sufficient to sustain a weight of 6000lbs. supposing the pitch, or distance between two contiguous threads of the screw to be $\frac{3}{4}$ of an inch?

Here we have given, $\delta = \frac{3}{4}$ or $= 0.75$; $R = 24$, and $w = 6000$; consequently, by equation (c), or the rule derived from it, we have

$$p = \frac{0.75 \times 6000}{24 \times 6.2832} = 29.87\text{lbs.}$$

EXAMPLE 2. What power acting at the extremity of a lever 72 inches long, will produce a pressure of $22619\frac{1}{2}$lbs., supposing the threads of the screw to be an inch asunder?

Here, by the rule, we have

$$p = \frac{22619\cdot5}{72 \times 6\cdot2832} = 50\text{lbs.}$$

7. PROBLEM 3. *In the equation $\delta w = 6\cdot2832Rp$, there are given δ, w and p; to find the value of R.*

Here the required quantity is found in combination with the constant coefficient 6.2832, and also with p, one of the given

quantities; consequently, the analysis will be performed by dividing both sides of the given equation by the product 6.2832 p, in the following manner.

$$R = \frac{\delta w}{6.2832 p}. \qquad (d)$$

From this equation it appears, that if the resistance and the pitch of the screw are given, the length of the lever varies inversely as the power; but if the power and the resistance or weight are given, the length of the lever varies directly as the pitch of the screw.

The practical rule afforded by equation (d), may be expressed as follows.

RULE. *Multiply the given weight, by the pitch, or distance between two contiguous threads of the screw, and divide the product by* 6.2832 *times the magnitude of the power; for the length of the lever, or distance from the axis of the cylinder at which the power acts.*

EXAMPLE 1. If a man acts upon the screw of a printing press, with a force equivalent to 150 lbs.; what must be the length of the lever to produce a pressure of 4436 lbs. supposing the pitch, or distance between two contiguous threads of the screw, to be half an inch?

Here we have given, $\delta = \frac{1}{2} = 0\cdot5$; $p = 150$, and $w = 4436$; hence, by equation (d), or the rule derived from it, we have

$$R = \frac{0\cdot5 \times 4436}{6.2832 \times 150} = 2.34 \text{ inches.}$$

EXAMPLE 2. The screw of a vice has its pitch, or distance between the threads equal to one quarter of an inch; what must be the length of the handle or lever, so that a man acting with a force equivalent to 98 lbs. may produce a pressure of 8698 lbs.

By the rule, we have

$$R = \frac{0\cdot25 \times 8698}{6\cdot2832 \times 98} = 3\cdot5 \text{ inches.}$$

8. PROBLEM 4. *In the equation* $\delta w = 6\cdot2832 Rp$, *there are given* p, w *and* R; *to find the value of* δ.

In this instance, the required quantity is simply combined with w, the representative of the weight; hence, if both sides of the equation be divided by w, we shall have

$$\delta = \frac{6\cdot2832 Rp}{w}. \qquad (e)$$

From this equation it is manifest, that if the power, and the distance at which it acts be given, the pitch of the screw, or the distance between the threads, varies inversely as the weight; but if the weight and the power be given, the variation of the pitch is directly as the length of the lever on which the power acts, and if

the weight and the length of the lever be given, the variation of the pitch is directly as the power.

The practical rule afforded by equation (e), may be expressed as follows.

RULE. *Multiply together the power, the length of the lever on which the power acts, and the constant number* 6·2832; *then divide the product by the given weight, pressure or resistance, and the quotient will give the pitch, or distance between two contiguous threads of the screw.*

EXAMPLE 1. A power equivalent to 324lbs. acting at the extremity of a lever 5 feet in length, is found to balance a resistance of 212 tons, or 474880lbs.; what is the pitch, or distance between the threads of the screw?

Here we have given, $p=324$; $w=474880$, and $R=60$; consequently, by equation (e), or the rule derived from it, we have

$$\delta=\frac{6.2832\times 60\times 324}{474880}=0.25+,$$

or something more than one quarter of an inch.

EXAMPLE 2. A pressure of 900 tons, or 2016000lbs. is produced by a power equivalent to 864lbs. acting at the extremity of a lever 72 inches long; what is the pitch, or distance between the threads of the screw?

Here by the rule, we have

$$\delta=\frac{6\cdot 2832\times 72\times 864}{2016000}=0\cdot 193+,$$

or very nearly two-tenths of an inch.

9. From these examples it appears, how immensely the power of the screw is augmented by decreasing the distance between the threads; for since the proportion between the power and the resistance, or the mechanical efficacy of the screw, depends on the relation between the pitch and the circumference of the circle described by the power, it is manifest that the energy of the instrument may be increased or decreased at pleasure, by altering for that purpose, either the length of the lever on which the power acts, or the distance between two contiguous threads of the screw: but in practice, it is necessary that the thread or spiral line of the screw should attain a certain strength, in order to withstand the pressure to which it is exposed; for if it be made very small and fine, it will be torn off by the vast resistance in passing through the nut; it is therefore probable, that the pitches or distances between the threads, as derived from the preceding examples, are far too small for resisting the pressures specified in the questions; but an inconsistency in this respect, has nothing to do with the accuracy or inaccuracy of theoretical principle, and it is evident, that (setting aside the consideration of friction), the foregoing theory is rigorously correct.

It is the province of the practical engineer, to adjust the thickness of the thread and the length of the lever to the magnitude of the strain to be withstood, the analyst can do nothing more than establish the maxims for such adjustments, but the establishment of these maxims must be deferred, till we come to treat on the strength of materials.

10. If, instead of instituting a separate equation for each case of the analysis, as we have done in the equations marked (*b*), (*c*), (*d*) and (*e*), we had substituted the given quantities for each case in the original equation marked (*a*), the theory would have been greatly circumscribed; for it is evident, that *that* equation alone, involves every circumstance respecting the equilibrium, when the screw is the instrument through whose means the equilibrium has been produced.

11. In order to illustrate this observation, let us assume $\delta=2$ inches, $R=84$ inches, $p=28$ lbs. and $w=7389.0432$ lbs.; then it is manifest, that any three of these quantities being given, the remaining one can always be found from the general equation (*a*), in the following manner.

1. *Given* δ, R *and* p; *to find the value of* w.

Let the numerical values of these letters be substituted for them in equation (*a*), and it becomes

$$2w=6\cdot 2832\times 84\times 28,$$

dividing both sides of the equation by 2, we get

$$w=6\cdot 2832\times 84\times 14=7389\cdot 0432 \text{ lbs.}$$

2. *Given* δ, R *and* w; *to find the value of* p.

Substitute the numerical values of these letters instead of them in equation (*a*), and it becomes

$$6.2832\times 84\times p=2\times 7389.0432,$$

divide both sides of this equation by the product 6.2832×84, and we get

$$p=\frac{7389\cdot 0432}{6\cdot 2832\times 42}=28 \text{lbs.}$$

3. *Given* δ, p *and* w; *to find the value of* R.

Substitute the numbers corresponding to these letters in the general equation, and it becomes

$$6.2832\times 28\times R=2\times 7389\cdot 0432,$$

let both sides of this equation be divided by 6.2832×28, and it is

$$R=\frac{7389.0432}{6.2832\times 14}=84 \text{ inches, or 7 feet.}$$

4. *Given* p, R *and* w; *to find the value of* δ.

Substitute the numerical values of the given letters, as in the preceding cases, and equation (*a*) becomes

$$7389.0432\delta=6.2832\times 84\times 28,$$

divide both sides of the equation by $7389\cdot 0432$, and it is

$$\delta=\frac{6\cdot 2832\times 84\times 28}{7389\cdot 0432}=2 \text{ inches.}$$

These four cases, exhibit all the varieties of questions that can be proposed respecting the screw in the manner we have considered it; that is, on the supposition of its being an instrument combining the principles of the lever and the inclined plane, which is the obvious characteristic of the simple screw and its kindred nut; but there is another arrangement of the screw of very frequent occurrence in practice, and which is much more powerful than the one we have considered, we mean *the endless or perpetual screw;* which, instead of urging forward or pressing backward a nut as represented in the preceding diagram, is turned by a handle or winch, and has its spiral thread acting on the teeth of a wheel, round the axle of which the rope is wound that elevates the weight.

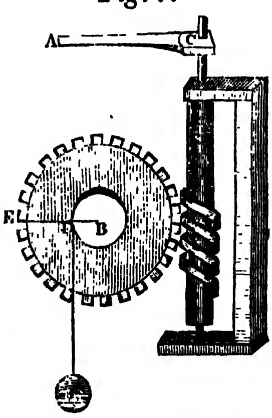
Section Second.

OF THE ENDLESS OR PERPETUAL SCREW.

12. A combination of this kind is represented in the annexed figure, where it is evident that the mechanical efficacy of the screw, must be augmented in proportion to the number of teeth in the wheel on which it acts.

In addition to the notation which we employed in establishing the theory of the simple screw,

Fig. 7.



Let d=BD, the radius of the axle on which the rope is coiled,
D=BE, the radius of the wheel on which the screw acts,
W=the weight or resistance to be kept in equilibrio,
and R=AC, the radius of the circle described by the power, the same as in the case of the simple screw.

Then by equation (b) we have seen, that the effect of the power p in maintaining the equilibrium on the screw, is expressed by

$$w=\frac{6.2832\text{R}p}{\delta},$$

and the same equation expresses the magnitude of the power, which operates on the circumference of the wheel, as transmitted from the handle of the winch by means of the screw; but by the property of the wheel and axle elsewhere announced, we have

$$d\text{W}=\frac{6.2832\text{DR}p}{\delta};$$

consequently, the equilibrated expression for the perpetual screw, or, as it is sometimes called, the endless screw, when combined with the wheel and axle, becomes

$$d\delta\text{W} = 6.2832\text{DR}p. \qquad (f)$$

This equation implies, that

When a power sustains a weight in equilibrio by means of a screw in combination with the wheel and axle; the power is to the weight, as the pitch, or distance between the threads of the screw drawn into the radius of the axle, is to the circumference described by the power drawn into the radius of the wheel.

The complete analysis of this equation is more extensive than that of equation (a), for the simple screw; because, here there are six quantities to be determined, all of which may, in their turn, be considered variable, and consequently, six separate equations will be required to exhibit the solution in its full extent; and the method of obtaining these equations will become manifest from the resolution of the following problems.

13. PROBLEM 5. *In the equation* $d\delta\text{W} \pm 6.2832\text{DR}p$ *there are given,* d, δ, D, R *and* p, *to find the value of* W.

Let W be disengaged from the quantities with which it is combined, by division, and we shall obtain

$$\text{W} = \frac{6.2832\text{DR}p}{\delta d}. \qquad (g)$$

Therefore the practical rule which this equation affords is as follows.

RULE. *Multiply all together, the radius of the wheel, the radius of the circle described by the power, the magnitude of the power, and the constant number* 6.2832; *then divide the product by the radius of the axle drawn into the distance between the threads of the screw, and the quotient will give the magnitude of the weight sought.*

EXAMPLE 1. What weight will be sustained in equilibrio by a power equivalent to 56lbs. acting on a winch of 2 feet radius, the radius of the axle on which the rope is coiled being 6 inches, that of the wheel 12 inches, and the pitch or distance between the threads of the screw half an inch?

Here we have given, $d=6$; $\text{D}=12$; $\delta=\frac{1}{2}$; $p=56$, and $\text{R}=24$; let these numerical values of the several letters be substituted in equation (δ), and it becomes

$$\text{W} = \frac{6.2832 \times 12 \times 24 \times 56}{6 \times \frac{1}{2}},$$

which, by performing the multiplication and dividing by 3, gives

$$\text{W} = 33778.4832\text{lbs}.$$

The effect of the screw alone, as determined from the equation marked (b), is

$$w=6.2832\times24\times56\times2=16889.2416\text{lbs.},$$

and this being applied at the circumference of the wheel, operates as a power to sustain the weight in equilibrio; but the radius of the wheel in the present instance is double the radius of the axle; hence, the effect of the wheel and axle is to double the energy of the screw.

EXAMPLE 2. In elevating a block of marble to the summit of a building, by means of an endless screw combined with the wheel and axle, it required a force of 286 lbs. applied to the handle of the screw to maintain the block in a state of quiescence; what was its weight, the radius of the wheel being 24 inches, that of the axle 8 inches, that of the winch or handle of the screw 30 inches, and the distance between the threads of the screw one inch?

In this example there are given, $d=8$; $D=24$; $\delta=1$; $p=286$, and $R=30$; consequently, by equation (g), or the rule derived from it, we have

$$W=\frac{6.2832\times24\times30\times286}{8\times1},$$

which, by actually performing the multiplication and dividing by 8, gives

$$W=161729.568\text{lbs. or } 72.2 \text{ tons very nearly.}$$

14. PROBLEM 6. *In the equation $d\delta W=6.2832DRp$, there are given d, δ, D, R and W; to find the value of p.*

Let both sides of the given equation be divided by $6.2832DR$, and we shall obtain

$$p=\frac{d\delta W}{6.2832DR}. \qquad (h)$$

And the practical rule derived from this equation is as follows.

RULE. *Multiply together the radius of the axle, the distance between the threads of the screw and the magnitude of the given weight; then divide the product by the radius of the wheel drawn into the circumference of the circle described by the power, and the quotient will give the magnitude of the power required.*

EXAMPLE 1. What power will be sufficient to balance a weight of 56 tons, by means of an endless screw in combination with the wheel and axle, the radius of the axle being 12 inches, that of the wheel 56 inches, the radius of the circle described by the power 28 inches, and the distance between the threads of the screw $\frac{3}{4}$ of an inch?

Here we have given, $d=12$; $D=56$; $\delta=\frac{3}{4}$; $R=28$, and $W=56$ tons; consequently, by equation (h), or the rule derived from it, we have

$$p=\frac{12\times\frac{3}{4}\times 56}{6.2832\times 56\times 28},$$

which, by actually performing the process, gives

$p=0.05115$ tons, or 112 lbs. very nearly.

EXAMPLE 2. A mass of granite, weighing 258 cwt. is just balanced by a power acting on an endless screw in combination with the wheel and axle; what is the magnitude of the power, the radius of the axle being 4 inches, that of the wheel 18 inches, the radius of the winch or handle of the screw 20 inches, and the pitch, or distance between the threads, half an inch?

In this example there are given, $d=4$; $D=18$; $\delta=\frac{1}{2}$; $R=20$, and $W=258$ cwt.; consequently, by the rule derived from equation (h), we have

$$p=\frac{4\times\frac{1}{2}\times 258}{6.2832\times 18\times 20},$$

which, by performing the process, gives

$p=0.228$ cwt. or 25.536 lbs very nearly.

15. PROBLEM 7. *In the equation $d\delta W=6.2832DRp$, there are given d, δ, D, W and p; to find the value of R.*

Let both sides of the equation be divided by $6.2832Dp$, and we shall obtain

$$R=\frac{d\delta W}{6.2832Dp}. \qquad (i)$$

The practical rule afforded by this equation is as follows.

RULE. *Multiply together the radius of the axle, the distance between the threads of the screw, and the magnitude of the given weight; then divide the product by the radius of the wheel, drawn into 6.2832 times the given power, and the quotient will give the radius of the winch required.*

EXAMPLE 1. Required the radius of the winch, in a combination of the endless screw with the wheel and axle, such, that a power of 5 lbs. may balance a weight of 3368 pounds; the radius of the axle being 2 inches, that of the wheel 14 inches, and the pitch, or distance between the threads of the screw, $\frac{1}{4}$ of an inch?

Here we have given, $d=2$; $D=14$; $\delta=\frac{1}{4}$; $p=5$, and $W=3368$ lbs.; consequently, by equation (i), or the rule derived from it, we shall have

$$R=\frac{2\times\frac{1}{4}\times 3368}{6.2832\times 14\times 5},$$

which, by performing the process, gives

$R=3.82$ inches, very nearly.

Example 2. In a combination of the endless screw with the wheel and axle, what must be the radius of the winch, so that a man pressing with a force equivalent to 150 lbs. shall sustain a load of 7896 lbs., the radius of the axle being $9\frac{1}{2}$ inches, that of the wheel 20 inches, and the distance between the threads of the screw $\frac{3}{4}$ of an inch?

Here by the rule, we have

$$\text{R}=\frac{9\frac{1}{2}\times\frac{3}{4}\times 7896}{6\cdot 2832\times 20\times 150}.$$

and by performing the process, we get

$$\text{R}=2\cdot 98 \text{ inches very nearly.}$$

16. Problem 8. *In the equation $d\delta\text{W}=6\cdot 2832\text{DR}p$, there are given, d, δ, p, R and W, to find the value of D.*

Let both sides of the equation be divided by $6\cdot 2832\text{R}p$, and we shall obtain

$$\text{D}=\frac{d\delta\text{W}}{6.2832\text{R}p}. \qquad (k)$$

And the practical rule afforded by this equation, is as follows.

Rule. *Multiply together, the radius of the axle, the distance between the threads of the screw, and the magnitude of the given weight; then divide the product by the radius of the winch, drawn into 6·2832 times the given power, and the quotient will give the radius of the wheel required.*

Example 1. A power equivalent to 10 lbs. acting on an endless screw combined with the wheel and axle, is found to balance a load of 36398 lb.; what would be the radius of the wheel, when that of the axle is 4 inches, the radius of the circle described by the power 12 inches, and the pitch, or distance between the threads of the screw $\frac{1}{2}$ an inch?

In this example there are given, $d=4$; $\delta=\frac{1}{2}$; $\text{R}=12$; $p=10$, and $\text{W}=36398$ lbs.; consequently, by equation (k), or the rule derived from it, we have

$$\text{D}=\frac{4\times\frac{1}{2}\times 36398}{6.2832\times 12\times 10},$$

let the process here indicated, be actually performed, and we get

$$\text{D}=96\cdot 54 \text{ inches,}$$

or something more than 8 feet and half an inch.

Example 2. A power of 156 lbs. being applied to the handle of an endless screw, combined with the wheel and axle, is found to equipoise a load of 67200 lbs. or 30 tons; what is the radius of the wheel, when the radius of the axle is 3 inches, that of the winch 36 inches, and the distance between the threads of the screw one inch?

Here by the rule to equation (k), we have

$$D=\frac{3\times 1\times 67200}{6\cdot 2832\times 36\times 156},$$

or performing the process, we obtain

$$D=\frac{67200}{11762\cdot 1504}=5\tfrac{3}{4}\text{ inches, very nearly.}$$

17. Problem 9. *In the equation* $d\delta W=6\cdot 2832DRp$, *there are given,* δ, D, p, R *and* W; *to find the value of* d.

Let both sides of this equation be divided by δW, the factors with which the required term is combined, and we shall obtain

$$d=\frac{6\cdot 2832DRp}{\delta W}. \qquad (l)$$

The practical rule which this equation affords, is as follows.

Rule. *Multiply all together, the radius of the wheel, the radius of the circle described by the power, the magnitude of the power, and the constant number* 6.2832; *then, divide the product by the weight, drawn into the distance between the threads of the screw, and the quotient will give the radius of the axle required.*

Example 1. A power equivalent to 28lbs., acting on an endless screw, in combination with the wheel and axle, is found to balance a weight of 12684lbs.; what is the radius of the axle, that of the wheel being 18 inches, of the winch 12 inches, and the distance between the threads of the screw one inch and a half?

Here we have given, $\delta=1\frac{1}{2}$; D$=18$; R$=12$; $p=28$, and W$=12684$lbs.; consequently, by equation (l), or the rule derived from it, we have

$$d=\frac{6.2832\times 18\times 12\times 28}{1\frac{1}{2}\times 12684},$$

or by performing the process, we shall have

$$d=\frac{3166\cdot 7328}{1585\cdot 5}=1\cdot 997\text{ or very nearly 2 inches.}$$

Example 2. If a weight of 8000lbs. is held in equilibrio, by a force equivalent to 6lbs. acting on an endless screw in combination with the wheel and axle: what is the radius of the axle, supposing that of the wheel to be 24 inches, the radius of the winch 16 inches, and the pitch, or distance between the threads of the screw 2 inches?

Here by the rule to equation (l), we have

$$d=\frac{6.2832\times 24\times 16\times 6}{2\times 8000},$$

let the process here indicated be actually performed, and we get

$$d=\frac{113.0976}{125}=09047\text{ inches.}$$

18. PROBLEM 10. *In the equation $d\delta W=6.2832DRp$, there are given, d, D, R, p and W; to find the value of δ.*

Let both sides of the equation be divided by dW, the factors which constitute the coefficient of the required term, and we shall obtain

$$\delta=\frac{6.2832DRp}{dW}. \qquad (m)$$

And the practical rule which this equation affords is as follows.

RULE. *Multiply together, the radius of the wheel, the radius of the circle described by the power, the magnitude of the power, and the constant number 6.2832; then, divide the product by the weight drawn into the radius of the axle, and the quotient will give the distance between the threads of the screw.*

EXAMPLE 1. If a weight of 2240lbs. is held in equilibrio, by a power of 9lbs. acting on an endless screw in combination with the wheel and axle; what is the pitch, or distance between the threads of the screw, the radius of the axle being 2 inches, that of the wheel 8 inches, and the radius of the circle described by the power 6 inches?

Here we have given, $d=2$; D$=8$; R$=6$; $p=9$, and W$=2240$ lbs.; consequently, by equation (m), or the rule derived from it, we have

$$\delta=\frac{6\cdot2832\times8\times6\times9}{2\times2240},$$

which, by performing the process, gives

$$\delta=\frac{21\cdot2058}{35}=0\cdot6058 \text{ inches.}$$

EXAMPLE 2. A power equivalent to 200lbs. acting on an endless screw, in combination with the wheel and axle, is found to equipoise a weight of 6000lbs.; what is the pitch, or distance between the threads of the screw, the radius of the axle being 8 inches, that of the wheel 10 inches, and the radius of the circle described by the power 14 inches?

Here by the rule to equation (m), we have

$$\delta=\frac{6.2832\times10\times14\times200}{8\times6000}$$

or by performing the process, we shall have

$$\delta=\frac{10.9956}{3}=3.6652 \text{ inches, for the distance}$$

between the threads of the screw.

19. In the developement of the foregoing theory relative to the endless, or perpetual screw, considered in combination with the principles of the wheel and axle, we have given a separate analysis for each quantity that enters into the composition of the original equation, and from the resulting formula to each analysis, we have

deduced a rule in words at length, by which the examples under the several problems are resolved.

We have adopted this plan with the view of accommodating the subject to the capacities of such of our readers, as are wholly unacquainted with the reduction of algebraic equations; but the theory may be established in a more concise and equally comprehensive manner, by substituting the several given quantities for their representatives in the original equation, by which means, a new equation will be obtained, involving only one unknown letter, combined with the numerical values of all the others; and from this equation, the value of the required term may be found by performing such a process as the composition of the derivative formula indicates.

We pursued this course in establishing the theory of the simple screw, and for the sake of uniformity, we proceed in a similar manner, to unfold the doctrine of the endless screw, considered in connexion with the wheel and axle; for which purpose we shall take the following

20. EXAMPLE. Let us suppose that $d=4$ inches, $D=10$ inches, $\delta=2$ inches, $R=16$ inches, $p=28$lbs., and $W=3518\cdot692$lbs.; then it is manifest from the nature of equations, that of these six quantities, any five being given, the sixth can easily be found.

1. *To find the value of* W, *in terms of the rest.*

Let the given numerical values of the several quantities, be substituted in equation *(f)*, and it becomes

$$8W=6.2832\times10\times16\times28,$$

or, dividing both sides of the equation by 8, we get

$$W=6.2832\times560=3518.592\text{lbs.}$$

2. *To find the value of* p, *the other quantities being given.*

Substitute the numerical values of the given quantities instead of them, in equation *(f)*, and it becomes

$$6\cdot2832\times10\times16\times p=4\times2\times3518\cdot592,$$

or dividing by 10×16, we obtain

$$6.2832p=175.9296,$$

and finally, dividing by 6.2832, we shall have

$$p=\frac{175\cdot9296}{6\cdot2832}=28\text{lbs.}$$

3. *To find the value of* R, *the other quantities being given.*

Let the given numerical values, be substituted for the respective letters in equation *(f)*, and it becomes

$$6\cdot2832\times10\times28\times R=4\times2\times3518\cdot592,$$

or, dividing by 10×28, we obtain

$$6\cdot2832R=100\cdot5312,$$

and finally, dividing by $6\cdot2832$, we shall have

$$R=\frac{100\cdot5312}{6\cdot2832}=16 \text{ inches.}$$

4. *To find the value of* D, *the other quantities being given.*

Substitute the given numerical values for their representatives in equation *(f)*, and it becomes

$$6.2832 \times 16 \times 28 \times D = 4 \times 2 \times 3518.592,$$

or, dividing by 16×28, we obtain

$$6.2832 D = 62.832,$$

and finally, dividing by 6.2832, we get

$$D = \frac{62.832}{6.2832} = 10 \text{ inches.}$$

5. *To find the value of d, the other quantities being given.*

Let the given numerical values of the several quantities be substituted in equation *(f)*, and it becomes

$$2 \times 3518.592 \times d = 6.2832 \times 10 \times 16 \times 28,$$

or, dividing by 3518·592, we obtain

$$2d = 8,$$

and finally, dividing by 2, we have

$$d = \frac{8}{2} = 4 \text{ inches.}$$

6. *To find the value of* δ, *the other quantities being given.*

Substitute the given numerical values of the quantities, for their representatives in equation *(f)*, and it becomes

$$4 \times 3518.592 \times \delta = 6.2832 \times 10 \times 16 \times 28,$$

or, dividing by 3518·592, we obtain

$$4 = 8,$$

and finally, dividing by 4, we get

$$\delta = \frac{8}{4} = 2 \text{ inches for the pitch, or distance}$$

between the threads of the screw.

21. Such then is the theory of the screw considered as a mechanical instrument, whether it be used in combination with the wheel and axle, for the purpose of elevating or sustaining vast loads, or simply in connexion with its nut for producing intense pressure; and from the equation of equilibrium in either case, it is manifest, that the less is the value of δ, or the distance between the threads of the screw, the greater is the efficacy of the engine; but it has already been observed, that beyond a certain limit, the distance between the contiguous threads cannot be decreased, for then, the instrument would become too slender and delicate for withstanding the resistance to which it may happen to be exposed, and consequently, in a practical point of view, it would be rendered entirely useless.

If, on the other hand, a proper distance between the threads of the screw be maintained, so as to give sufficient strength to the machine, and it be required to produce a very high effect, it is obvious, that the leverage of the power must be increased for that

purpose, but if it be increased beyond a certain extent, the power will have to travel through an inconveniently large space, and besides, the machine itself will be rendered very unmanageable, by reason of the great length of leverage that may be requisite for attaining a sufficient power.

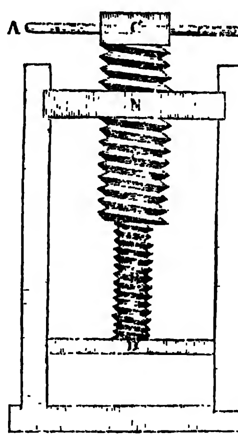
In order therefore, to produce a competent degree of mechanical energy, while at the same time, the parts of the instrument are preserved of a substantial and commodious form, we must have recourse to some other contrivance, such as that represented in the annexed diagram, which, on account of the minuteness and delicacy of its operation, we have thought proper to designate *the differential screw.*

Section Third.

OF THE DIFFERENTIAL SCREW.

22. Let N be a fixed nut or interior screw, through which the screw S is made to pass, by means of the lever AC, at the extremity of which the power is applied, and where it operates to produce an equilibrium or motion in the machine.

Fig. 8.



The screw S, is a hollow cylinder having a spiral groove cut on its concave surface, and fitted to receive the screw B, the thread of which is finer than that of the screw S, operating in the nut at N.

Now, if the cylinder B, on which the thread of a screw is raised, be supposed fixed into the board D in such a manner, that it is prevented from turning on its axis, but suffered to rise and fall in the direction of its length; then it is manifest, that when the power acting at A, the extremity of the lever AC, has made one complete revolution, the screw S has moved downwards through a space equal to the distance between two contiguous threads, while the screw B has moved in the contrary direction, through a space also equal to the distance between two of its contiguous threads; but because the threads of the screw B, are finer and closer than those of the screw S, the effect of this compound motion, during one revolution of the power, will obviously be, to depress the board D, through a space equal to the difference between the distances of the threads of the two screws S and B.

Put n = the number of threads in one inch of the moveable screw S, and

$n+1$ = the number of threads in an equal space of the screw B;

then [illegible] and [illegible] the respective distances between two contiguous threads of the screws S and B [illegible] consequently,

$$\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)},$$

is the space through which the board D is moved, during a complete revolution of the power.

Hence it appears, that the mechanical efficacy of this machine, is the same as that of the simple screw, when the distance between the threads is represented by the quantity $\frac{1}{n(n+1)}$, or which is the same thing, when

$$\delta=\frac{1}{n(n+1)}.$$

If therefore, instead of δ in equation (a), we substitute its equivalent as obtained above, the equilibrated expression for the differential screw becomes

$$\frac{w}{n(n+1)}=6.2832\text{R}p. \qquad (n)$$

This equation implies, that

When a power sustains a weight in equilibrio, or balances a certain resistance, by means of the differential screw, constructed as represented in the diagram; the power is to the resistance, as the difference between the distances of the threads of the two screws, to the circumference of the circle described by the power.

This enunciation expressed in symbolical language, is as follows; viz.

$$p : w :: \frac{1}{n(n+1)} : 6.2832\text{R},$$

where the term $\frac{1}{n(n+1)}$ denotes the difference of the distances between the threads of the screws, and 6.2832R the circumference of the circle described by the power.

The following problems will suffice to exhibit the analysis of equation (n), and the examples under each problem, will amply illustrate the mode of their application.

23. PROBLEM 11. *In the equation* $\frac{w}{n(n+1)}=6\cdot2832\text{R}p$, *there are given n,* R *and p, to find the value of w.*

Let both sides of equation (n), be multiplied by the denominator $n(n+1)$, and the power of the screw will be expressed as follows; that is,

$$w=6\cdot2832\text{R}pn(n+1). \qquad (o)$$

And the practical rule deduced from this equation, may be expressed in words in the following manner.

RULE. *To the number of threads in one inch of the revolving screw add unity; then, multiply together, the sum, the number of threads in one inch of the revolving screw, the magnitude of the power, and the circumference of*

the circle described by the power; then, the product arising from the continued multiplication, will express the mechanical efficacy of the screw, or the intensity of pressure which it can generate.

EXAMPLE 1. A power equivalent to 4 lbs. acting on a lever of 16 inches in length, is found to balance a certain weight, or produce a certain pressure on a body placed under the board D, as in the diagram; what is the weight of the body so balanced, or the intensity of the pressure produced, there being 8 threads in one inch of the moveable screw?

Here, by the rule to equation (o), we have

$$8(8+1)=8\times 9=72, \text{ and}$$

the circumference of the circle described by the power $\Big\} = 6\cdot 2832\times 16 = 100.5312$;

consequently, we obtain $w = 100.5312\times 4\times 72 = 28952.9856$ lbs. for the weight of the body, or the measure of the pressure produced.

A simple screw having the distance between the threads $\frac{1}{72}$th part of an inch, would produce the same effect theoretically, but practically it would not, for the extreme fineness of the threads would render the instrument unfit to resist the pressure which the power produces.

EXAMPLE 2. What pressure will be created in the differential screw, by a power of 84 lbs. acting at the extremity of a lever 4 inches long, there being 30 threads in one inch in length of the moveable screw?

Here, by the rule to equation (o), we have

$$30(30+1)=30\times 31=930, \text{ and}$$

the circumference of the circle described by the power $\Big\} = 6\cdot 2832\times 4 = 25\cdot 1328$;

consequently, we obtain $w = 25.1328\times 84\times 930 = 1963374\cdot 336$ lbs. or $876\frac{1}{2}$ tons very nearly, for the measure of the pressure required.

24. PROBLEM 12. *In the equation* $\frac{w}{n(n+1)} = 6.2832\text{RP}$, *there are given,* n, R *and* w; *to find the value of* p.

Divide both sides of the equation by 6.2832R, the circumference of the circle described by the power, and we shall have

$$p = \frac{w}{6\cdot 2832\text{R}n(n+1)}, \qquad (p)$$

The practical rule afforded by this equation, may be expressed as follows.

RULE. *To the number of threads in one inch of the revolving screw add unity, and multiply together, the sum, the number of threads in one inch of the revolving screw, and the circumference of the circle described by the power; then, divide the weight of the body, or the pres-*

sure to be produced, by the continued product of the three factors, and the quotient will express the magnitude of the required power.

EXAMPLE 1. What power acting at the extremity of a lever 6 inches long, will equipoise a weight of 864lbs. attached to the board at D, there being 6 threads in one inch of the revolving screw?

In this example there are given, $n=6$; $R=6$, and $w=864$; consequently, by equation *(p)*, or the rule derived from it, we have

$$6(6+1)=6\times7=42, \text{ and}$$

the circumference of the circle described by the power $\left.\right\} =6.2832\times6=37.6992$;

consequently, we obtain $p=\dfrac{864}{37.6992\times42}=0.545$lbs., or very nearly $8\frac{3}{4}$ ounces, for the power required.

EXAMPLE 2. A pressure equivalent to 4480lbs. is produced by means of a power acting at the extremity of a lever 12 inches long; what is the magnitude of the power, supposing that there are 12 threads in one inch in length of the moveable screw?

Here we have given, $n=12$; $R=12$, and $w=4480$; consequently, by equation *(p)*, or the rule derived from it, we have

$$12(12+1)=12\times13=156, \text{ and}$$

the circumference of the circle described by the power $\left.\right\} =6.2832\times12=75.3984$;

consequently, we obtain $p=\dfrac{4480}{75.3984\times156}=0.38$lbs. or $6\frac{1}{10}$ ounces, very nearly, for the magnitude of the power required.

25. PROBLEM 13. *In the equation* $\frac{w}{n(n+1)}=6.2832Rp$, *there are given, n, p and w, to find the value of* R.

Let both sides of the equation be divided by $6.2832p$, and we shall obtain

$$R=\frac{w}{6.2832pn(n+1)}. \qquad (q)$$

And the practical rule afforded by this equation may be expressed as follows.

RULE. *To the number of threads in one inch of the revolving screw add unity, and multiply together the sum, the number of threads in one inch of the revolving screw, the magnitude of the power, and the constant number* 6.2832; *then divide the weight of the body, or the pressure to be produced by the product of the factors, and the quotient will give the radius of the circle described by the power.*

EXAMPLE 1. A power of 2lbs. acting at the extremity of a lever fixed into the head of a screw, as represented in the diagram, is found to balance a weight of 9000lbs. attached to the board

at D; what is the length of the lever, supposing there are 15 threads in one inch in length of the moveable screw?

In this example there are given, $n=15$; $p=2$, and $w=9000$ lbs.; consequently, by equation (q), or the rule derived from it, we have

$$15(15+1)=15\times 16=240,$$
$$\text{and } 6\cdot 2832\times 2=12.5664;$$

consequently, we obtain by division

$$R=\frac{9000}{12.5664\times 240}=2.984 \text{ inches, for the radius}$$

of the circle described by the power.

EXAMPLE 2. A pressure equivalent to a weight of 125620 lbs. is produced by a power of 8 lbs. acting at the extremity of a lever fixed into the head of the screw, in the manner represented in the diagram; what is the length of the lever, supposing the number of threads in one inch in length of the moveable screw to be 23?

Here we have given, $n=23$; $p=8$, and $w=125620$ lbs.; consequently, by equation (q), or the rule derived from it, we have

$$23(23+1)=23\times 24=552,$$
$$\text{and } 6.2832\times 8=50.2656;$$

consequently, by division we obtain

$$R=\frac{125620}{50.2656\times 552}=4.528 \text{ inches, for the radius}$$

of the circle described by the power.

20. PROBLEM 14. *In the equation* $\frac{w}{n(n+1)}=6.2832Rp$, *there are given,* R, p *and* w; *to find the value of* n.

Reciprocate the terms of the equation, and we get

$$\frac{n^2+n}{w}=\frac{1}{6.2832Rp},$$

multiply both sides of the equation by w, and it becomes

$$n^2+n=\frac{w}{6.2832Rp},$$

complete the square, and we have

$$n^2+n+\tfrac{1}{4}=\frac{w}{6.2832Rp}+\tfrac{1}{4},$$

extract the square root, and it is

$$n+\tfrac{1}{2}=\tfrac{1}{2}\sqrt{\frac{4w}{6.2832Rp}+1},$$

transpose, and we finally obtain

$$n=\tfrac{1}{2}\left\{\sqrt{\frac{4w}{6.2832Rp}+1}-1\right\}. \qquad (r)$$

And the practical rule derived from this equation may be expressed in words at length in the following manner.

RULE. *Divide four times the given weight or resistance by the magnitude of the power drawn into the circumference*

of the circle which it describes, and add one to the quotient; then, from the square root of the sum subtract unity, and half the remainder will give the number of threads in one inch in length of the moveable screw.

EXAMPLE 1. A power equivalent to 7lbs. acting at the extremity of a lever 8 inches long, is found to balance a weight of 2428lbs.; how many threads are there in one inch of the revolving screw?

Here we have given, R=8; $p=7$, and $w=2428$lbs.; consequently, by equation (r), or the rule derived from it, we have

$$4\times 2428=9718,$$

$$\text{and } 6.2832\times 8\times 7=351.8592;$$

therefore, we have

$$\sqrt{\frac{9718}{351.8592}+1}=\sqrt{28.619}=5.349;$$

consequently, by subtraction and division, we get

$n=\frac{1}{2}(5.349-1)=2.1745$ threads to an inch in length of the revolving screw.

EXAMPLE 2. A pressure equivalent to 21 tons, or 47040lbs. is produced by a power of 4lbs. acting at the extremity of a lever 18 inches long fixed in the head of a screw, as represented in the diagram; how many threads are there in one inch in length of the revolving screw, or that which is turned by the lever?

In this example there are given, R=18; $p=4$, and $w=47040$lbs.; consequently, by equation (r), or the rule derived from it, we have

$$4\times 47040=188160,$$

$$\text{and } 6\cdot 2832\times 18\times 4=452\cdot 3904;$$

therefore, we have

$$\sqrt{\frac{188160}{452\cdot 3904}+1}=\sqrt{416\cdot 92}=20\cdot 42;$$

consequently, by subtraction and division, we get

$n=\frac{1}{2}(20\cdot 42-1)=9\cdot 71$ threads to an inch in length of the revolving screw, and $9\cdot 71+1=10\cdot 71$ threads in one inch of the screw B.

Having thus determined the value of each quantity, which enters into the composition of the original equation, in terms of the rest, and illustrated the application of the resulting formula by appropriate examples under each problem, we proceed in the next place, to show in what manner the theory may be abbreviated, by substituting the given quantities in the equilibrated equation, and disengaging the required term, by such an operation as the arrangement of the quantities so substituted may indicate.

27. EXAMPLE. Let us in this case put $n=6$; $p=4$; R=8, and $w=8444\cdot 6208$; then it is manifest, from the nature of the subject under consideration, that, if any three of these quantities are given, together with the constant number, the value of the fourth quantity can easily be assigned. The process for each is briefly as follows.

1. *To find the value of w, the other quantities being given.*

Substitute the numerical values of the given quantities instead of them in equation (n), and it becomes

$$\frac{w}{6(6+1)}=6\cdot2832\times8\times4,$$

or, multiplying both sides by $6(6+1)$, we get

$$w=6\cdot2832\times32\times42=8444\cdot6208\text{lbs.}$$

2. *To find the value of p, the other quantities being given.*

Let the given numerical values of the quantities be substituted for the respective letters in equation (n), and it becomes

$$6\cdot2832\times8\times p=\frac{8444\cdot6208}{6(6+1)}$$

or dividing by 8, we obtain

$$6\cdot2832p=25\cdot1328,$$

and finally, dividing by $6\cdot2832$, we shall have

$$p=\frac{25\cdot1328}{6\cdot2832}=4\text{lbs.}$$

3. *To determine the value of* R, *the other quantities being given.*

Let the numerical values of the given quantities be substituted for their representatives in equation (n), and it becomes

$$6\cdot2832\times4\times\text{R}=\frac{8444\cdot6208}{6(6+1)}$$

or dividing by 4, we obtain

$$6\cdot2832\text{R}=50\cdot2656,$$

and finally, dividing by $6\cdot2832$, we shall have

$$\text{R}=\frac{50\cdot2656}{6\cdot2832}=8\text{ inches.}$$

4. *To find the value of n, the other quantities being given.*

Substitute the numerical values of the given quantities for their representatives in equation (n), and it becomes

$$\frac{8444\cdot6208}{n(n+1)}=6\cdot2832\times8\times4,$$

or, by reciprocating the members of the equation, we have

$$\frac{n^2+n}{8444\cdot6208}=\frac{1}{201\cdot0624},$$

and multiplying both sides of the equation by $8444\cdot6208$, we get

$$n^2+n=\frac{8444\cdot6208}{201\cdot0624},$$

or, by actually dividing, it is

$$n^2+n=42,$$

complete the square, and we have

$$n^2+n+\tfrac{1}{4}=42\tfrac{1}{4},$$

extract the square root, and it is

$$n+\tfrac{1}{2}=6\tfrac{1}{2},$$

transpose, and we finally obtain

$n=6$, the number of threads in one inch in length of the revolving screw.

28. Having thus unfolded the theory of the screw, and exemplified its application by numerous examples, under the three following forms, viz.

1. The simple screw, or the inclined plane combined with the lever,
2. The endless screw, combined with the wheel and axle,
3. The differential or double screw, invented by Mr. Hunter, and applied by him to the admeasurement and subdivision of very minute spaces; it therefore only remains, to place before the eye of the reader the theory in a contracted form, or, which is the same thing, the equations of equilibrium from whence the whole doctrine has been drawn; they are as follows.

1. For the simple screw, $\delta w = 6\cdot 2832Rp$. (a)
2. For the endless screw, $d\delta w = 6.2832DRp$. (f)
3. For the differential screw, $\dfrac{w}{n(n+1)} = 6\cdot 2832Rp$. (n)

And these three equations involve every particular that can be proposed respecting the screw, both in its simplest and most complicated applications, but it must be kept in mind, that in the foregoing theory no notice has been taken of the effects of friction, which in the screw is very great, and if taken into the account would render the subject extremely intricate. It must therefore be understood, that our results have been obtained on the supposition, that the operation of the screw is not affected by friction, nor influenced by the form of the spiral thread which communicates the action; indeed, the theory throughout is wholly independent of this latter consideration, the form of the spiral having nothing to do with the mechanical effect.

6. OF THE WEDGE.

INTRODUCTION.

In our definition of the wedge we have made remarks which this introduction ought not to anticipate. We shall, therefore, in this notice direct the attention of our readers to a few leading facts, which may serve as the preliminaries of the subsequent mathematical discussion. And first we have to observe that in order to determine the conditions of equilibrium in the theory of the wedge, we are compelled to abandon all idea of the percussive force employed in the practical application of this implement, and to consider the power which is engaged to overcome the resistance, precisely as if it were a passive force. With this limitation the theory of the wedge is susceptible of a very rigid demonstration. It is well known, however, that in practice, the potential force we employ is the blow of a hammer, the action of which is percussion, while the resistance to be overcome is a pressive force, whether it be the tenacity of wood or the adhesive mass of a granite rock. We can understand the vast difference which exists between the modifications of pressive and percussive forces, and the impossibility of establishing the conditions of equilibrium of a machine in which the weight or resistance is a force of one, and the power applied to overcome it a force of another species. In other words, one of the bodies is moveable and the the other at rest. And owing to this circumstance, we presume, exists the difference to be found in the rules given by some mechanical writers for determining the power of the wedge.

Playfair thinks the best way of considering the subject is to resolve all the forces that act upon the wedge into parts parallel to two axes, at right angles to one another, and one of them parallel to the base of the wedge on the side to which the power is applied. In the case of equilibrium the opposite forces, in the direction of these axes must be equal between themselves. Though it be perfectly true that in practice, pressure is never used in moving the wedge home to rend a tree or a rock, yet we consider it as necessary to determine the relation which subsists between the resistances on the side of the wedge, and their counteracting power on the back, as to investigate the theory of the inclined

plane, which is nothing more than a rectangular wedge in which one of the planes containing the right angle coincides with the horizon, to which the other is perpendicular. We have, however, treated each of these mechanical powers as a separate machine.

In looking at a man splitting a tree with wedges, it must occur to the most inexperienced in mathematical knowledge, that the blow of the hammer or mallet moves through the whole extent of the wedge, while the resistance yields to its breadth. Hence the common rule, which is not however perfectly correct, that the power will be to the resistance, if one of the bodies is moveable, as the back of the wedge to its length. And the more acute the angle, the greater the power of the instrument, or the smaller the force required to overcome a given resistance. In other words, the harder the substance to be divided, the greater will be the angle of the wedge. It is on this principle that chisels for cutting soft woods are thinner or sharper than those used for the harder species; and these again are sharper and thinner than chisels employed for cutting metals. Though the common wedge be generally used for cleaving and severing cohesive masses, it may sometimes be employed very advantageously in raising great weights to a small height.

There is no instrument more common or more useful than the wedge, which we see employed by shipcarpenters, quarriers &c., who having great resistances to overcome, use heavy hammers to accomplish their purpose of driving it home. And the only instance perhaps in which the wedge is used for the purposes of equilibrium, is in the construction of arches which are built of truncated wedges. To form an idea of these, we have only to describe two concentric circles, and divide the circumference of the outer into any number of equal parts, from each of which to the centre as we draw lines, we shall form so many isosceles wedges, from which if we cut off the portions between the inner circle and the centre, we shall form a circular arch with truncated wedges.

The theory of equilibrium of the wedge which we here present, branches into two sections: one, under the conditions of figure and force, which we have assigned to the machine: the other to particular cases that flow immediately from the general theory; and we have concluded our article with a series of equilibrative equations, the illustration of which we have left for exercise to the reader.

Having in the first section expounded the theorem that established the general equation (c) of equilibrium for the wedge in the form of a triangular prism, we have drawn out in words a rule for ascertaining the magnitude of the power, and this rule we have illustrated by appropriate examples, which we have resolved graphically and numerically.

When, however, instead of the segments of the vertical angle the sides of the wedge themselves are given, the general equation

assumes a different form, and becomes an eligible formula for investigating the case of oblique action, that is to say, when all the forces are supposed to act obliquely on the sides to which they are respectively applied. The equation in this case we have divested of its algebraic trappings and expressed by a general rule in words. To this rule we have subjoined a couple of examples, which we have resolved geometrically and numerically, thus furnishing all the illustration which the reader can desire.

These two theorems with their rules and examples complete the first section of our subject: the second proceeds with the conditions of equilibrium in the particular cases that flow from the general theory; and the first theorem that we investigate, involves the conditions of equilibrium in terms of the forces, their angles of direction and the angles of the wedge.

To exemplify the application of the formula when composed of these data, we have thrown the equation into a general rule, illustrated by two examples that have been resolved graphically and numerically. In the second theorem of this section, the equation expresses the conditions of equilibrium in terms of the forces, their angles of direction and the sides of the wedge; and the rule which applies generally in the case of an isosceles prism, as deduced from this equation, is illustrated, like the former, with two examples accompanied by their graphical and numerical solutions.

The graphical solutions we may be allowed to consider as desiderata in mechanical science, as they enable the practical mechanic to verify his calculations; nor is there any sound or stable mode of proceeding but this, if we would desire to place at the reader's disposal all the materials upon which our own knowledge of the subject is built.

OF THE WEDGE.

SECTION FIRST.

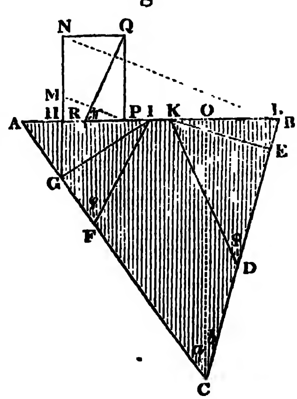
Definitions and Conditions of Equilibrium.

1. THE WEDGE is a solid piece of wood or metal, generally made in form of a triangular prism, of which the two ends or bases are equal and similar plane triangles, and the three sides rectangular parallelograms; and it is called *rectangular*, *isosceles*, or *scalene*, according as its equal and similar bases are composed of right angled, isosceles, or scalene triangles.

As a mechanical power, the wedge performs its office, sometimes in raising heavy bodies, but more frequently in dividing or cleaving them; hence, all those instruments which are used in separating the parts of bodies, such as axes, adzes, knives, swords, coulters, chisels, planes, saws, files, nails, spades, &c. are only different modifications that fall under the general denomination of the wedge; but these instruments are made in so many shapes, and operated on by forces applied in such a variety of ways, that of all the mechanical powers, the action of the wedge is the most difficult of being brought under a strict mathematical calculation; indeed, in a general point of view this is probably impossible, but in the particular case of a triangular prism according to our definition, whose sides are perfectly hard, inflexible and smooth, if the resistances on the back and sides of the wedge can be considered as pressive forces acting in assignable directions, the conditions of equilibrium, or the relation that subsists between the resistances on the sides of the wedge, and their counteracting power on the back, may be determined in the following manner.

2. Let ABC represent a transverse section, perpendicular to the axis of the triangular prism of which the wedge is composed, and suppose the sides AB, AC and BC to be perfectly smooth, hard, and inflexible.

Fig. 1.



Then, let a power P represented in magnitude and direction by the line RQ, be supposed to act at the point R on AB the back of the wedge, and conceive it to be counteracted by two other forces or resistances R and r, whose magnitude and directions are respectively represented by the straight lines FI and DK, acting on the sides AC and BC at the points F and D.

Resolve the force FI into its equivalent component forces IG and GF, respectively perpendicular and parallel to the side AC; then, since AC is supposed to be perfectly smooth, the

force IG, which is perpendicular to AC, only operates to prevent the progress of the wedge, the part GF being parallel has no effect.

Again, resolve the effective force IG into its equivalent component forces IH and GH, respectively parallel and perpendicular to AB the back of the wedge; then is GH the only portion of the resistance R, which is directly opposed to that part of the power P, acting perpendicularly on AB the back of the wedge, in the point P.

In like manner, resolve the force DK into its equivalent component forces KE and DE, respectively perpendicular and parallel to the side BC; then, since BC is supposed to be perfectly smooth, the force KE, which is perpendicular to BC, is only effectual in opposing the progress of the wedge, the force ED acting in the direction of the side BC produces no effect whatever in resisting the power.

Next, resolve the effective force KE into its equivalent component forces KL and EL, respectively parallel and perpendicular to AB the back of the wedge; then is EL that part of the resistance r, which directly opposes the power P when reduced to a direction perpendicular to AB.

Finally, resolve the power RQ into its equivalent component powers PQ and RP, respectively perpendicular and parallel to AB the back of the wedge; then PQ, which is perpendicular to AB, is the only part of the power P that becomes effectual in counterbalancing the reduced resistances GH and EL.

3. Now, it has been shown in treating of parallel forces acting in the same plane, that

> *If three parallel forces acting perpendicularly upon a right line keep it in equilibrio, one of them will act in a direction opposite to the other two, and it will be equal to their sum, dividing the line of application into two parts, which are to one another reciprocally as the magnitudes of its opposing forces.*

Consequently, if we suppose every part of the wedge to be perfectly smooth, hard and inflexible, an equilibrium will obtain between the power P, and the two resistances R and r, when

$$PQ = GH + EL. \qquad (a)$$

That is, when the magnitudes of the two resistances reduced to a direction perpendicular to the back of the wedge are conjointly equivalent to their equipoising force reduced in an opposite direction.

4. It must however be observed, that the above equation of equilibrium will not hold, unless the resistances R and r, or rather their reduced equivalents GH and EL, balance themselves about the point P considered as the fulcrum of a straight inflexible lever, for otherwise there will manifestly be a tendency to a vibratory or rotatory motion about that point.

Therefore, to determine the position of the point P, so that the equilibrium may be maintained, and this tendency to a vibratory or rotatory motion prevented; produce GH to the point N, making HM and MN respectively equal to EL and GH, the reduced equivalents of the resistances r and R; join NL, and through the point M draw MP parallel to NL cutting AB the back of the wedge in the point P; then is P the point where the reduced equivalent of the power p must be applied, in order that the foregoing equilibrated equation may obtain, and that the resistances R and r, acting at the points F and D in the directions FI and DK, may have no tendency to produce a vibratory or rotatory motion in the wedge of which ABC is a section.

Through the point P thus determined draw the straight line PQ parallel to HN, and through the point N draw NQ parallel to AB the back of the wedge; then is PQ the reduced equivalent of the power P, which, operating at the point P, balances the united energies of the reduced resistances GH and EL acting at the points H and L.

5. To determine the respective magnitudes of the reduced resistances GH and EL, in terms of the given resistances and their angles of direction, together with the angles at the vertex of the wedge, made by a straight line CO in the plane of the section, drawn from C the vertex perpendicular to AB the back of the wedge.

Let R$=$FI, the magnitude of the resistance, acting at the point F on the side AC,

$r=$DK, the magnitude of the resistance, acting at the point F on the side BC,

P$=$RQ, the magnitude of the power, acting at the point R on the back AB, and balancing the joint effects of the resistances R and r, acting on the sides AC and BC,

$p=$PQ, the reduced equivalent of the power P, acting perpendicularly at the point P on the back of the wedge, and balancing the united effects of the reduced resistances GH and EL, acting at the points H and L,

$a=$ACO, the angle of the vertex of the wedge, contained between the altitude CO, and the side AC,

$b=$BCO, the angle at the vertex of the wedge, contained between the altitude CO, and the side BC,

$\varphi=$AFI, the angle contained between the side AC, and FI, the direction of the resistance R,

$\varphi'=$BDK, the angle contained between the side BC, and DK, the direction of the resistance r.

Then, by Plane Trigonometry, we have

rad. : sin. φ :: FI : GI,

rad. : sin. φ' :: DK : EK.

Therefore, by taking the radius equal to unity, and equating the products of the extreme and mean terms, we obtain

$$\text{GI} = \text{FI} \sin.\varphi,$$
$$\text{EK} = \text{DK} \sin.\varphi'.$$

But because the triangles AGI and BEK are respectively similar to the triangles AOC and BOC, we have, by using the above values of GI and EK,

$$\text{rad.} : \sin. a :: \text{FI} \sin.\varphi : \text{GH},$$
$$\text{rad.} : \sin. b :: \text{DK} \sin.\varphi' : \text{EL}.$$

Therefore, by taking radius as before, equal to unity, and equating the products of the extreme and mean terms, we shall have

$$\text{GH} = \text{FI} \sin. a \sin.\varphi,$$
$$\text{EL} = \text{DK} \sin. b \sin.\varphi'.$$

Consequently, by introducing the literal representatives of FI and DK, and adding the equations, we obtain

$$\text{GH} + \text{EL} = \text{R} \sin. a \sin.\varphi + r \sin. b \sin.\varphi';$$

that is, from the equation (a),

$$p = \text{R} \sin. a \sin.\varphi + r \sin. b \sin.\varphi'. \qquad (b)$$

6. This equation has been obtained on the supposition, that the power acts in a direction perpendicular to the back of the wedge, while the resistances R and r may be conceived to act in any degree of obliquity; it may however happen, that the power also acts in an oblique direction, such as QR, in which case it must be reduced to its perpendicular equivalent as represented by PQ, and then the equation of equilibrium will become

$$\text{P} \sin.\pi = \text{R} \sin. a \sin.\varphi + r \sin. b \sin.\varphi'. \qquad (c)$$

Where π is the angle which AB the back of the wedge makes with QR the direction of the power.

The equation as it now stands, is general for the wedge in form of a triangular prism, when its sides are considered to be perfectly smooth, hard and inflexible, and when both the resistances acting on the sides, and the counterbalancing power acting on the back, are conceived to be pressive forces, exerting their energies in directions that are assignable; under any other conditions the equations will not hold; it would therefore be useless to attempt its developement on the consideration of other forms, or when subjected to the influence of other forces.

7. If both sides of the general equation of equilibrium be divided by $\sin.\pi$, we shall obtain for the magnitude of the power

$$\text{P} = \{\text{R} \sin. a \sin.\varphi + r \sin. b \sin.\varphi'\} \operatorname{cosec}.\pi. \qquad (d)$$

And the rule which will apply generally in this case may be expressed in words at length in the following manner.

RULE. *Multiply the magnitude of each resistance separately by the sine of the angle of its direction, and again by the sine of that segment of the vertical angle adjacent to the*

side of the wedge on which the respective resistance acts; then multiply the sum of the products by the cosecant of the angle which the direction of the power makes with the back of the wedge, and the result will express the magnitude of the power required.

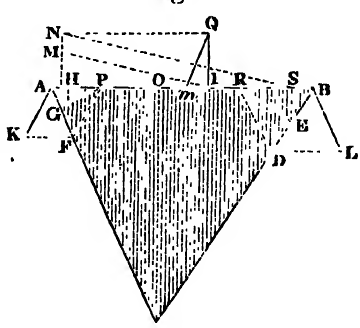
EXAMPLE 1. Two resistances, respectively equivalent to pressures of 30 and 40 tons, acting in angles of 52 and 60 degrees on the sides of a perfectly smooth wedge, are sustained in equilibrio by a certain power acting on its back in an angle of 68 degrees; what is the magnitude of the power, supposing the segments of the vertical angle made by a perpendicular on the back of the wedge, to be respectively equal to 25 and 33 degrees?

The graphical solution of this question is as below.

Let AC and BC represent the sides of the wedge, considered as being perfectly smooth, hard and inflexible, of which AB represents the back, and in which the angle ACO and BCO, are respectively equal to 25 and 33 degrees.

Then, since it can make no difference as regards the magnitude of the opposing power, at what point of the sides the resistances may be applied, we shall consider them as acting at the points A and B, immediately under the shoulders of the wedge, these points being the most convenient for the purpose of construction.

Fig. 2.



At the points A and B make the angles FAK and DBL respectively equal to 52 and 60 degrees, the angles which the directions of the resistances make with the sides of the wedge; take AK and BL proportional to the numbers 30 and 40 from a scale of equal parts of any dimensions whatever, these numbers denoting the magnitude of the resistances R and *r*; then, through the points K and L, draw the straight lines KF and LD parallel to AB the back of the wedge, and meeting the sides AC and BC in the points F and D.

Through the points F and D, draw FP and DR parallel to AK and BL, meeting AB the back of the wedge in the points P and R; then shall FP and DR also denote the magnitudes of the resistances R and *r*, being respectively equal and parallel to AK and BL.

Resolve the forces FP and DR, into their equivalent component forces PG, FG and RE, DE respectively perpendicular and parallel to AC and BC, the sides of the wedge; and again, resolve the effective forces PG and RE, into their equivalent components GH, PH and ES, RS respectively perpendicular and parallel to AB, the

back of the wedge; then are GH and ES, the parts of the resistances that become effectual in opposing the power P, when reduced to a parallel but opposite direction, and it is manifest, that in the case of an equilibrium, the reduced power must be equal to their sum; that is,

$$P \sin. \pi = p = GH + ES.$$

And moreover, to prevent a tendency to a vibratory or rotatory motion, the reduced power p must be applied at the point I, so situated in the line HS, that

$$HI : IS :: ES : GH.$$

Therefore, to determine the position of the point I; produce GH to N, making HM equal to ES and MN equal to GH; join NS, and through the point M, draw MI parallel to NS and meeting AB in I the required point.

Through the point I determined as above, draw IQ parallel and equal to HN, the sum of the reduced resistances, and at the point Q make the angle IQm equal to 22 degrees, the compliment of the angle which the direction of the power makes with the line AB; then are mQ and IQ, respectively equal to the magnitudes of P and p the required power and its reduced equivalent.

Numerical Calculation. In calculating examples of this nature, where so many angular magnitudes enter, it is better to employ logarithms than to perform the process arithmetically, because the number of decimal places that occur in the several angular factors, renders the entire operation exceedingly laborious; we shall therefore, in order to avoid a multiplicity of tedious and prolix computations, perform the process logarithmically, in the following manner.

Resistance R	$=30$ . . .	log. . .	1.477121
Direction of R	$=52°$ . . .	log. sin. .	9.896532
Seg. vert. angle	$=25°$ . . .	log. sin. .	9.625948
Nat. number	$=9.991$ . .	log. . .	0.999601
Resistance r	$=40$ . . .	log. . .	1.602060
Direction of r	$=60°$ . . .	log. sin. .	9.937531
Seg. vert. angle	$=33°$ . . .	log. sin. .	9.736109
Nat. number	$=18.866$.	log. . .	1.275700

Here then, the magnitude of the resistance R, when reduced to a direction perpendicular to the back of the wedge, is

$$GH = 30 \sin. 25° \sin. 52° = 9 \cdot 991 \text{ tons};$$

and the magnitude of the resistance r, when reduced to the same direction, is

$$ES = 40 \sin. 33° \sin. 60° = 18 \cdot 866 \text{ tons};$$

but by equation (b), we have shown, that the sum of the reduced resistances, is equal to the magnitude of the reduced power; that is,

$$p = 9 \cdot 991 + 18 \cdot 866 = 28 \cdot 857 \text{ tons}:$$

consequently, by equation (d), or the rule derived from it, we obtain

$$P=28\cdot857\times1\cdot078=31\cdot12 \text{ tons.}$$

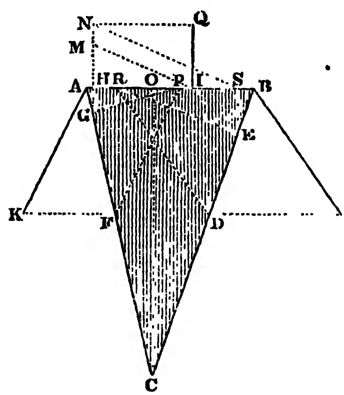
EXAMPLE 2. Two resistances, respectively equivalent to 382 and 428lbs., acting on the sides of a perfectly smooth wedge in angles of 40° 36′ and 56° 20′, are balanced by a power acting at right angles to the back of the wedge; what is the magnitude of the power, supposing the segments of the vertical angle made by a perpendicular on the back of the wedge, to be respectively equal to 13 and 20 degrees?

In this example, the power is supposed to act perpendicularly on the back of the wedge; consequently, cosec. ϖ in equation (d), becomes equal to unity, and the question is resolved by the expression represented in equation (b), or in other words, the solution dispenses with the last precept in the rule derived from equation (d), this renders the computation somewhat simpler, but the principles of construction are nearly the same, as will become manifest from the following operation.

Graphical construction and solution.

At the points A and B, the extremities of the back of the wedge, make the angles FAK and DBL equal respectively to 40° 36′ and 56° 20′, the angles which the directions of the resistances make with AC and BC, the sides of the wedge; take AK and BL proportional to the numbers 382 and 428, from a scale of equal parts of any convenient dimensions, these numbers representing the magnitudes, or expressing the measures of the resistances R and r.

Fig. 3.



Through the points K and L, draw the straight lines KF and LD parallel to AB the back of the wedge meeting the sides AC and BC in the points F and D; again, through the points F and D just determined, draw FP and DR parallels to AK and BL meeting AB the back of the wedge in the points P and R; then shall the lines FP and DR, also represent the magnitudes of the resistances R and r, being respectively equal and parallel to AK and BL.

Resolve the forces FP and DR into their equivalent component forces PG, FG and RE, DE respectively perpendicular and parallel to AC and BC the sides of the wedge.

Again, resolve the forces PG and RE into their equivalent component forces GH, PH and ES, RS respectively perpendicular and parallel to the line AB; then are GH and ES, the parts of the resist-

ances R and r, that become effectual in opposing the power and maintaining the equilibrium.

Produce GH to N, making HM equal to ES and MN to GH; join NS, and through the point M draw MI parallel to NS, meeting AB the back of the wedge in the point I; then is I the point where the power must act to preserve the stability of the wedge, or to prevent a vibratory or rotatory motion about the point of application, which would, under the same circumstances, obviously take place in any other position.

Through the point thus determined, draw the straight line IQ equal and parallel to HN the sum of the reduced resistances; then shall IQ represent the magnitude of the power required, which being taken in the compasses and applied to a scale of equal parts, will indicate $177\frac{3}{4}$ very nearly.

The logarithmic operation is as follows.

Resistance R	$=382$ . . .	log. . .	2.582063
Direction of R	$=40^\circ\,36'$ . .	log. sin. .	9.813430
Seg. vert. angle	$=13^\circ\,0'$ . .	log. sin. .	9.352088
Nat. number	$=55.92$ . .	log. . .	1.747581
Resistance r	$=428$ . . .	log. . .	2.631444
Direction of r	$=56^\circ\,20'$.	log. sin. .	9.920268
Seg. vert. angle	$=20^\circ\,0'$ . .	log. sin. .	9.534052
Nat. number	$=121.83$.	log. . .	2.085764

From the first of the preceding steps, it appears, that the magnitude of the resistance R, when reduced to a direction perpendicular to the back of the wedge, is

$$\text{GH} = 382 \sin. 40^\circ\,36' \sin. 13^\circ = 55.92 \text{ lbs},$$

but its original oblique magnitude is 382lbs.; consequently, the direct opposition to the power is very little more than oneseventh part of the oblique resistance, a circumstance which should remind the mechanic, of the vast advantage to be gained, from a direct application of power.

From the second step, we find that the magnitude of the resistance r, when reduced to a direction perpendicular to the back of the wedge, is

$$\text{ES} = 428 \sin. 56^\circ\,20' \sin. 20^\circ = 121 \cdot 83 \text{lbs}.$$

consequently, by equation (b), we have

$$p = 55 \cdot 92 + 121 \cdot 83 = 177 \cdot 75 \text{ lbs}.$$

9. These two questions exemplify the method of ascertaining the magnitude, or the intensity of the power, in terms of the resistances and the angles of their direction, together with the segments of the vertical angle, made by a perpendicular on the back of the wedge; but it may very frequently happen, that instead of the segments of the vertical angle, the sides of the wedge themselves may be given, in which case, the general equation will assume a different form; we shall therefore, in the next place,

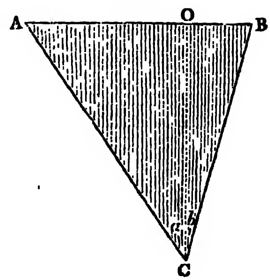
endeavour to express the conditions of equilibrium, in terms of the resistances R and r, their angles of direction φ and φ', together with the sides and the back of the wedge; for which purpose.

Put $\beta=$ AB, the back of the wedge, on which the power acts.

$d=$ AC, the side of the wedge, on which the resistance R is applied.

$\delta=$ BC, the side of the wedge operated on by the resistance r.

Fig. 4.



Then, by a well known theorem in Plane Trigonometry, we have

$$\beta : d+\delta :: d \backsim \delta : \text{AO} \backsim \text{BO},$$

which, by equating the products of the mean and extreme terms, becomes

$$\beta(\text{AO} \backsim \text{BO})=(d+\delta)\,(d \backsim \delta),$$

But the product of the sum and difference of any two quantities is equal to the difference of their squares; therefore, and by division, we have for the difference of the segments AO and BO, as follows, viz.

$$\text{AO} \backsim \text{BO}=\frac{d^2 \backsim \delta^2}{\beta}.$$

Now, if half the sum of any two quantities be increased and diminished by half their difference, the sum in the one case, and the difference in the other will give respectively the greater and lesser quantity; therefore, if d be supposed greater than δ, we shall have

$$\left.\begin{aligned} \text{AO}&=\frac{\beta^2+d^2-\delta^2}{2\beta}. \\ \text{BO}&=\frac{\beta^2+\delta^2-d^2}{2\beta}. \end{aligned}\right\} \qquad (c)$$

Consequently, by Plane Trigonometry, we get

$$d : \frac{\beta^2+d^2-\delta^2}{2\beta} :: \text{rad.} : \sin. a.$$

$$\delta : \frac{\beta^2+\delta^2-d^2}{2\beta} :: \text{rad.} : \sin. b.$$

From which, by taking radius equal to unity, and equating the products of the extremes and means, we have

$$\sin. a=\frac{\beta^2+d^2-\delta^2}{2\beta d},$$

$$\sin. b=\frac{\beta^2+\delta^2-d^2}{2\beta\delta}.$$

Let these values of sin. a and sin. b be substituted for them in equation (b), and it becomes

$$p=\text{R} \sin. \varphi \left\{\frac{\beta^2+d^2-\delta^2}{2\beta d}\right\}+r \sin. \varphi' \left\{\frac{\beta^2+\delta^2-d^2}{2\beta\delta}\right\},$$

which, by reducing the substituted fractions, and multiplying both sides of the equation by $2\beta d\delta$, the common denominator, reduces to

$$2\beta\delta dp = \delta R \sin.\varphi(\beta^2+d^2-\delta^2)+dr\sin.\varphi'(\beta^2+\delta^2-d^2). \quad (f)$$

This equation determines the magnitude of the power, when it is supposed to act perpendicularly to the back of the wedge; but in the case of oblique action, it becomes transformed into

$$2\beta d\delta P = \{\delta R\sin.\varphi(\beta^2+d^2-\delta^2)+dr\sin.\varphi'(\beta^2+\delta^2-d^2)\}\operatorname{cosec}.\pi. \quad (g)$$

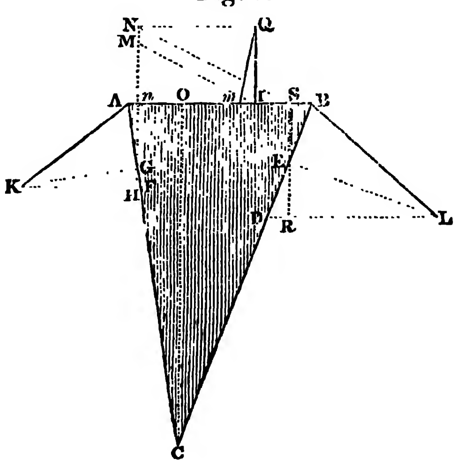
Where, as we have before observed, π is the angle which the direction of the power makes with AB, the back of the wedge.

10. This last is the general form of the equation in terms of the forces, their angles of direction, and the sides of the wedge, when all the forces are supposed to act obliquely on the sides to which they are respectively applied; consequently, for a triangular wedge under these circumstances, the general rule in words may be expressed as follows.

RULE. *Multiply the magnitude of each resistance by the sine of the angle of its direction, and again by the remote side of the wedge, or that on which the opposite resistance acts; then multiply the result corresponding to each resistance by the adjacent segment of the back of the wedge, made by a perpendicular from the vertical angle; add the products, and multiply the sum by the cosecant of the angle, which the direction of the power makes with the back of the wedge; and finally, divide this last product by that of the two sides, for the magnitude of the power required.*

EXAMPLE 1. Two resistances, whose magnitudes are respectively represented by 36 and 48 tons, acting at angles of 60 and 70 degrees on the sides of a wedge supposed to be perfectly hard and smooth, are sustained in equilibrio by a power acting on the back in an angle of 80 degrees; what is the magnitude of the power, supposing the sides of the wedge to be $9\frac{1}{2}$ and 10 inches respectively, and its back 5 inches?

Fig. 5.



The geometrical construction of this question is effected in the following manner.

Construct the triangle ABC to represent a transverse section of the triangular prism perpendicular to the axis, and having the

sides AB, AC and BC respectively equal to 5, $9\frac{1}{2}$ and 10 inches, as specified in the question.

At the points A and B, or at any other convenient positions on the sides AC and BC, make the angles FAK and DBL, equal respectively to 60 and 70 degrees, being the angles at which the directions of the resistances R and r, or the lines AK and BL which represent them, are inclined to the sides AC and BC.

Make AK and BL respectively proportional to the numbers 36 and 48, taken from a scale of equal parts of any convenient magnitude; through the points K and L draw the straight lines KF and LD parallel to AB; and KG and LE perpendiculars to AC and BC, the sides of the wedge; and moreover, from the points G and E let fall GH and ER perpendiculars on KF and LD, and produce HG and RE to meet AB in the points n and s, the points where the reduced resistances GH and ER are conceived to act in opposition to the power.

Produce Hn to N, making nM equal to ER and MN to GH; join NS, and through the point M draw MI parallel to NS, meeting AB the back of the wedge in the point I; then is I the point where the reduced power must be applied, in order to counterbalance the resistances GH and ER, which are conceived to act at the points n and s, in parallel but opposite directions to the power which is applied at the point I.

Through the point I determined as above, draw IQ parallel and equal to nN the sum of the reduced resistances; then will IQ represent the magnitude of the reduced power.

At the point Q make the angle IQm equal to the complement of the angle which the direction of the power makes with AB the back of the wedge; then is mQ the magnitude of the original or oblique power, and m the point at which it is applied; consequently, the three lines AK, BL and mQ represent the magnitudes and directions of the three forces R, r and P, whose united but opposite energies maintain the wedge in a state of rest.

We now proceed to the calculation, and, as we observed in the solution of the foregoing examples, it will be preferable to employ the logarithms, at least in so far as the continued simple factors in the several members of the general equation will admit.

The logarithmic operation is as follows.

Resistance R	$=36$ . . .	log. .	1.556303
Direction of R	$=60°$ . . .	log. sin.	9.937531
Value of the side δ	$=10$ . . .	log. .	1.000000
Nat. number	$=311.77$.	log. .	2.493834
Resistance r	$=46$ . . .	log. .	1.681241
Direction of r	$=70°$ . . .	log. sin.	9.972986
Value of the side d	$=9\frac{1}{2}$ . . .	log. .	0.977724
Nat. number	$=428.5$ . .	log. .	2.631951

We have next to determine the numerical values of the segments AO and BO, made by the perpendicular CO from the vertical angle C; these segments are determined directly by the equations marked (e), but for the sake of a little variety, we shall here endeavour, by means of algebra, to discover the values indicated in these equations. Thus,

Put x = the segment AO,
and y = the segment BO;

then, retaining the numerical values of the sides β, d and δ, we have

$$x+y=5,$$

and by the property of the right angled triangle, we get

$$90.25-x^2=100-y^2,$$

which by transposition becomes

$$y^2-x^2=9.75,$$

divide this last equation by the first, and we obtain

$$y-x=1.95;$$

consequently, we now have given

$$y+x=5,$$

and $y-x=1.95$;

therefore, by addition we have

$$2y=6.95,$$

and by subtraction it is

$$2x=3.05;$$

hence, by division, the values of the required segments are as below, viz.

$$\text{BO}=y=3.475,$$
$$\text{AO}=x=1.525.$$

Then, by proceeding according to the rule, we get

$$\text{GH}=\frac{311.77\times 1.525}{9.5\times 10}=5.005,$$

$$\text{ER}=\frac{428.5\times 3.475}{9.5\times 10}=15.674$$

therefore, by addition, IQ the magnitude of the reduced power is

$$p=5.005+15.674=20.679.$$

To find mQ, the magnitude of the oblique power, it is

sin. 80° : 20.679 :: rad. : 20.679 cosec. 80°,

but the natural cosecant of 80° is 1.015;

therefore, we finally obtain

$$\text{P}=20.679\times 1.015=21 \text{ tons very nearly.}$$

Such are the geometrical constructi . and the trigonometrical and analytical solutions, for determining the magnitude of the power, which acting in a given angle, shall keep in equilibrio two resistances whose magnitudes are given, together with the angles which the faces of the wedge make with its back.

Example 2. Two resistances, whose magnitudes are respectively equal to pressures of 400 and 300 lbs., acting at angles of 54° 30′

and 66° 10′, on the sides of a perfectly smooth wedge, are held in equilibrio by a power acting on the back in an angle of 51° 30′; what is the magnitude of the power, supposing the sides of the wedge to be 40 and 49 inches, its back being 18 inches?

In the solution of this example, we shall invert the order of the operation, by giving the numerical calculation before the graphical construction; we prefer this method in the present instance, in order that the geometrical process, which is a very singular one, may have its several steps verified by a comparison with the corresponding numerical results; this is the more necessary in the present case, as from the peculiar relation of the several data, the reduced directions of some, or probably of all the forces, may fall without the limits of the wedge; which circumstance may to many of our readers create a difficulty, unless the results can be compared with others which must inevitably be true.

The logarithmic operation is as follows.

Resistance R	$=400$	log.	2·602060
Direction of R	$=54° 30'$	log. sin.	9·910686
Value of the side δ	$=49$	log.	1·690196
Nat. number	$=15956·6$	log.	4·202942
Resistance r	$=300$	log.	2·477121
Direction of r	$=66° 10'$	log. sin.	9·961290
Value of the side d	$=40$	log.	1·602060
Nat. number	$=10976·7$	log.	4·040471

We have next to determine the numerical values of the segments of the back of the wedge, made by a perpendicular from the vertical angle; but it is manifest, from the relation that subsists between the back and the two sides, with respect to magnitude, that in this case the perpendicular will fall entirely without the wedge, and meet the production of its back in a direction beyond the shortest side; therefore,

Let $x=$ the extent of production, then will

$18+x=$ the distance from the remote extremity of the back to the point where the perpendicular falls;

therefore, by the property of the right angled triangle, we obtain

$$40^2-x^2=49^2-(18+x)^2,$$

which expression reduces to

$$1600-x^2=2077-36x-x^2;$$

consequently, by expunging x^2 and transposing, we have

$$36x=477,$$

from which, by division, we get

$$\text{AO}=x=13·25 \text{ inches},$$

$$\text{and BO}=13·25+18=31·25 \text{ inches}.$$

Then, proceeding according to the rule, the magnitudes of the resistances, when reduced to a direction perpendicular to the back of the wedge, are respectively as below.

$$\text{R reduced} = \frac{15956 \cdot 6 \times 13 \cdot 25}{49 \times 40} = 107 \cdot 88 \text{ nearly.}$$

$$r \text{ reduced} = \frac{10976 \cdot 7 \times 31 \cdot 25}{49 \times 40} = 175 \cdot 5.$$

Consequently, by addition, IQ the reduced power, is

$$p = 107 \cdot 88 + 174 \cdot 5 = 282 \cdot 38,$$

and the magnitude of the oblique power is

$$\text{P} = 282 \cdot 38 \times 1 \cdot 278 = 360 \cdot 88 \text{lbs.},$$

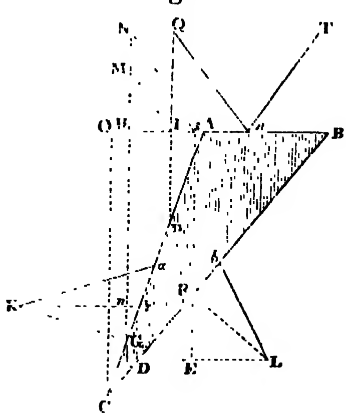
where the multiplier 1·278 is the natural cosecant of 51° 30′.

Geometrical Construction.

If the results obtained above be compared with those which the following construction affords, the coincidence of the measurements with the calculation, will have a tendency to confirm the truth of both.

Let ABC represent a transverse section, perpendicular to the axis of the triangular prism which constitutes the wedge.

Fig. 6.



Then, from a scale of equal parts of any dimensions at pleasure, the larger the better, take AB, AC and BC, equal respectively to 18, 40 and 49 inches.

In the sides AC and BC, take any points a and b, no matter how posited, and at the points a and b, make the angles Kac and Lbc, equal respectively to 54° 30′ and 66° 10′; then shall Ka and Lb represent the directions in which the resistances R and r are supposed to exert themselves.

Take aK and bL respectively equal, or proportional to the numbers 400 and 300, which numbers measure the magnitudes or intensities of the resistances R and r, acting at the points a and b on the side of the wedge.

Through the points K and L, draw KF and LD parallel to AB, meeting AC and BC, the sides of the wedge, in the points F and D, and from the same points K and L, let fall the perpendiculars KG and LR, also meeting AC and BC in the points G and R. From C the vertex of the triangle, let fall the perpendicular CO, meeting BA produced in O, and through the points G and R, draw GH and RS

parallels to CO, meeting KF and LD in the points n and E, and AB the back of the wedge produced, in the points H and S; then shall Gn and RE represent the reduced values of the resistances R and r, which being taken in the compasses, and applied to the same scale as aK and bL, will indicate respectively as below, viz.

$$Gn = 107.88,$$
$$\text{and } RE = 174.5,$$

corresponding with the results of calculation.

Produce GH to N, making HM and MN respectively equal to RE and Gn; join NS, and through the point M draw MI parallel to NS, meeting BA produced in I; then is I the point where the reduced power must be applied corresponding to the point P on the side of the wedge.

Make IQ equal and parallel to HN, the sum of the reduced resistances, and at Q make the angle IQm equal to the complement of the angle, which the direction of the power makes with the back of the wedge; then is mQ the magnitude of the oblique power P, which being taken in the compasses, and applied to the same scale as Gn and RE, will be found to indicate 360.88lbs., the same as was determined by calculation.

From an inspection of the figure however, it is manifest, that although the line mQ may correctly represent the magnitude of the oblique power, yet it by no means indicates the direction in which that power is supposed to act, especially when the resistances R, r are applied at the points a and b, so near to the face of the wedge; for in that case, the tendency of the power acting in the direction Qm, would evidently be to turn the wedge about a certain point as a centre, instead of urging it forward on the resistances pressing on its sides; but since, by the conditions of the question, its direction must be inclined to AB, the back of the wedge, in a given angle, its true effective direction may be assigned as below.

At the point m in the line AB, make the angle BmT equal to ImQ the given angle of direction, and make mT equal to mQ, the magnitude of the oblique power; then shall mT represent the direction in which the effort of the power must be exerted, and the three lines aK, bL and mT represent the magnitudes and directions of the three forces R, r and P, whose united but opposite energies exerted simultaneously, maintain the wedge in a state of quiescence.

Section Second.

PARTICULAR CASES OF EQUILIBRIUM FLOWING IMMEDIATELY FROM THE GENERAL THEORY.

11. What we have now done, is sufficient in our opinion for a general illustration of the principles of the wedge, under the conditions of figure and force which we have assigned to it; but there are several particular cases that flow immediately from the general

theory, to the developement and exemplification of which, our attention shall in the next place be directed.

If a and b, the segments of the vertical angle, made by a perpendicular from the vertex C on the back AB, be supposed equal to one another; then, by a well known geometrical property, the transverse section of the wedge ABC is isosceles, and the general equation of equilibrium (c) becomes

$$P \sin. \pi = \sin. a(R \sin. \varphi + r \sin. \varphi'). \qquad (h)$$

Where it must be understood, that a denotes half the vertical angle of the wedge.

If both sides of equation (h) be divided by sin. π, the coefficient of the power P, we shall obtain

$$P = \sin. a \operatorname{cosec.} \pi(R \sin. \varphi + r \sin. \varphi'). \qquad (i)$$

From this equation, the rule which applies generally in the case of an isosceles wedge, may easily be derived, and when enunciated in words at length, is read as follows.

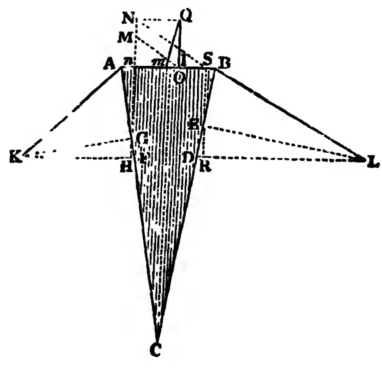
RULE. *Multiply the magnitude of each resistance separately, by the sine of the angle of its direction; then multiply the sum of the products by the sine of half the vertical angle, and again by the cosecant of the angle, which the direction of the power makes with the back of the wedge, and this last product shall be the magnitude of the power required.*

EXAMPLE 1. Two resistances, respectively equivalent to pressures of 19 and 25 tons, acting on the sides of a perfectly smooth isosceles wedge, in angles of 56 and 70 degrees, are balanced by a certain power acting on the back in an angle of 78 degrees; what is the magnitude of the counteracting power, supposing the angle at the vertex of the wedge to be 18 degrees?

Graphical Construction and Numerical Solution.

The principle of construction in this case being precisely the same as that which we have hitherto employed, renders it almost unnecessary to repeat the operation; but, that nothing may be wanting to place the subject in a clear and comprehensive light, we shall illustrate the whole of the examples by an appropriate construction.

Fig. 7.



Let ABC be a transverse section of the triangular prism constituting the wedge, of which the sides AC and BC are equal to one another.

At the points A and B, the

extremities of the section's base, make the angles KAC and LBC, respectively equal to 56 and 70 degrees, the given angles of direction.

Make the lines AK and BL equal or proportional to the numbers 19 and 25, being the respective measures of the resistances R and r, which are supposed to act on AC and BC, the sides of the wedge; through the points K and L, draw KF and LD parallels to AB, and let fall the perpendiculars KG and LE, meeting AC and BC in the points G and E.

From the points G and E, let fall the perpendiculars GH and ER, meeting the lines KF and LD in the points H and R; then are GH and ER the effective portions of the resistances R and r, when reduced to a direction perpendicular to AB the back of the wedge.

Produce HG and RE to meet AB in the points n and s, and let HG be continued beyond n, till nM equals ER and MN equals GH; join NS, and through the point M draw MI parallel to NS, meeting AB in I, the point where the reduced power must be applied to counteract the united resistances GH and ER.

Through the point I, draw IQ parallel and equal to nN, and at Q make the angle IQm equal to 12 degrees, the complement of the angle which the direction of the power makes with AB the back of the wedge; then is mQ the magnitude of the power required, and m the point at which it is applied. If mQ be taken in the compasses, and applied to a scale of equal parts, it will be found to measure 6.275 tons.

The logarithmic operation is as follows.

Resistance R $=19$	. . . .	log. ·	1.278754
Direction of R$=56°$	. . . .	log. sin.	9.918574
Nat. number $=15.75$	. . .	log. .	1.197328
Resistance r $=25$	. . . .	log. .	1.397940
Direction of $r=70°$	. . . .	log. sin.	9.972986
Nat. number $=23.49$	. . .	log. .	1.370926

Consequently, we have

KG$+$LE$=15\cdot75+23.49=39.24$	log. .	1.593729
Half the vertical angle $=9°$ . .	log. sin.	9.194332
Direction of the power P$=78°$ . .	log. cosec.	0.009596
Natural number $=6.275$	log. .	0.797657

Here, then, the magnitude of the oblique power mQ is 6.275 tons, and the magnitude of the reduced power IQ is

$$\text{GH}+\text{ER}=\text{IQ}=p=6.275 \sin. 78°=1.138 \text{ tons}.$$

The foregoing process has been conducted according to the rule, which the general equation (i) for the isosceles wedge affords; but the same result may be obtained directly from the diagram with the greatest facility, as follows.

From the vertex C let fall the perpendicular CO, bisecting AB in the point O; then, by reason of the parallel lines Hn, CO and RS, it is manifest, that the triangles KGF and LED are similar; consequently, we have

$$KG : GH :: LE : ER,$$

but by Plane Trigonometry it is

$$\text{rad.} : \sin. 56^\circ :: 19 : KG = 19 \sin. 56^\circ,$$

$$\text{and rad.} : \sin. 70^\circ :: 25 : LE = 25 \sin. 70^\circ,$$

and moreover

$$\text{rad.} : \sin. 9^\circ :: 19 \sin. 56^\circ : GH = 19 \sin. 9^\circ \sin. 56^\circ;$$

then, from the first of the above analogies, by employing the respective numerical values of the quantities, we obtain

$$19 \times .82904 : 19 \times 82904 \times .15643 :: 25 \times .93969 : ER,$$

which by reduction gives

$$ER = 25 \times .15643 \times .93969 = 3.675$$

$$GH = 19 \times .15643 \times .82904 = 2.464$$

wherefore, by addition, we have IQ or $p = \overline{6.139}$;

consequently, from the triangle mQI we get

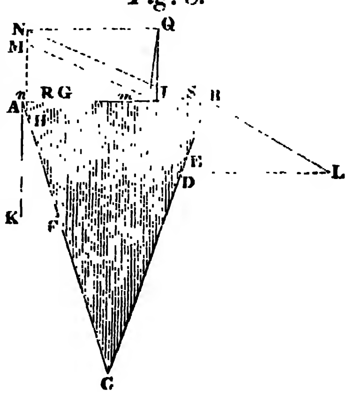
$$mQ = 6.139 \times \text{cosec. } 78^\circ = 6.275 \text{ tons, the same as before.}$$

Such are the graphical construction and numerical solution of a question, that proposes to find the magnitude of a power, which acting at a given angle shall counteract two resistances or pressures, acting on the sides of a wedge, whose inclinations to the back are elements of the proposition.

EXAMPLE 2. Two resistances, respectively equivalent to pressures of 649 and 844 lbs., acting on the sides of a smooth isosceles wedge, in angles of 20 and 79 degrees, are kept in equilibrio by a certain power acting on the back of the wedge in an angle of 84 degrees; what is the magnitude of the opposing power, the angle at the vertex of the wedge being 37 degrees?

The geometrical construction of this example, is effected in the following manner, viz.

Fig. 8.



Let ABC be a transverse section, pependicular to the axis of the triangular prism which constitutes the wedge; and let the sides AC and BC be equal, and inclined to each other in an angle of 37 degrees.

At the points A and B, the extremities of the line AB, make the angles FAK and DBL respectively equal to 20 and 79 degrees, the given angles of direction.

Make the lines AK and BL respectively equal, or proportional to the numbers 649 and 844, being

the numbers that express the measures of the resistances R and r, which are supposed to operate on AC and BC, the sides of the wedge. Through the points K and L, draw KF and LD parallels to AB, meeting AC and BC in the points F and D; through F and D, the points just found, draw FG and DR respectively parallel to AK and BL, meeting AB in the points G and R.

From the points G and R, let fall the perpendiculars GH and RE on the sides AC and BC, and from H and E, draw Hn and ES perpendicular to AB; then are Hn and ES the portions of the resistances R and r, that become effectual in opposing the power, when reduced to a direction perpendicular to the line AB.

Produce Hn to N, making nM equal to ES, and MN equal to Hn; join NS, and through M draw MI parallel to NS, meeting AB in the point I; then is I the point where the reduced power IQ or p must be applied, to produce the equilibrium, and to prevent the vibratory tendency that would take place, if the reduced power should be applied at any other point.

Through the point I, draw IQ parallel and equal to nN, the sum of the reduced resistances, and at the point Q, make the angle IQm equal to the compliment of the angle, which the direction of the power P makes with AB the back of the wedge; then is mQ the magnitude of the power required, and m the point where it is applied, when the resistances R and r are supposed to act at the points A and B.

If the resistances be conceived to be applied at any other points in the lines AC and BC, the position of the points I and m will vary accordingly, but the magnitudes of the powers represented by IQ and mQ will remain the same.

If the lines IQ and mQ be taken in the compasses, and applied to the same scale of equal parts, from which the resistances R and r, or the lines AK and BL were taken, their numerical values will be indicated as follows, viz.

$$\text{IQ} = 333.31 \text{ lbs.}$$
$$m\text{Q} = 335.2 \text{ lbs.}$$

The numerical computation of this question is as under.

Resistance R $=649$	. . . .	log. .	2.812245
Direction of R $=20°$	. . . .	log. sin.	9.534052
Nat. number $=221.97$	. . .	log. .	2.346297
Resistance r $=844$	. . . .	log. .	2.926342
Direction of $r=79°$	. . . .	log. sin.	9.991947
Nat. number $=828.49$	. . .	log. .	2.918289

Consequently, we have

GH and RE, respectively equal to 221·97 and 828·49; therefore,

221·97+828·49	=1050·46	.	log. .	3·021379
Half vert. angle	=18° 30′	.	log. sin.	9·501476
Direction of P	=84° . .	.	log. cosec.	0·002386
Nat. number	=335·15	.	log. .	2·525241

The magnitude of the oblique power $mQ=P$ being 335·15 lbs., its reduced equivalent IQ is

$$Hn+ES=p=335\cdot15 \sin. 84°=333\cdot31.$$

The calculation deduced directly from the construction, being similar to that exhibited for the last example, we think proper to omit it here, as the reader may very easily supply a solution from the figure, by attentively considering the steps of that which has already been performed.

12. If for sin. a, in the general equation of equilibrium for the isosceles wedge, we substitute its value $\frac{\beta}{2d}$, we shall obtain

$$2dP=\beta \text{ cosec. } \varpi(R \cos. \varphi+r \cos. \varphi'). \qquad (k)$$

The foregoing equation (i) involves the conditions of equilibrium in terms of the forces, their angles of direction, and the angles of the wedge; and these two examples have been resolved for the purpose of exemplifying the application of the formula, when composed of these data; but the present equation (k), expresses the conditions of equilibrium in terms of the forces, their angles of direction, and the sides of the wedge; and the rule which applies generally, in the case of an isosceles prism, as deduced from this equation, may be expressed in words at length, in the following manner.

RULE. *Multiply the magnitude of each resistance separately, by the sine of the angle of its direction; then add the products, and multiply the sum by the back of the wedge, and again by the cosecant of the angle, which the direction of the power makes with the back; then divide this last product by twice the side, for the magnitude of the power required.*

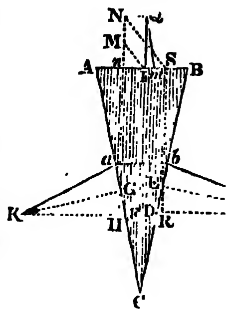
EXAMPLE 1. Two resistances, equivalent to pressures of 58 and 70 tons, acting on the sides of a smooth isosceles wedge, in angles of 74 and 80 degrees respectively, are balanced by a power acting on the back in an angle of 80 degrees; what is the magnitude of the power, the side of the wedge being 56 and its back 22 inches; the resistances being each applied at the distance of 24 inches from the shoulders of the wedge?

The graphical solution of this example is as follows.

Let ABC be a transverse section, perpendicular to the axis of the triangular prism which constitutes the wedge, and let the sides AC and BC be each equal to 56 inches, and AB equal to 22 inches.

Upon the equal sides AC and BC, set off A*a* and B*b*, each equal to 24 inches; then are *a* and *b* the points on the sides of the wedge, where the resistances R and *r* exert themselves.

Fig. 9.



At the points *a* and *b*, make the angles K*a*C and L*b*C, respectively equal to 74 and 80 degrees, the given angles of direction, and from a scale of equal parts, of any convenient dimensions, set off *a*K and *b*L, equal or proportional to the numbers 58 and 70, which numbers express the measures of the resistances applied at the points *a* and *b*.

Through the points K and L, draw the straight lines KF, KG and LD, LE, respectively parallel to AB and perpendicular to AC and BC; then are KG and LE the portions of the resistances R and *r*, that become effectual in stopping the progress of the wedge.

From the points G and E, let fall the perpendiculars GH and ER, meeting KF and LD in the points H and R; then are GH and EF the portions of the resistances R and *r*, which directly oppose the effects of the power, or that portion of it which operates in a direction perpendicular to the back of the wedge.

Produce HG and RE, to meet AB the back of the wedge in the points *n* and S, and let H*n* be continued directly forward, till *n*M is equal to ER and MN to GH; then is *n*N equal to that portion of the power which counteracts GH and ER, the reduced equivalents of the resistances.

Join NS, and through the point M, draw MI parallel to NS, meeting AB the back of the wedge in I, the point where the reduced power must be applied to maintain the equilibrium.

Through the point I draw IQ equal and parallel to *n*N, and at Q make the angle *m*QI equal to the complement of the angle, which the direction of the power P makes with the back of the wedge; then is *m*Q the magnitude of the required power, and *m* the point of application.

If *m*Q the oblique power, and IQ its reduced equivalent, be each taken in the compasses, and applied to the same scale of equal parts on which *a*K and *b*L were measured, they will be found to indicate as below, viz.

$$mQ = 24 \cdot 87$$
$$IQ = 24 \cdot 49.$$

The numerical operation for this example is as below.

Resistance R $=58$	. . . . .	log. .	1.763428
Direction of R $=74^\circ$	. . . . .	log. sin.	9.982842
Nat. number $=55.75$	. . . .	log. .	1.746270
Resistance r $=70$	. . . . .	log. .	1.845098
Direction of $r=80^\circ$	. . . . .	log. sin.	9.993351
Nat. number $=68.94$	. . . .	log. .	1.838449

Consequently, we have

KG + LE $=55.75+68.94=124.69$		log. .	2.095832
Value of the back β	$=22$ . .	log. .	1.342423
Direction of	P $=80^\circ$ . .	log. cosec.	0.006649
Value of twice	$d=112$ . .	ar. co. log.	7.950782
Natural number	$=24.87$.	log. .	1.395686

The magnitude of P, the oblique power being 24.87 as determined above; the magnitude of p its reduced equivalent is

$$\text{IQ}=p=24.87 \sin. 80^\circ=24.49 \text{ tons.}$$

In the example, since the positions of the points a and b, at which the resistances are applied, are precisely fixed, it is easy to assign the positions of the points I and m with respect to A and B, the extremities of the back; and since this has not been done in any of the preceding examples, we shall avail ourselves of the opportunity now offered, to show in what manner the point of application of the power is to be ascertained, when the points at which the resistances act are given.

PROBLEM.—*To show in what manner the point of application of the power is to be ascertained, when the points are given at which the resistances act.*

13. It is manifest from the preceding construction, and from the nature of the figure, that the triangles FKG, LED, GnA and ESB are similar among themselves; therefore we have

$$\text{KG} : \text{GH} :: \text{GA} : \text{A}n$$
$$\text{LE} : \text{ER} :: \text{EB} : \text{BS},$$
$$\text{but KG} : \text{GH} :: \text{AC} : \tfrac{1}{2}\text{AB},$$
$$\text{and LE} : \text{ER} :: \text{AC} : \tfrac{1}{2}\text{AB};$$

consequently, by equality of ratios, it is

$$d : \tfrac{1}{2}\beta :: \text{GA} : \text{A}n,$$
$$d : \tfrac{1}{2}\beta :: \text{EB} : \text{BS};$$

now GA $=$ Aa $+$ aG, and EB $=$ Bb $+$ bE,

but by Plane Trigonometry, we have

$$a\text{G}=\text{R} \cos. \varphi,$$
$$b\text{E}=r \cos \varphi';$$

therefore, by addition, we obtain

$$\text{GA} = \text{A}n + \text{R} \cos. \varphi,$$
$$\text{EB} = \text{B}b + r \cos. \varphi';$$

consequently, by restoring the numerical values of the several quantities, and substituting in the above proportions, we get

$$56 : 11 :: (24 + 58 \cos. 74^\circ) : \text{A}n,$$
$$56 : 11 :: (24 + 70 \cos. 80^\circ) : \text{BS},$$

and by equating the products of the extreme and mean terms, we obtain

$$56 . \text{A}n = 264 + 638 \cos. 74^\circ,$$
$$56 . \text{BS} = 264 + 770 \cos. 80^\circ;$$

but the natural cosine of 74° is 0.27564, and that of 80° is 0.17365; therefore, and by division, we get

$$\text{A}n = \frac{264 + 638 \times 0.27564}{56} = 7.854 \text{ inches},$$

$$\text{BS} = \frac{264 + 770 \times 0.17365}{56} = 7.102 \text{ inches};$$

but $n\text{S} = \text{AB} - (\text{A}n + \text{BS})$; therefore, we have

$$n\text{S} = 22 - 14.956 = 7.044 \text{ inches}.$$

Now, in order to find the point I, where the reduced equivalent of the power P is applied, we have to divide the line nS into two parts, nI and SI, which are to each other reciprocally as GH and ER, the reduced resistances; thus,

$$\text{GH} + \text{ER} : n\text{S} :: \text{ER} : n\text{I},$$
$$\text{GH} + \text{ER} : n\text{S} :: \text{GH} : \text{SI}.$$

Substituting for GH and ER their values $\frac{\beta \text{R}}{2d} \sin. \varphi$, and $\frac{\beta r}{2d} \sin. \varphi'$, it is

$$\frac{\beta}{2d}(\text{R} \sin. \varphi + r \sin. \varphi') : \frac{\beta r}{2d} \sin. \varphi' :: n\text{S} : n\text{I} = \frac{r \sin. \varphi' \times n\text{S}}{\text{R} \sin. \varphi + r \sin. \varphi'}$$

$$\frac{\beta}{2d}(\text{R} \sin. \varphi + r \sin. \varphi') : \frac{\beta \text{R}}{2d} \sin. \varphi :: n\text{S} : \text{SI} = \frac{\text{R} \sin. \varphi \times n\text{S}}{\text{R} \sin. \varphi + r \sin. \varphi'}$$

Consequently, by restoring the numerical values of these several quantities, these expressions for nI and SI will become

$$n\text{I} = \frac{7.044 \times 70 \sin. 80^\circ}{58 \sin. 74^\circ + 70 \sin. 80^\circ} = \frac{7.044 \times 68.94}{124.69} = 3.89 \text{ inches},$$

$$\text{SI} = \frac{7.044 \times 58 \sin. 74^\circ}{58 \sin. 74^\circ + 70 \sin. 80^\circ} = \frac{7.044 \times 55.75}{124.69} = 3.15 \text{ inches}.$$

Therefore, by addition, we have

$$\text{AI} = \text{A}n + n\text{I} = 7.854 + 3.89 = 11.744,$$
$$\text{BI} = \text{BS} + \text{SI} = 7.102 + 3.15 = 10.252.$$

Having thus determined the position of the point I, with respect to the points A and B, it only now remains to assign the position of the point m; now

$$Im = mQ \cos. 80°;$$

but it has already been shown, that

$$mQ = 24.87;$$

therefore, we obtain

$$Im = 24.87 \cos. 80° = 4.32.$$

But because aK and bL, are taken from a scale of only half the dimensions of that from which the sides of the triangle are measured, it is manifest, that the true value of Im is only half of what we have determined above; that is,

$$Im = 2.16 \text{ inches};$$

consequently, we have

$$Am = 11.744 + 2.16 = 13.904,$$
$$Bm = 10.252 - 2.16 = 8.096.$$

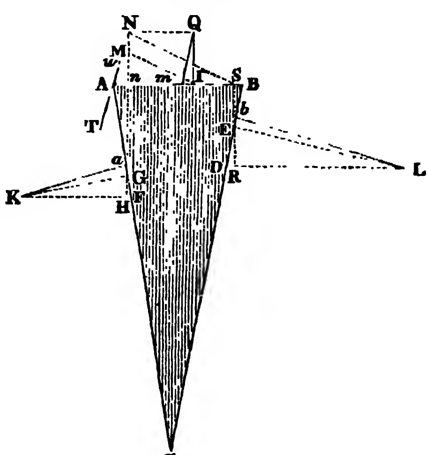
Such, then, is the method of finding the point at which the power is applied, when we know the points at which the resistance acts.

EXAMPLE 2. A certain power acting on the back of an isosceles wedge, in an angle of 75° 15′, is found to balance two resistances acting on its sides in angles of 84 and 86 degrees; what is the magnitude or intensity of the power, supposing the resistances to be respectively equivalent to pressures of 6000 and 10000 lbs., the side of the wedge being 108 inches, and its back 36 inches?

This example is constructed as follows.

Let ABC be a transverse section of the wedge, perpendicular to the axis of the triangular isosceles prism of which the wedge is constituted; and let the sides AC and BC be each equal to 108 inches, and AB equal to 36 inches.

Fig. 10



In AC and BC, the sides of the wedge, take a and b any points at pleasure, and at the points a and b make the angles FaK and DbL, equal respectively to 84 and 86 degrees, the given directions of the resistances R and r.

Make aK and bL respectively equal or proportional to 6000 and 10000, the numbers that express the measures of the resistances R and r, applied at the points a and b.

Through the points K and L draw KF, LD parallel to AB, and KG, LE perpendiculars to AC and BC; from the points G and E, demit the perpendiculars GH and ER, and produce HG and RE to meet AB, the back of the wedge, in the points n and S; then are GH and ER the reduced equivalents of the resistances R and r, or those portions of them that become effectual in counteracting the power.

Continue Hn directly forward to N, making nM equal to ER, and MN equal to GH; join NS, and through the point M draw MI parallel to NS, meeting AB the back of the wedge in I, the point where IQ the reduced equivalent of the power P must be applied. Through the point A, draw the indefinite right line Tw, making with AB the angle BAw equal to 75° 15′, the angle which the direction of the power makes with the back of the wedge; then through the point Q (IQ being equal and parallel to nN), draw Qm parallel to Tw, meeting AB in m, the point where the power P is applied, when the wedge is kept in equilibrio by the three forces Qm, Ka and Lb acting at the points m, a and b, and in those directions.

If Qm and QI be each taken in the compasses, and applied to the same scale as Ka and Lb were measured from, they will indicate as under.

$$Qm = 2747\cdot67 \text{ lbs.}$$
$$QI = 2657\cdot12 \text{ lbs.}$$

The numerical process for this example is as follows.

Resistance R $=6000$	. . .	log. .	3·778151
Direction of R $=84°$	. . . .	log. sin.	9·997614
Nat. number $=5967\cdot12$	. .	log. .	3·775765
Resistance r $=10000$	. .	log. .	4·000000
Direction of r $=86°$	. . . .	log. sin.	9·998941
Nat. number $=9975\cdot64$	. .	log. .	3·998941

Consequently, we have

KG+LE$=5967\cdot12+9975\cdot64$	$=15942\cdot76$	log. .	4·202563
Value of the back β	$=36$ . .	log. .	1·556303
Direction of P	$=75°\ 15'$.	log. cosec.	0·014553
Value of twice d	$=216$ . .	ar. co. log.	7·665546
Natural number	$=2747\cdot67$	log. .	3·438965

Therefore, the magnitude mQ$=$P the oblique power, being 2747·67 lbs., as determined above; the magnitude of IQ$=p$ the reduced equivalent, is

$$IQ = p = 2747\cdot67 \sin. 75°\ 15' = 2657\cdot12 \text{ lbs.}$$

In this last example, it would be difficult to assign the *positions* of the points m and I with respect to A and B, the extremities of the back of the wedge; indeed, since we are at liberty to assume the points a and b anywhere in AC and BC, it is easy to perceive, that the positions of m and I vary according to the positions of a and b, and therefore cannot be determined till the places of a and b have been ascertained.

If we recur to the original equation of equilibrium (c), and conceive one of the segments of the vertical angle as b to vanish; then the member r sin. b sin. φ', in which sin. b occurs, must also vanish, and the expression for the conditions of equilibrium becomes

$$P \sin. \pi = R \sin. a \sin. \varphi. \qquad (l)$$

The triangular prism of which the wedge is constituted, is in this case right angled, and the wedge is therefore rectangular.

If, moreover, we conceive the angles of direction φ and π each to become equal to 90 degrees or a right angle; then equation (l) becomes

$$P = R \sin. a. \qquad (m)$$

Again, if we suppose the angles of direction φ, φ' and π to be each equal to 90 degrees, then the general equation (c) becomes

$$P = R \sin. a + r \sin. b. \qquad (n)$$

And if the angles a and b become equal to one another, we have from equation (n)

$$P = \sin. a (R + r). \qquad (o)$$

And moreover, if the resistances R and r are equal to one another, then equation (o) becomes

$$P = 2R \sin. a. \qquad (p)$$

The equations (m), (n) and (o) express the relation that subsists between the power and the resistances, in the case of a *rectangular*, a *scalene* and an *isosceles* wedge, when the direction of the forces are perpendicular to the sides of the wedge, on which they respectively act.

Equation (p) expresses the same thing for the isosceles wedge, when the resistances are supposed to be equal.

If the resistances are equal in the scaline wedge, while their directions are perpendicular to the sides; then equation (n) becomes

$$P = R (\sin. a + \sin. b). \qquad (q)$$

If the angle $\pi = 90$, while a and b are respectively equal to φ and φ'; then it is manifest, that the power and the resistances all

act at right angles to the back of the wedge, in which case equation (c) becomes

$$\text{P} = \text{R}\sin^2 a + r\sin^2 b. \qquad (r)$$

This equation applies to the scalene wedge; but if the angles a and b are supposed equal to one another, then equation (r) becomes

$$\text{P} = \sin^2 a(\text{R} + r). \qquad (s)$$

And finally, if the resistances R and r are equal, then equation (s) becomes

$$\text{P} = 2\text{R}\sin^2 a. \qquad (t)$$

The illustration of all these equations, from (l) to (t) both inclusive, is left for exercise to the reader.

Procumbunt piceæ: sonat icta securibus ilex:
Fraxineæque trabes, cuneis et fissile robur
Scinditur: advolvunt ingentes montibus ornus.

Æn. lib. VI.

The reader will see in all that we have done, how the road to abstract science may be smoothed; but he may rest assured that any popular version thereof is quite illusory, for no portion of sound knowledge was ever acquired without some corresponding exertion of mind; and in all mechanical operations and mathematical investigations, men must be as clever with their hands as ingenious with their heads. In calculations, as in the practice of every art, learning and judgment are of little avail unless accompanied by manual dexterity. It is one of the improvements to be made in our systems of education for the various professions, if we would strive to retrieve the declining taste for science, that students in Mechanics, as we have written for their use, should cultivate more methodically the use of their hands, and work efficiently for themselves in the profitable but toilsome drudgery of computation.

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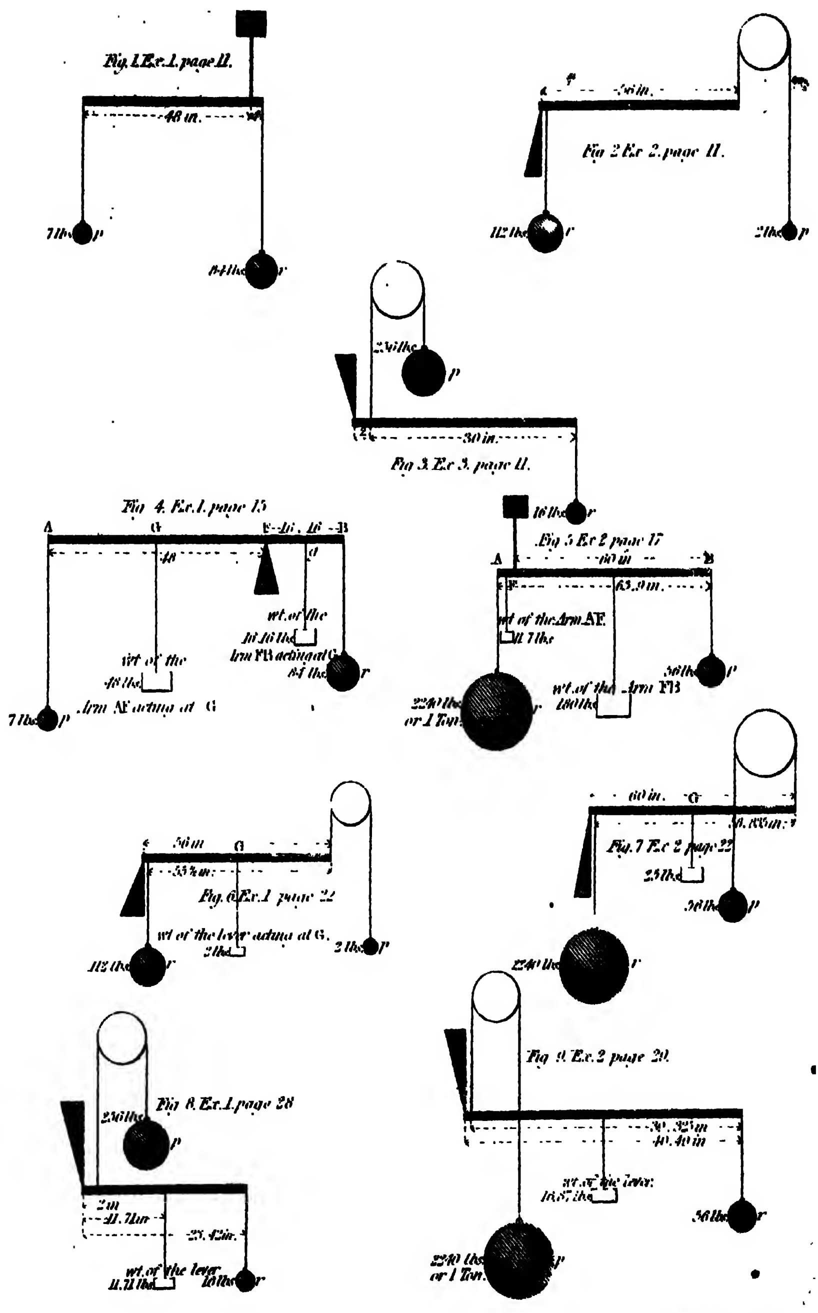
MECHANICAL POWERS.

Plate 1



J. W. Lowry sculp.

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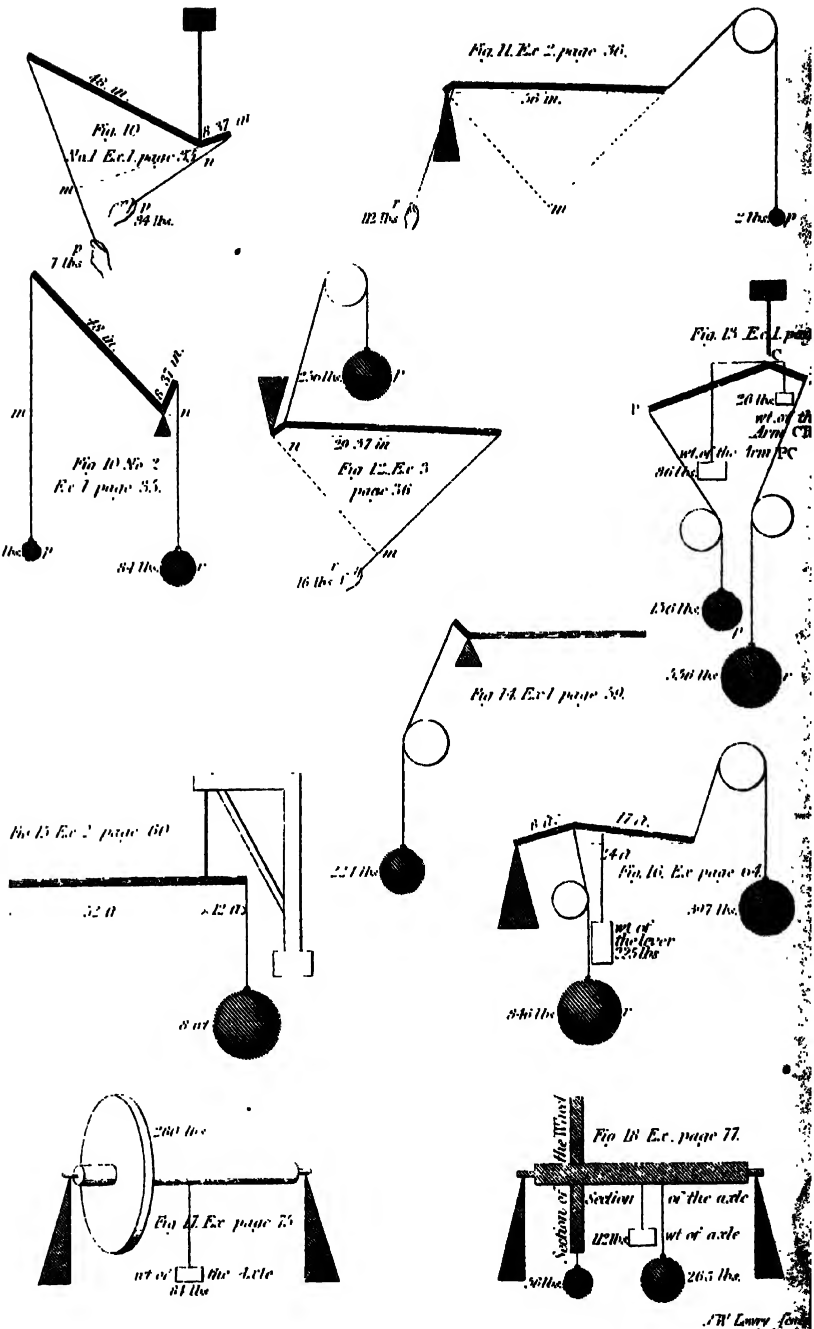
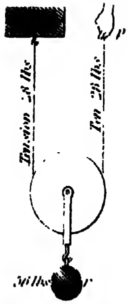
MECHANICAL POWERS.

Plate



MECHANICAL POWERS.



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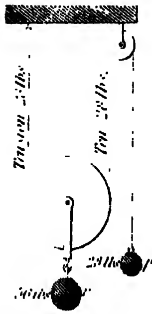

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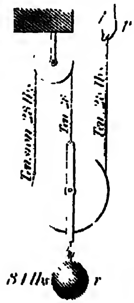

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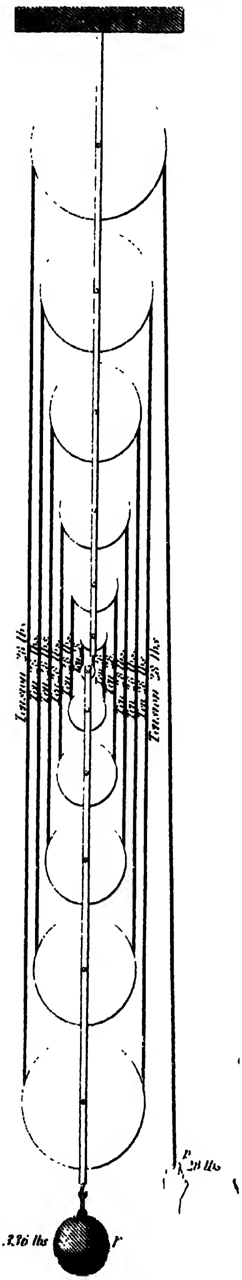

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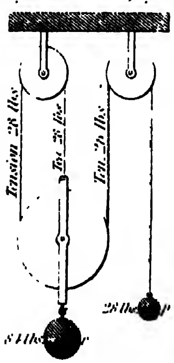

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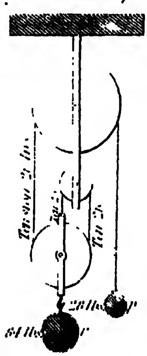

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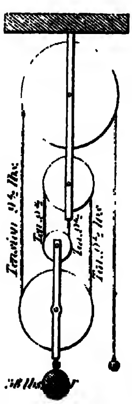

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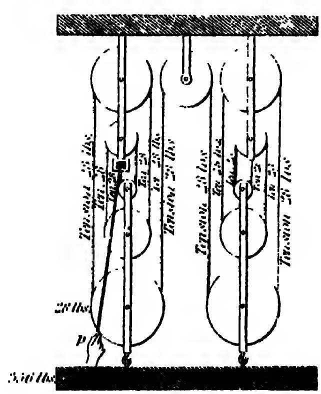
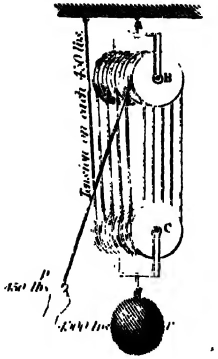


Fig 22 N° 2 Ex 2 page 96

J.W. Lowry Sculp

Fig. 23 Ex. 1 page 98



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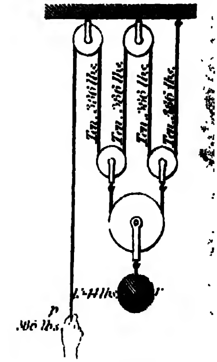

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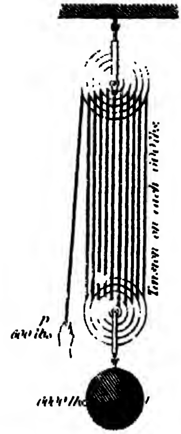

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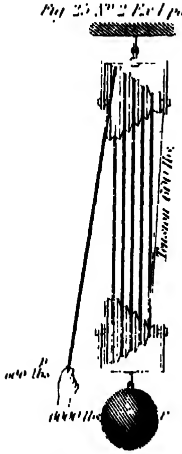

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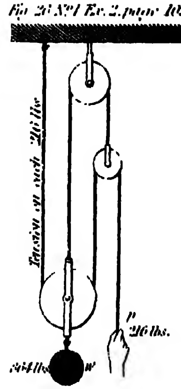

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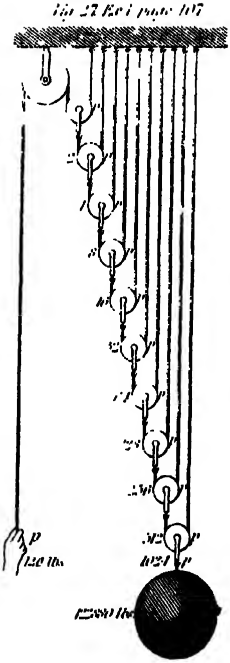
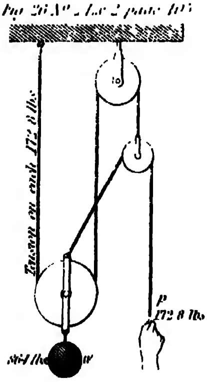


Fig. 26 No. 2 Ex. 2 page 105



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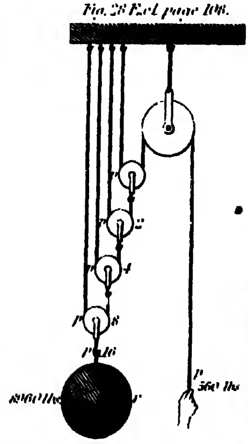
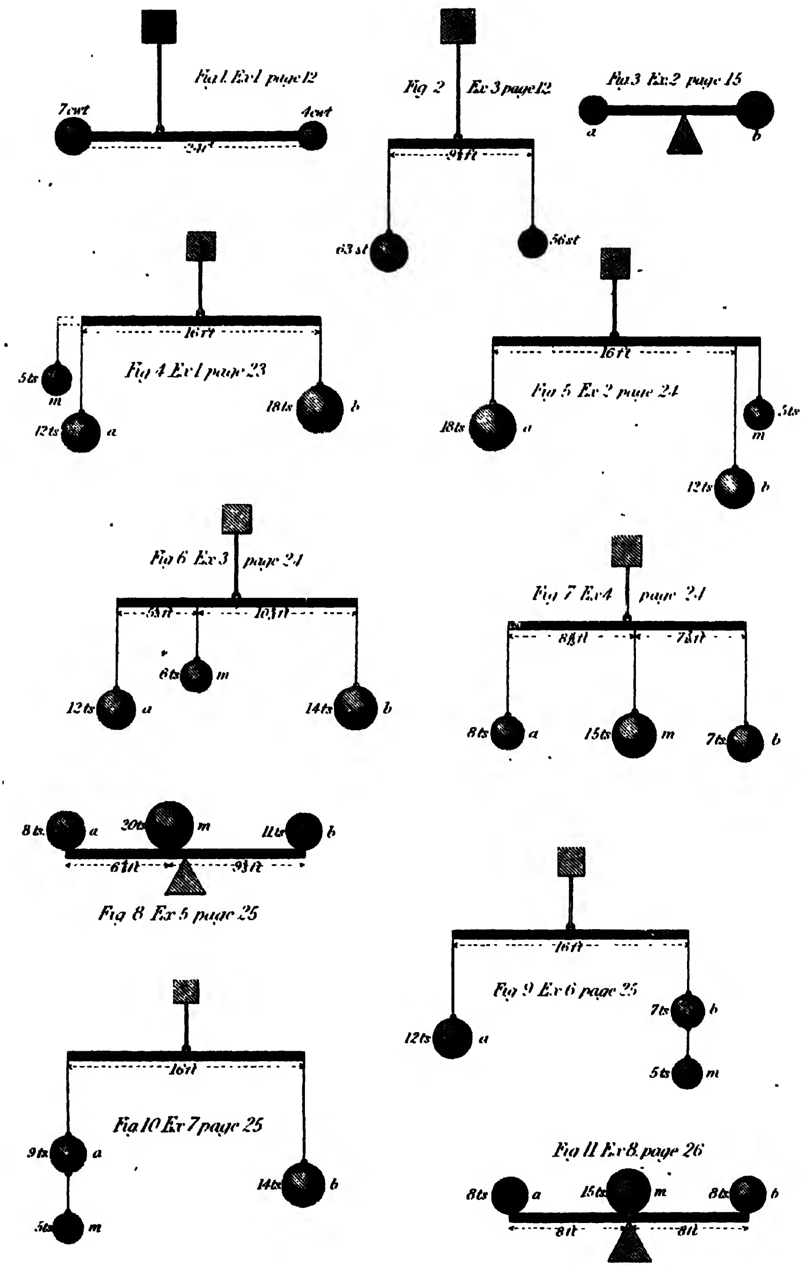


J. W. Lowry, delt

CENTRE OF GRAVITY OF PLANES AND SOLIDS.

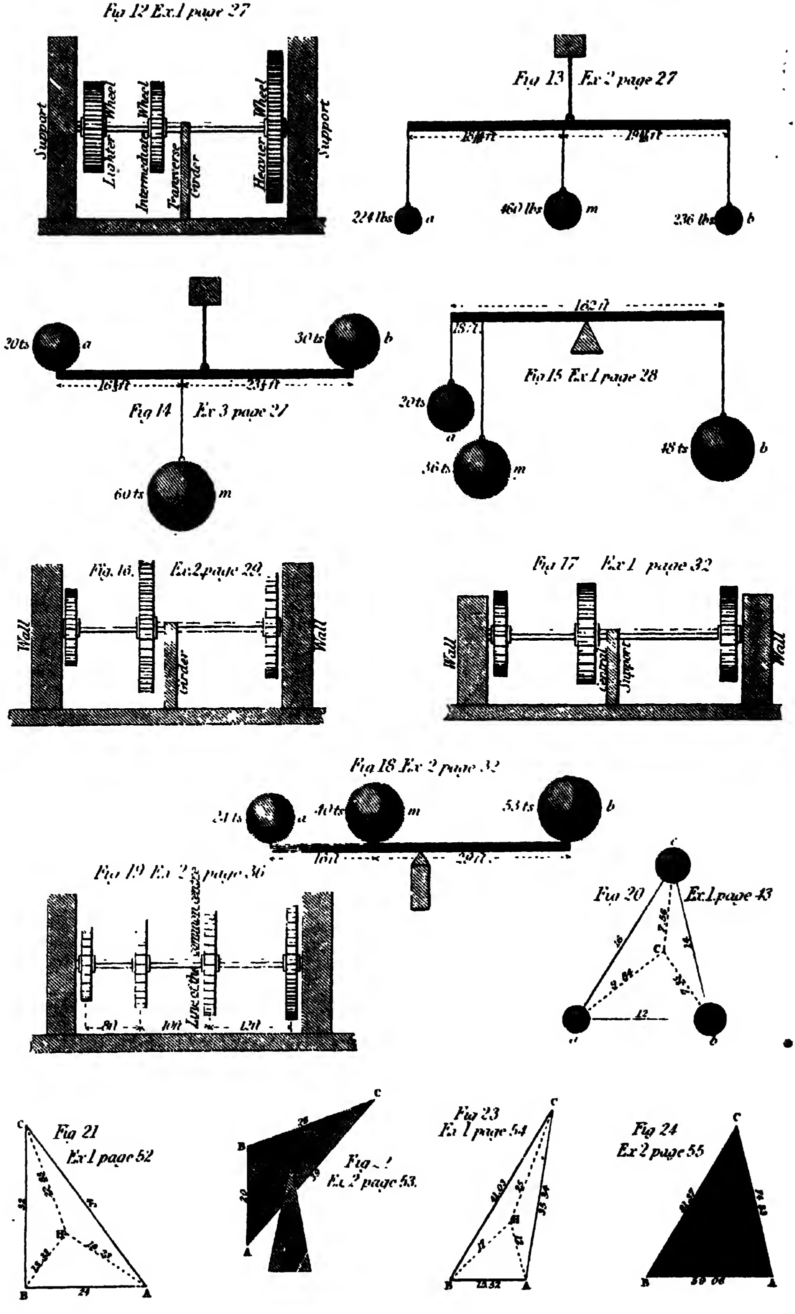
PLATE 1



J W Lowry, sculp.

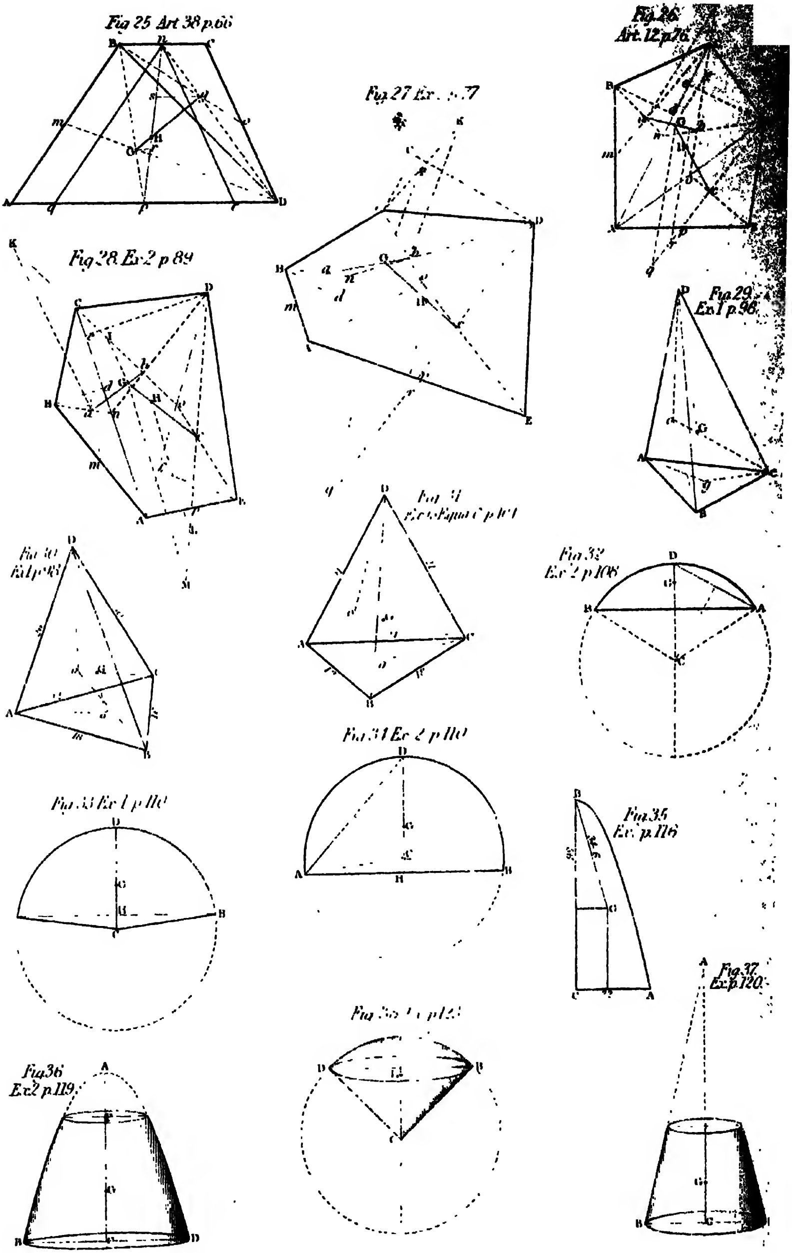
CENTRE OF GRAVITY OF PLANES AND SOLIDS.

PLATE 2

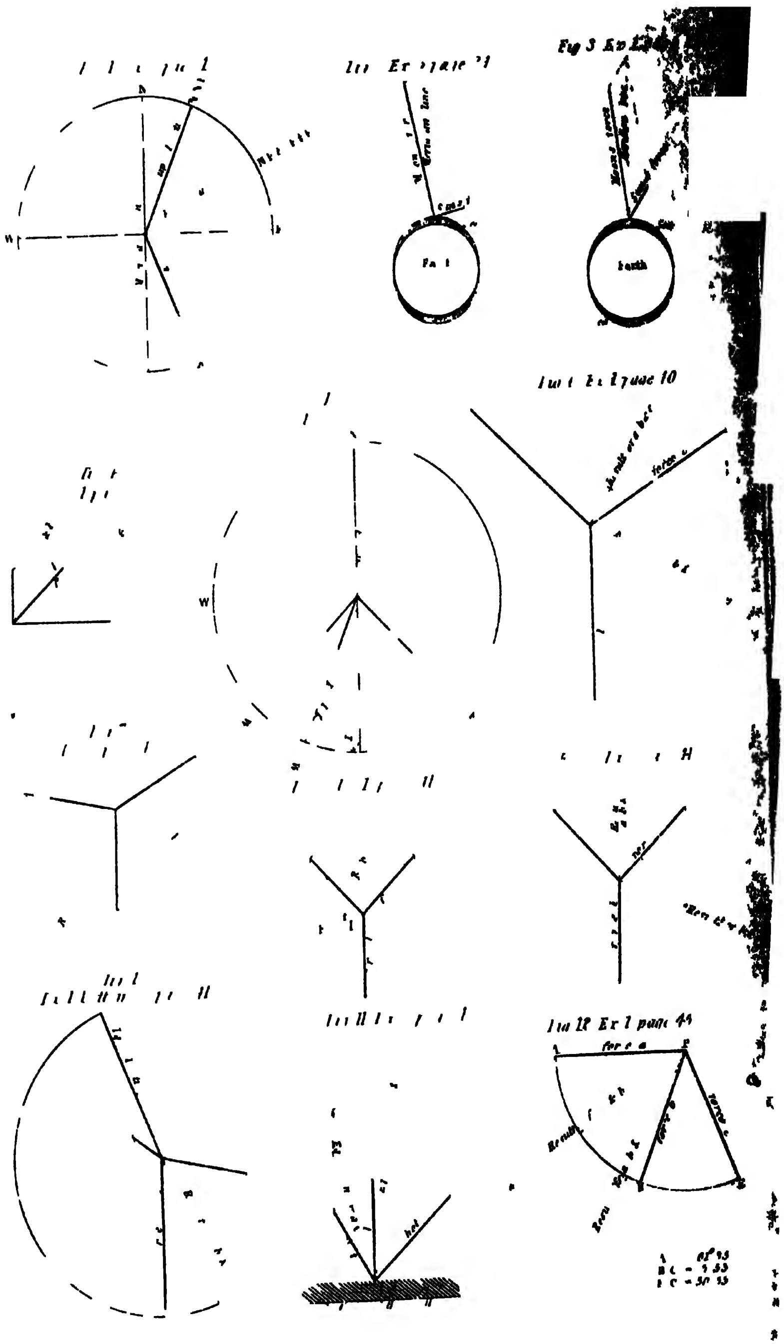


J W Lowry, sculp

CENTRE OF GRAVITY OF PLANES AND SOLIDS



PARALLELOGRAM OF FORCES.



PARALLELOGRAM OF FORCES.

Plate 2.